Mixed Intersection of Cevians and Perspective Triangles

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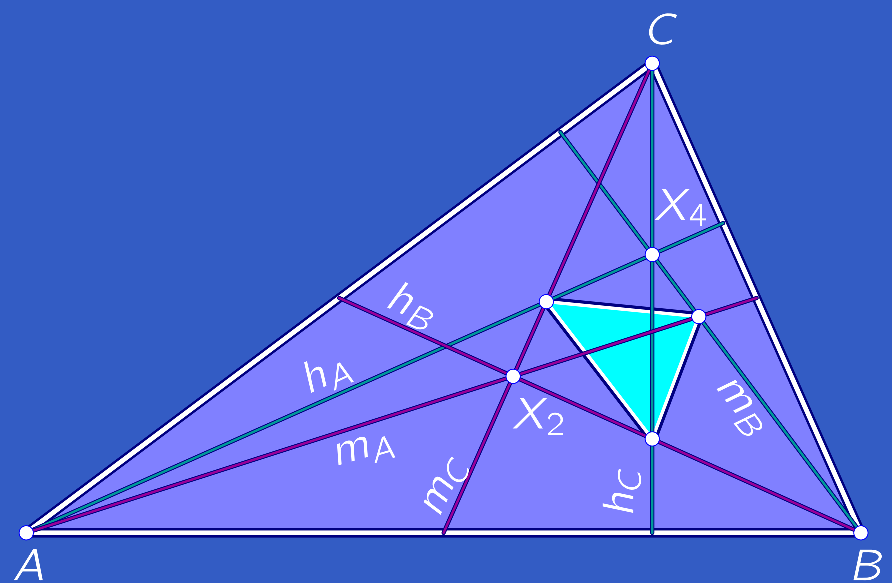
A simple construction - or a mistake?



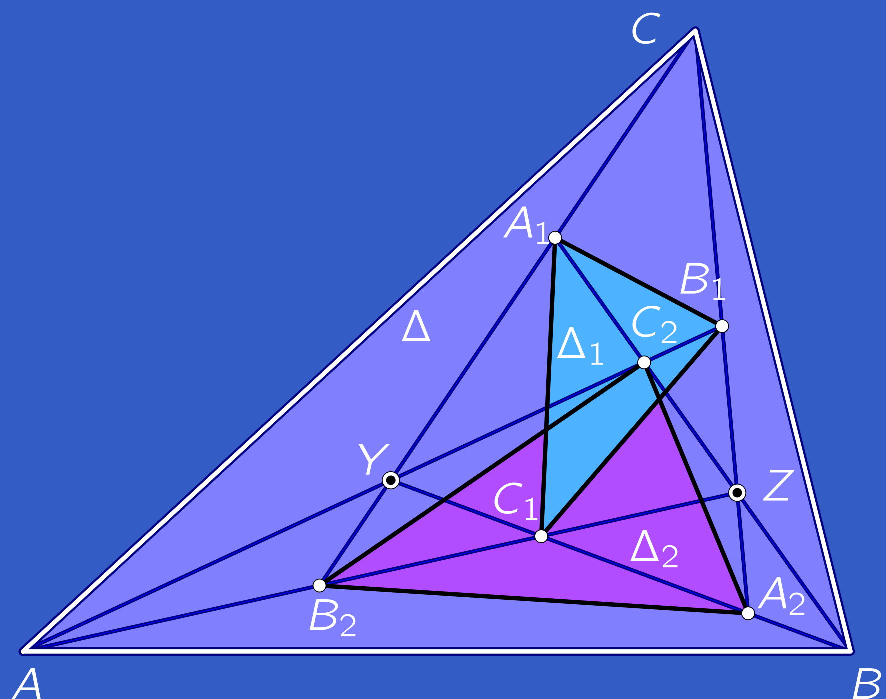
Triangle centers appear as intersections of certain Cevians.

What happens if we mix them?

Arbitrarily chosen Cevians of different centers do not concur in a single point!

X_i is the Kimberling notation for the i-th center.

No center - what now?



Y, Z . . . two points

define $\Delta_1 = (A_1, B_1, C_1)$

$A_1 = [Y, C] \cap [Z, B]$ cyclic

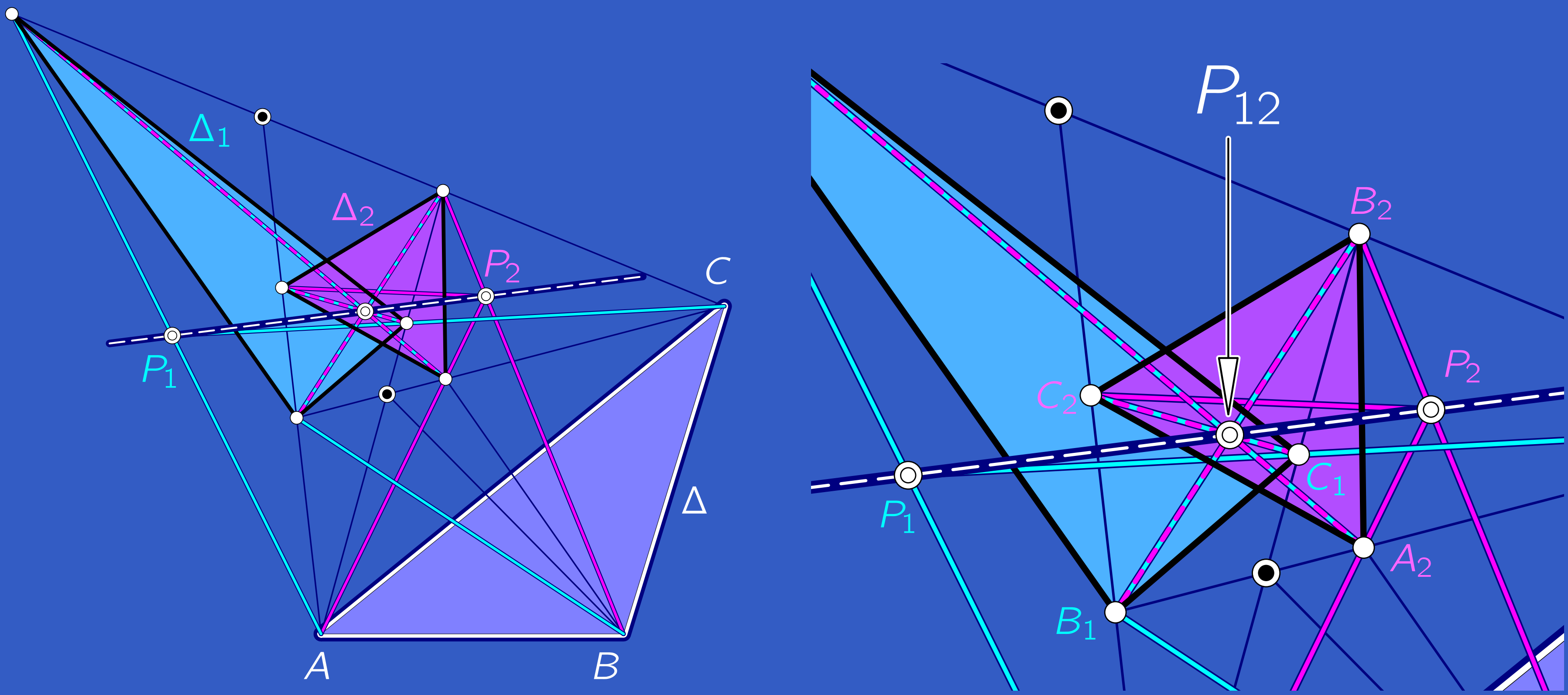
define $\Delta_2 = (A_2, B_2, C_2)$

$A_2 = [Y, B] \cap [Z, C]$ cyclic

Both Δ_1 and Δ_2 are obviously perspective to Δ.

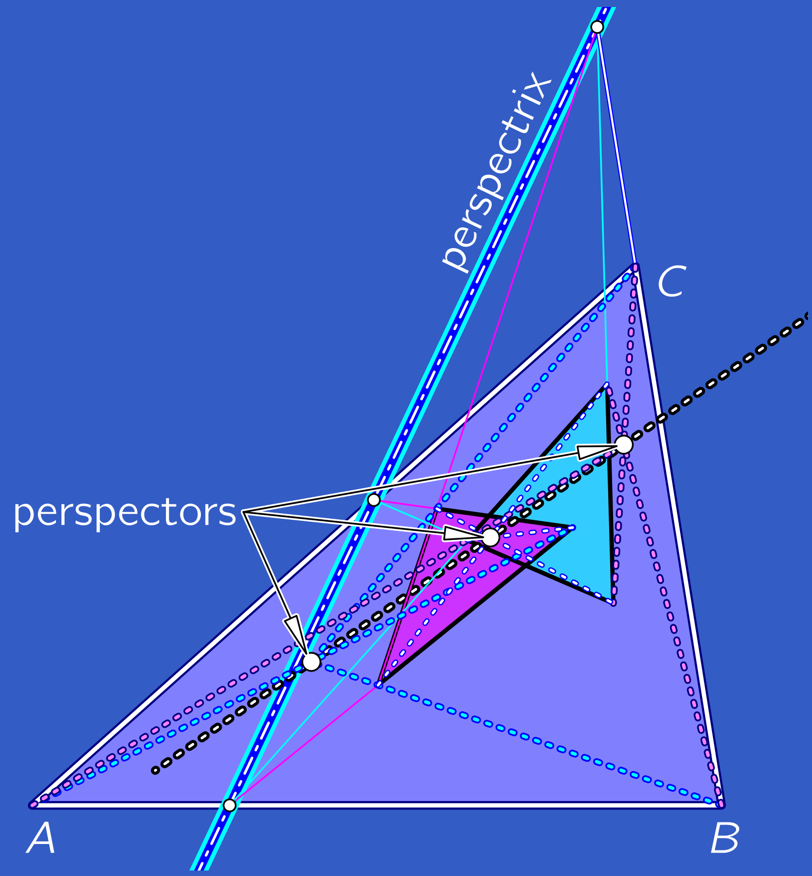
Theorem 1. $\Delta_1 \overset{P_1}{\underset{\wedge}{=}} \Delta, \quad \Delta_2 \overset{P_2}{\underset{\wedge}{=}} \Delta, \quad \Delta_1 \overset{P_{12}}{\underset{\wedge}{=}} \Delta_2,$
properly labeled: $A \longleftrightarrow A_1, \ldots$

Three collinear perspectors



Theorem 2. If Y and Z are triangle centers of Δ then P_1, P_2, and P_{12} are collinear centers of Δ.

Three perspectors and only one perspectrix

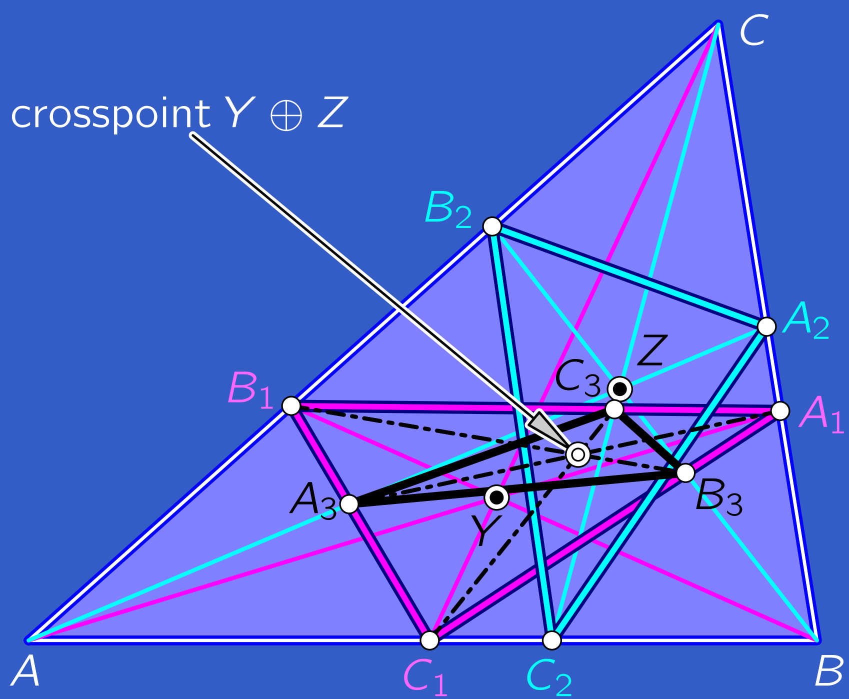


Desargues. If two triangles have a center of perspectivity then they also have a perspectrix.

Theorem 3. Any pair out of Δ, Δ_1, Δ_2 has the same perspectrix which is a central line.

$\Longrightarrow$ We obtain a closed chain of Desargues $(10_3, 10_3)$-configurations.

The crosspoint of Y and Z ...



Cevian triangle of Y

$\Delta_1 := (A_1, B_1, C_1)$

Cevian triangle of Z

$\Delta_2 := (A_2, B_2, C_2)$

define $\Delta_3 := (A_3, B_3, C_3)$

$A_3 \;:=\; [A, A_2] \cap [B_1, C_1]$

cyclic

Theorem 4. The crosspoint $Y \oplus Z$ of Y and Z is the perspector of Δ_1 and Δ_3. (Y, Z not necessarily centers)

Theorem 5. P_{12} is the crosspoint of Y and Z.

Proof. Let $Y = (\xi_0 : \xi_1 : \xi_2)$, $Z = (\eta_0 : \eta_1 : \eta_2)$ and compare the trilinear representations

$$P_{12} = (\xi_0\eta_0(\xi_1\eta_2 + \xi_2\eta_1) : \ldots : \ldots) = Y \oplus Z,$$

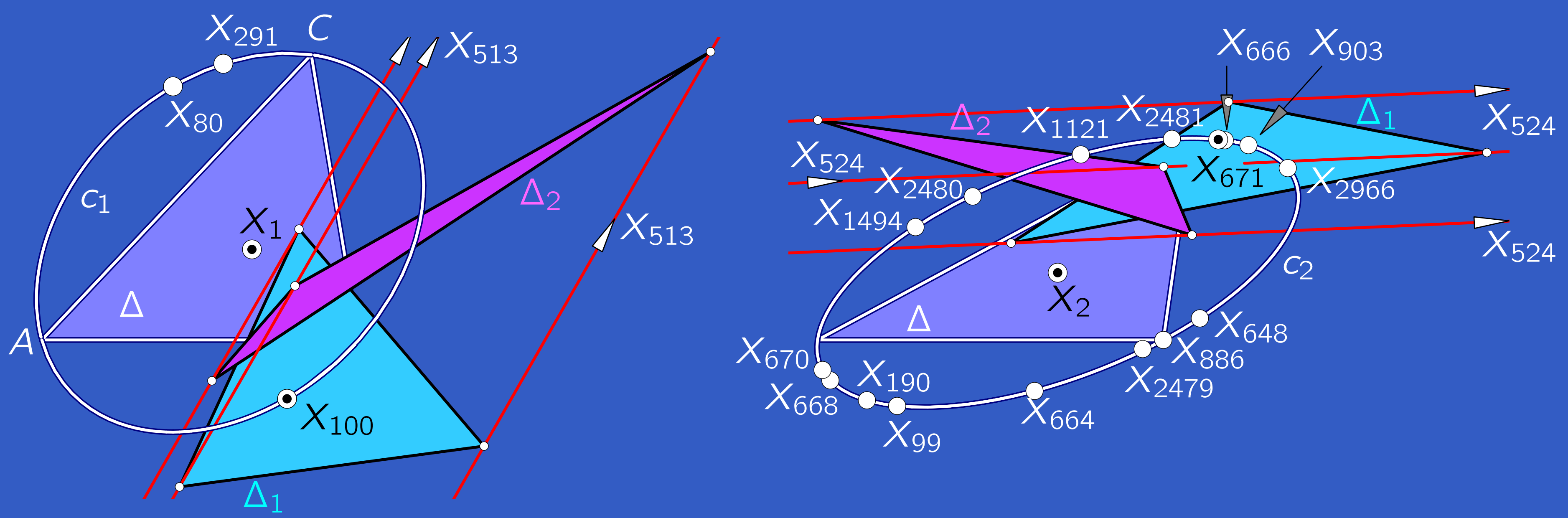
cf. C. Kimberling, ECT.

Note: The Cevian triangles do not appear in the construction of the perspectors P_1, P_2, and P_{12}.

$\oplus$-compositions of some centers yields a tool for an elementary construction of X_i with large i.

$\oplus$	1	2	3	4	5	6	7	8	9	10
1	1	37	73	65	2599	42	354	3057	55	2292
2		2	216	6	233	39	1	9	1212	1213
3			3	185	$\star$	184	$\star$	$\star$	$\star$	$\star$
4				4	3574	51	1836	1837	1864	1834
5					5	$\star$	$\star$	$\star$	$\star$	$\star$
6						6	$\star$	$\star$	2347	$\star$
7							7	497	$\star$	4854
8								8	210	$\star$
9									9	$\star$

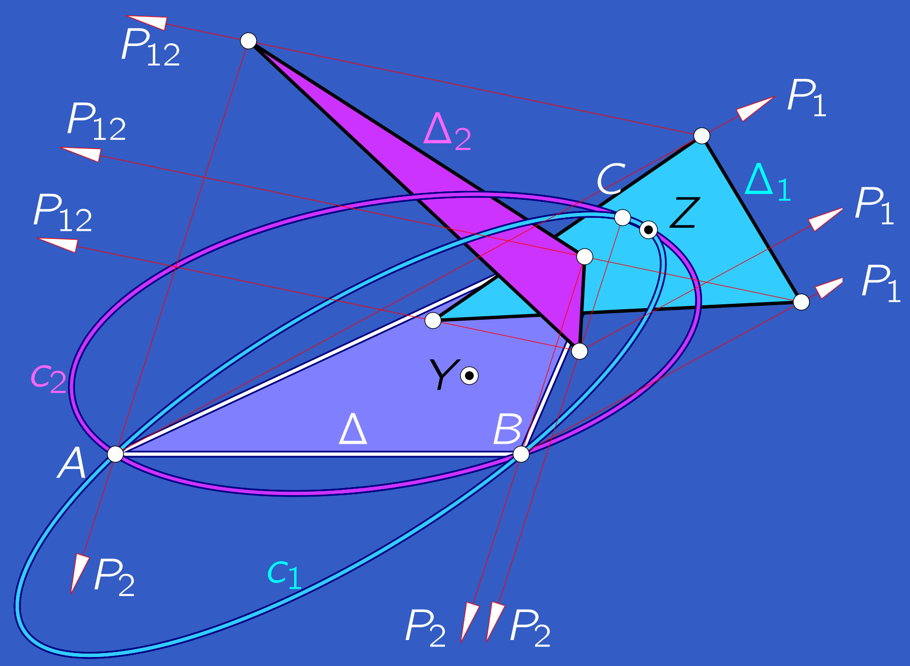
Special affine versions Desargues configurations



$$X_1 \oplus X_{100} = X_{513} \qquad X_2 \oplus X_{671} = X_{524}$$

c_2 . . . Steiner ellipse

For a fixed center Y there is a circumconic c_Y of points Z such that $P_{12} = Y \oplus Z$ is an ideal point.



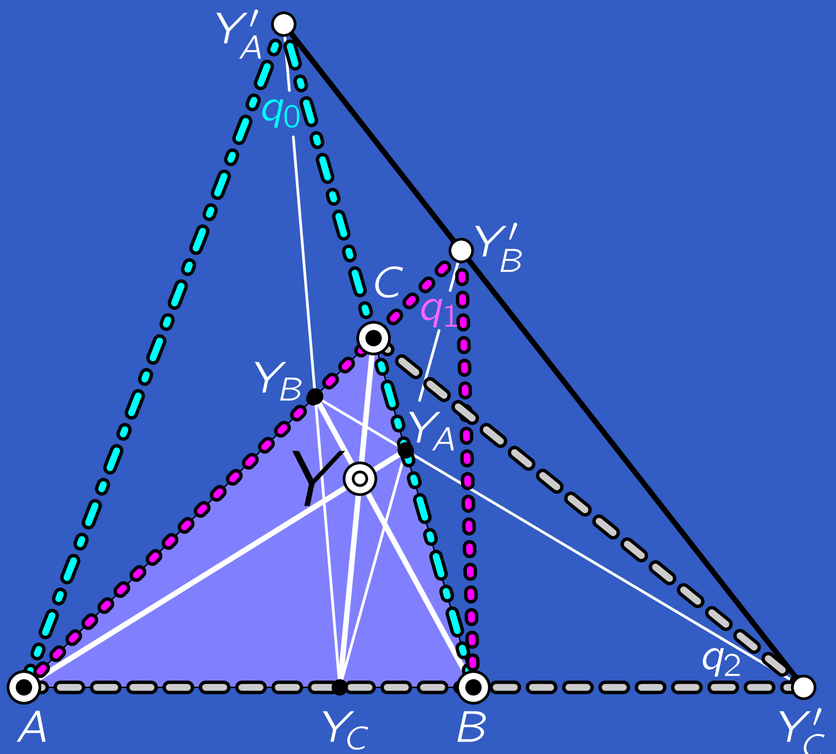
Theorem 6. For any fixed center Y there exists exactly one center Z such that P_1, P_2, and P_{12} are ideal points.

Proof. For fixed Y the set of Z such that P_1/P_2 is a circumconic c_1/c_2 of Δ.

These conics share A, B, C and a center of Δ

$$Z = \left(\frac{\xi_0}{\xi_0^2 a^2 - \xi_1 \xi_2 bc} : \ldots : \ldots \right).$$

A quadratic Cremona transformation q



For fixed Y the coordinate representation of P_{12} yields a QCT with basepoints A, B, C and exceptional lines $[A, B]$, $[B, C]$, $[C, A]$. Fundamental conics q_i are pairs of lines:

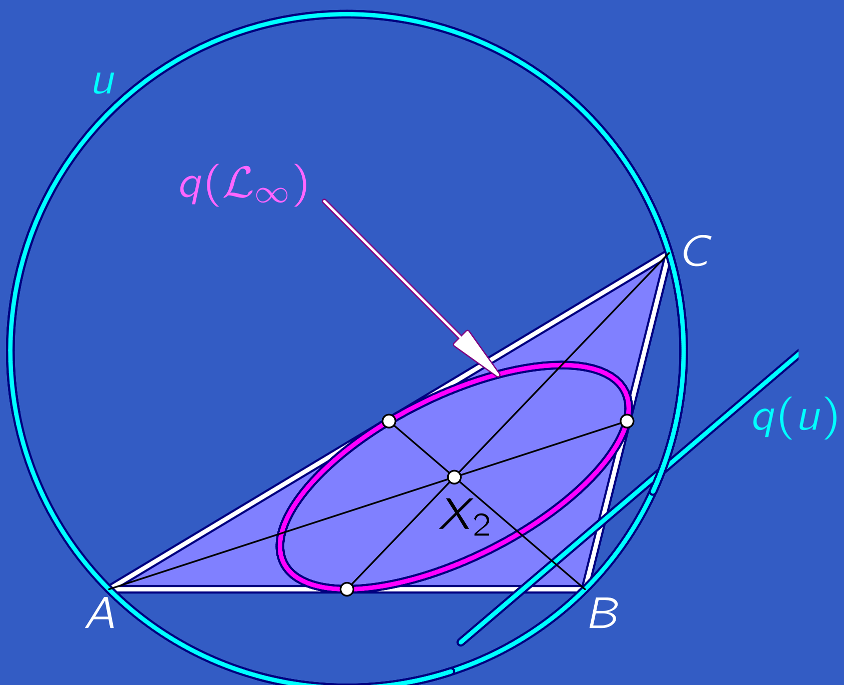
$q_0 = [B, C] \cup [A, Y'_A]$ with Y'_A as harmonic conjugate of Y_A w.r.t. (B, C). (cyclic)

q is QCT for ...

it has three base points and it is a composition of the isogonal conjugation and a collineation:

$$P_{12} = \underbrace{\begin{bmatrix} 0 & \xi_0\xi_1 & \xi_0\xi_2 \\ \xi_0\xi_1 & 0 & \xi_1\xi_2 \\ \xi_0\xi_2 & \xi_1\xi_2 & 0 \end{bmatrix}}_{\text{fixed for fixed } Y} \cdot \underbrace{\begin{bmatrix} \eta_1\eta_2 \\ \eta_2\eta_0 \\ \eta_0\eta_1 \end{bmatrix}}_{\substack{\text{isogonal} \\ \text{conjugation}}} .$$

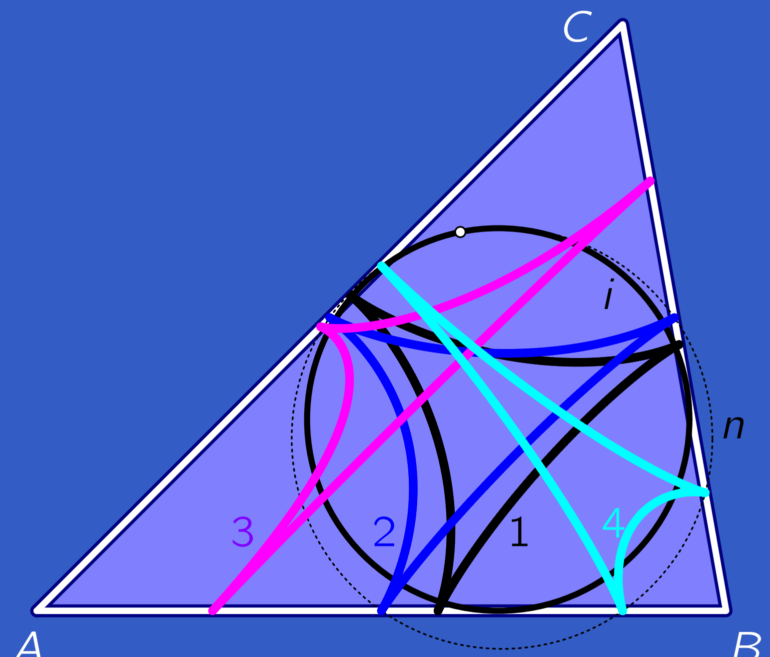
The action of q on . . .



- circumcircle $u \overset{q}{\mapsto}$ a collinear image of the line at infinity $\mathcal{L}_\infty$ (also a line)
- line at infinity $\mathcal{L}_\infty \overset{q}{\mapsto}$ Steiner inellipse, if $Y = X_2$

The QCT q maps central lines to central conics, and vice versa.

The action of q with different pivots Y on . . .



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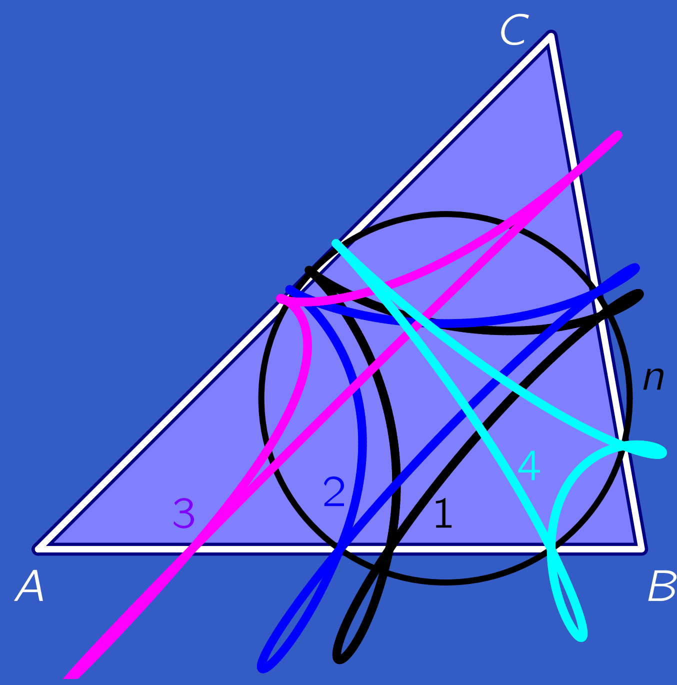

. . . the nine-point circle n.

Thank You For Your Attention!