Common Normals of Two Tori

Boris Odehnal

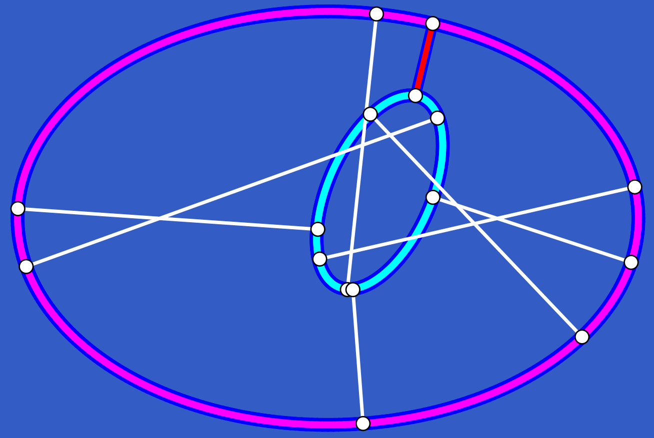
Dresden University of Technology

$$\text{dist}^2(X_1 - X_2) - \lambda_1 F_1(X_1) - \lambda_2 F_2(X_2) \longrightarrow \text{min / max}$$

$$\Longleftrightarrow$$

grad $F_1(X_1)$, grad $F_2(X_2)$ are linearly dependent if ∂O_i is described by an algebraic equation $F_i = 0$

$$\Longleftrightarrow$$



detect closest points of two objects O_1, O_2

$$\Longleftrightarrow$$

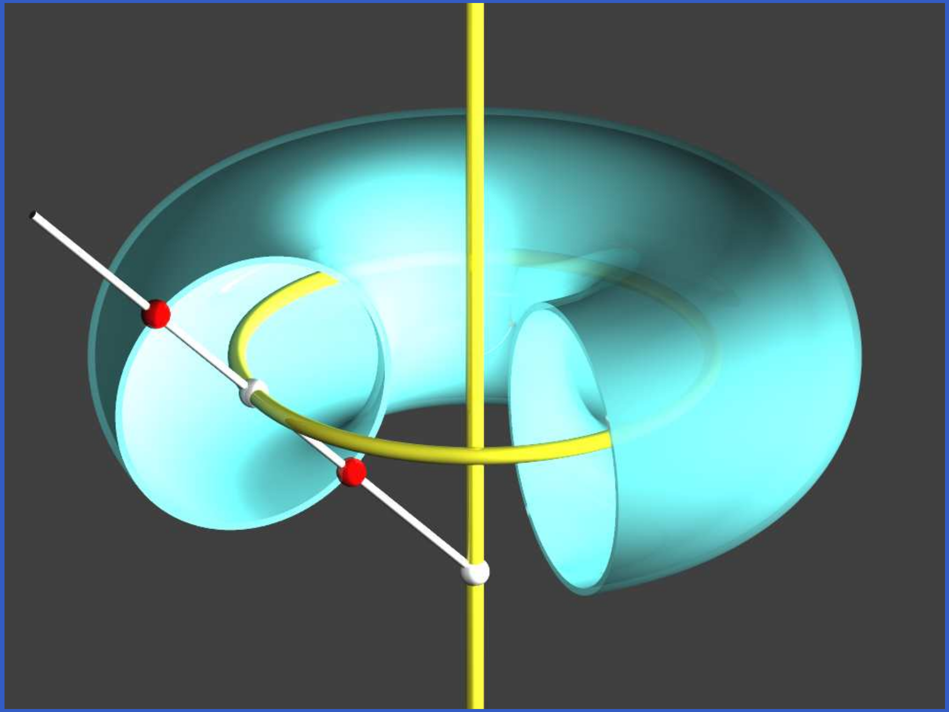
compute common normals

known algorithms

- efficient algorithms for polyhedral objects
- algorithms for rational tensor product surfaces
- replace objects with unconvenient representation by polyhedral approximations
- for ellipsoids: [Sun-Jüttler-Kim-Wang] used line geometric methods

$\Longrightarrow$ this method applies to any algebraic surface

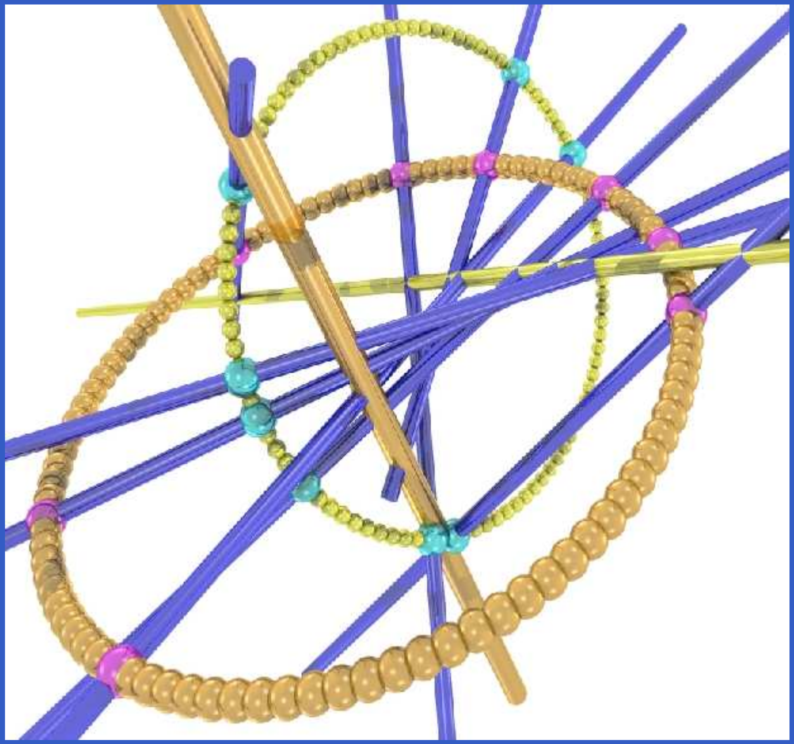
family of normals of a torus



Any torus has an axis and a spine curve.

Any surface normal of a torus meets the axis and the spine curve and is at least a **double normal**.

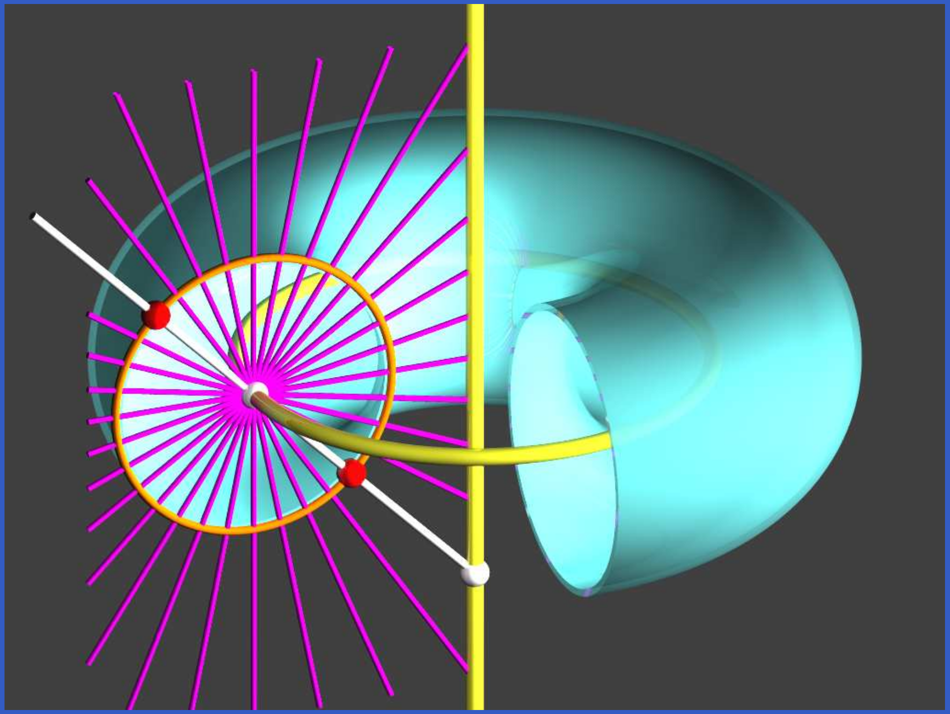
family of normals of a torus



$\Longrightarrow$ The common normals of two tori meet both axes and spines curves.

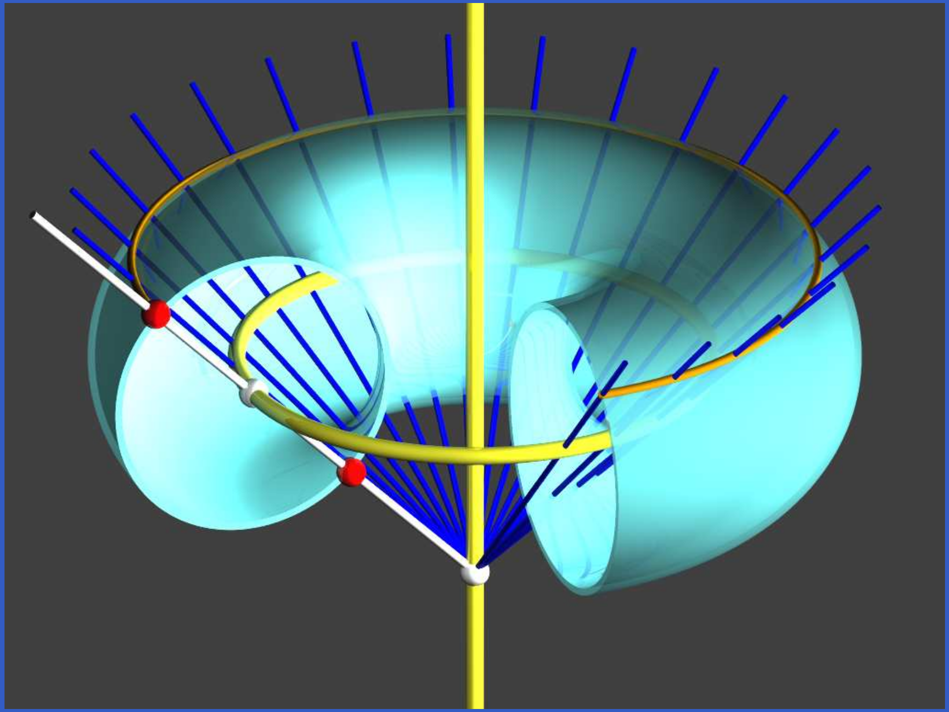
They are the common normals of the two circular spine curves, cf. [Zsombor-Murray-Hayes-Husty].

family of normals of a torus

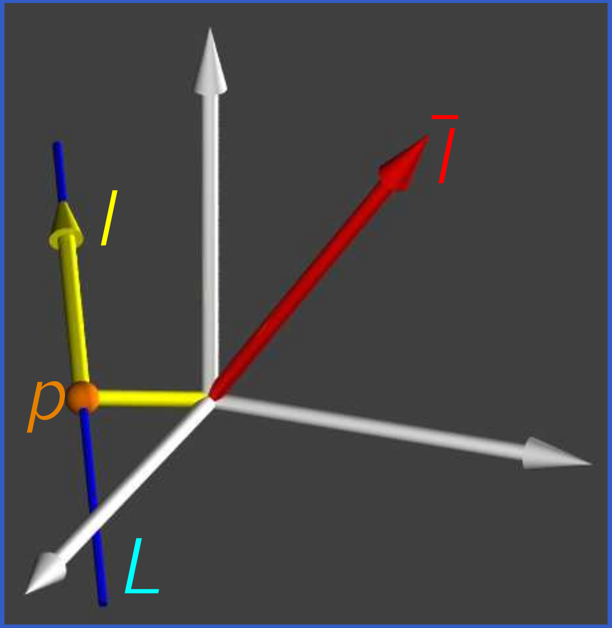


The surface normals along a meridian circle form a pencil of lines with vertex on the spine curve.

family of normals of a torus



The surface normals along a parallel circle form a cone of revolution with its vertex on the axis.



A point p and a direction (vector) l determine a line L.
Replace (p, l) by $(l, \bar{l})$ with

$$\bar{l} := p \times l$$

momentum vector $\bar{l} \perp [O, L]$.

Plücker coordinates $(l, \bar{l}) = (l_1, l_2, l_3; l_4, l_5, l_6)$ determine L, do not depend on p, and fulfil
$\langle l, \bar{l} \rangle = l_1 l_4 + l_2 l_5 + l_3 l_6 = 0.$

algebraic formulation - number of solutions

The Plücker coordinates of the torus normals in standard position with major radius R fulfil

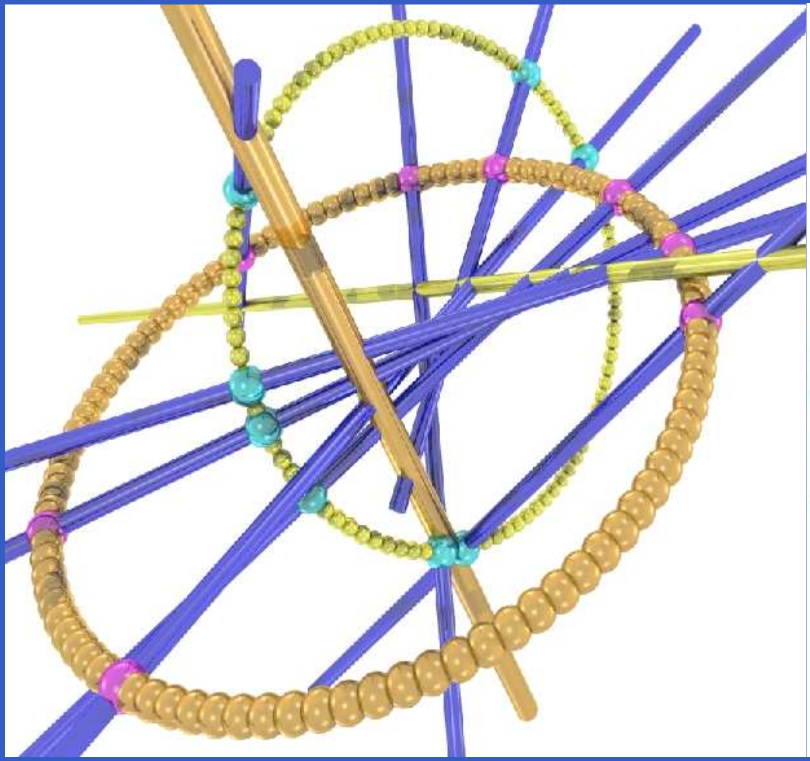
$$l_6 = 0 \quad \text{and} \quad l_4^2 + l_5^2 = R^2 l_3^2 .$$

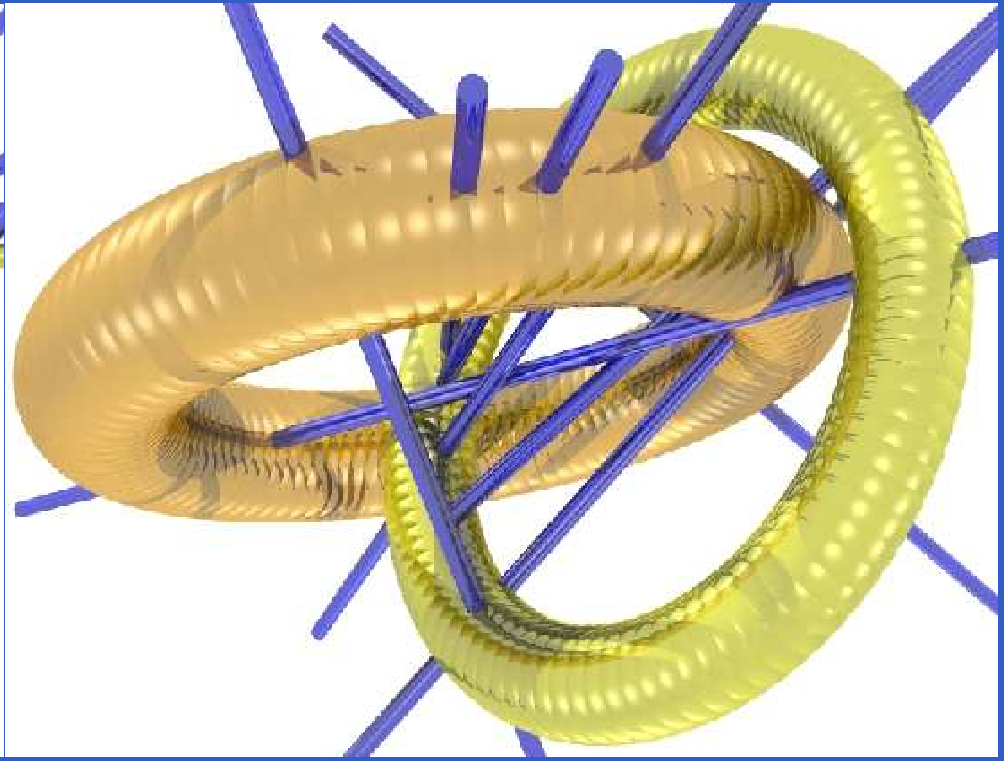
second torus in general position $\Longrightarrow$ further linear and quadratic quation for the common normals $\Longrightarrow$

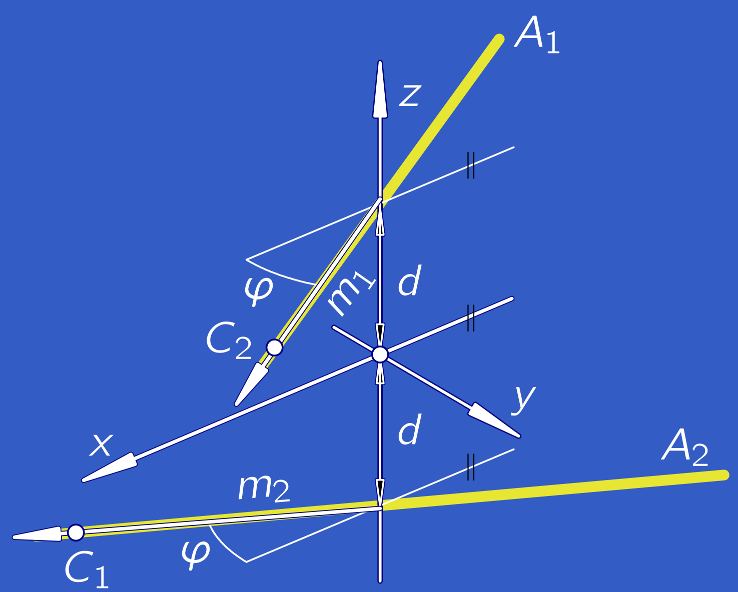
Theorem. If two tori have finitely many common normals then they have at most eight common normals.

$$\Longleftrightarrow$$

Theorem. If two circles have finitely many common normals then they have at most eight common normals.







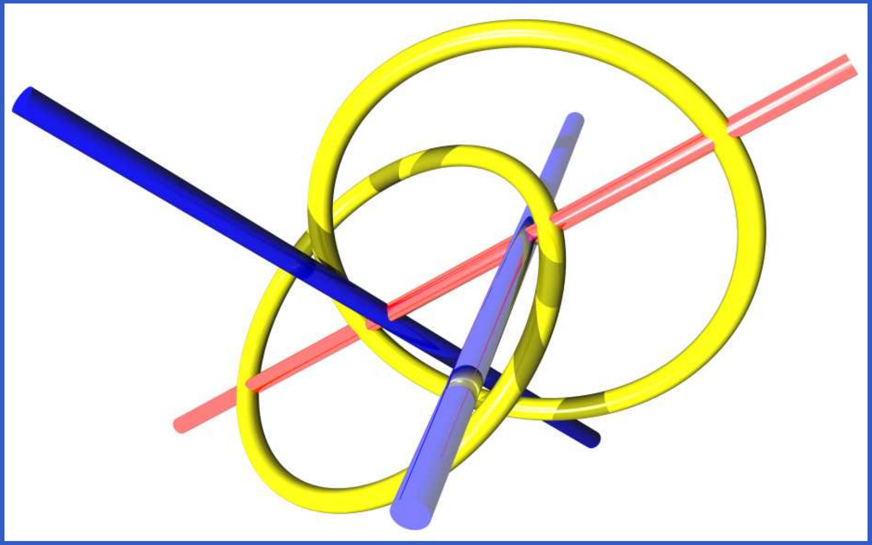
$\mathrm{dist}(A_1, A_2) = 2d$,
$\sphericalangle(A_1, A_2) = 2\phi$,
m_i offsets of centers C_i from feet of common perpendicle of A_i
Centers C_i of tori equal centers of spine curves.

Lines that meet A_1, A_2, and s_1 form a quartic ruled surface Φ_1. Lines that meet A_1, A_2, and s_2 form a quartic ruled surface Φ_2.

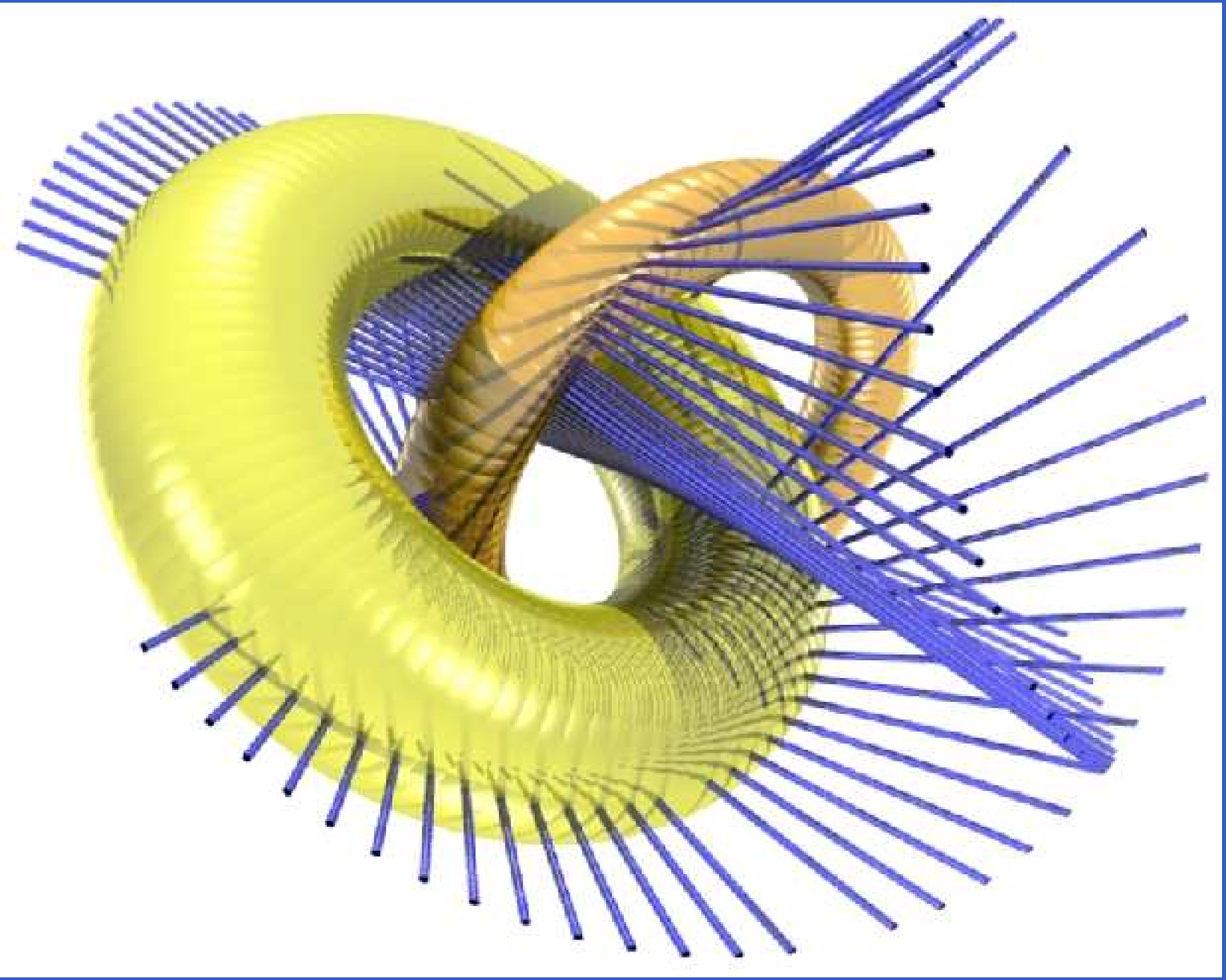
$\Phi_1 = \Phi_2 \iff$ infinitely many common normals

Theorem. Two tori with skew axes have infintely many common normals exactly if the axes and spine curves satisfy

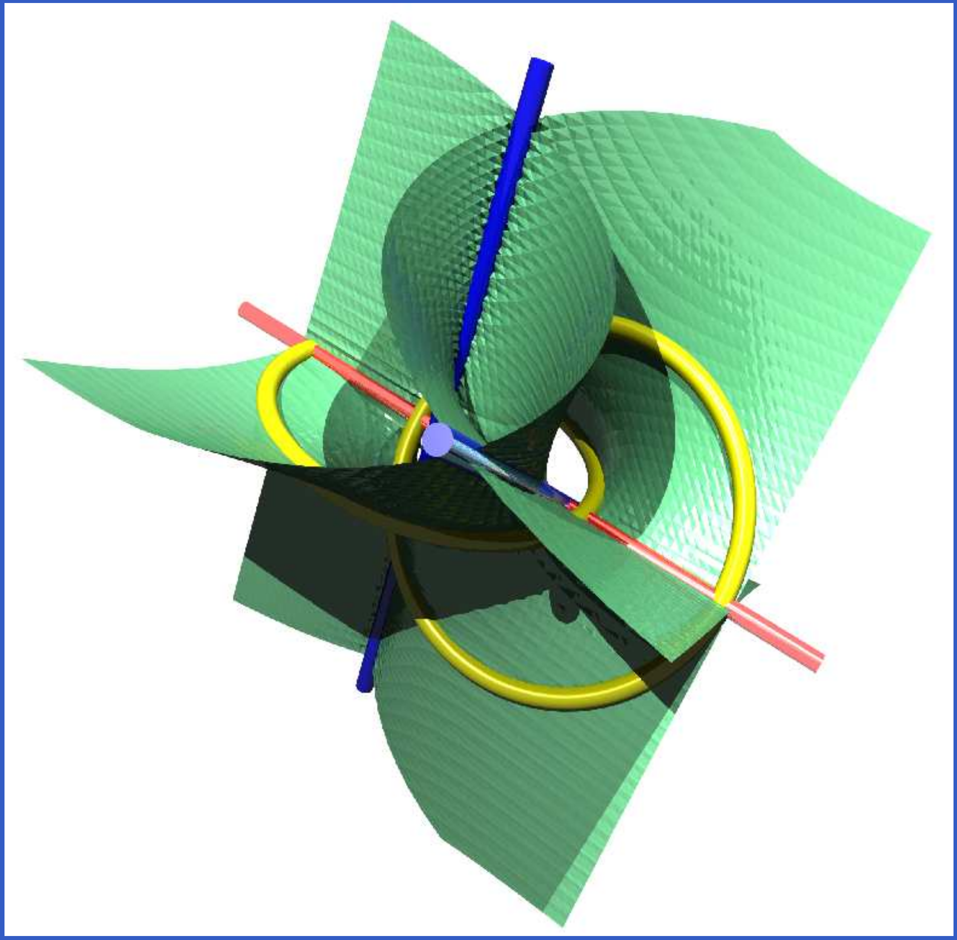
$$m_1 = m_2 = 0, \quad R_1 = R_2, \quad 2d = \pm R_i \sin 2\varphi.$$



configurations with infinitely many common normals



configurations with infinitely many common normals

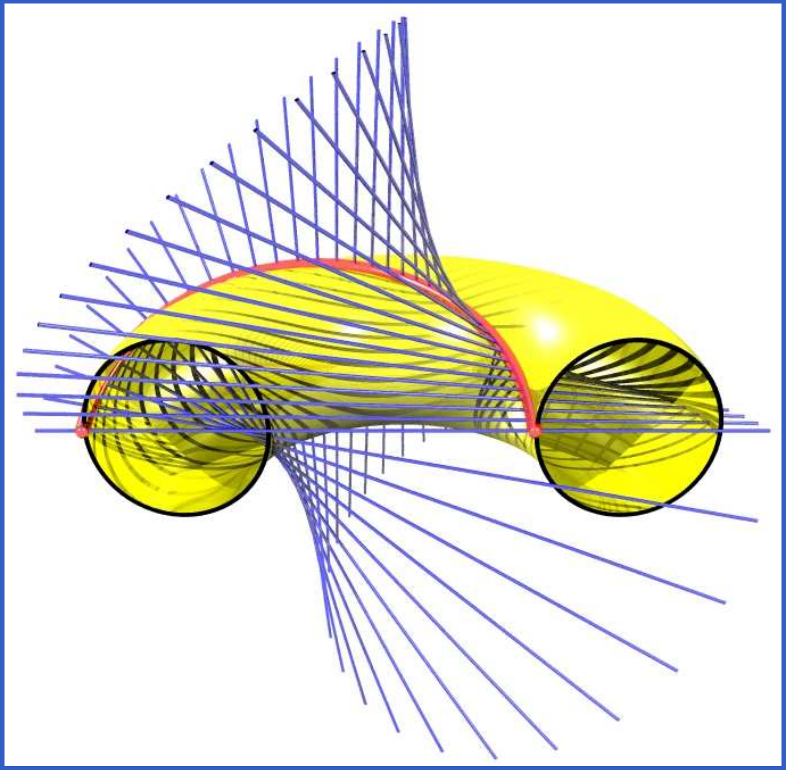


$\Phi_1 = \Phi_2$ is a quartic ruled surface of Sturm type 7.

Axes A_1 and A_2 are part of the double curve.

The common perpendicle of the axes is also contained in the ruled surface.

configurations with infinitely many common normals



common normals meet both spine curves orthogonally

$$\langle s_1 - s_2, \dot{s}_1 \rangle = \langle s_1 - s_2, \dot{s}_2 \rangle = 0$$

$$\Longleftrightarrow$$

$$\langle s_1 - s_2, s_1 - s_2 \rangle = \text{const.}$$

Theorem. The spine curve s_1 is contained in a torus with spine curve s_2, and vice versa.

complete list of cases 1

$2d$	2φ	R_1, R_2, m_1, m_2	$\#$ real solutions	shape
skew axes				
$\neq 0$	$\neq 0$		up to 8	-
$\neq 0$	$\neq 0$	$2d = R_i \sin 2\varphi,\ m_i = 0$	∞^1	quartic ruled surface
$\neq 0$	$\pi/2$	$2d = R_i \sin 2\varphi,\ m_i = 0$	$2\infty^1$	two two-fold pencils
coplanar axes (not parallel)				
0	$\neq 0$	$R_1^2 + m_1^2 \neq R_2^2 + m_2^2$	4, 6	- [a]
0	$\neq 0$	$R_1^2 + m_1^2 = R_2^2 + m_2^2$	∞^1	pencil + line [b]
0	$\neq 0$	$m_1 = m_2 = 0$	5	- [c]

complete list of cases 2

$2d$	2φ	R_1, R_2, m_1, m_2	# real solutions	shape
parallel axes				
$\neq 0$	$0, \pi$	$m \neq 0, \lvert R_1 \pm R_2 \rvert \neq 2d$	4,6	-
$\neq 0$	$0, \pi$	$m \neq 0, \lvert R_1 \pm R_2 \rvert = 2d$	5	-[d]
$\neq 0$	$0, \pi$	$m = 0, \lvert R_1 \pm R_2 \rvert \neq 2d$	1, 3	-[e]
$\neq 0$	$0, \pi$	$m = 0, \lvert R_1 \pm R_2 \rvert = 2d$	∞^1	pencil [f]
identical axes				
0	0	$m \neq 0, R_1 \neq R_2$	$2\infty^1$	two cones of revolution
0	0	$m \neq 0, R_1 = R_2$	$2\infty^1$	cone + cylinder of revolution
0	0	$m = 0, R_1 \neq R_2$	$4\infty^1$	four-fold pencil
0	0	$m = R_1 = R_2 = 0$	∞^2	-

the tiny details

[a] In case of four common normals there can be one line with multiplicity three.

[b] The pencil contains a line with multiplicity three. The further line is not contained in the pencil.

[c] One line is of multiplicity three.

[d] See footnote c.

[e] There is one common normal of multiplicity four.

[f] See footnotes b, e.

Thank You For Your Attention!