Curvature Functions on a one-sheeted Hyperboloid

Boris Odehnal

University of Applied Arts Vienna

Outline

motivation	—	what's done, what's left
Gaussian curvature	—	support function, tangent developables
Mean curvature	—	relation to circular sections
principal curvatures	—	not to be confused with principal lines
ratio of principal curvatures	—	minimal-surface-likeness of a hyperboloid

Motivation

Curvature analysis is often rendered by CAD systems.

Some times it produces strange results:
positive Gaussian curvature on ruled surfaces, . . .

Curvature functions on surfaces of revolution, helical surfaces, cylinders, cones, developables are well understood.

The ellipsoid was studied by W. Wunderlich in [Wu 1].

Hyperboloid - algebraic variety or ruled surface

hyperboloid as variety, given by an algebraic equation ($a, b, c \in \mathbb{R}^+$)

$$\mathcal{S}: \ \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

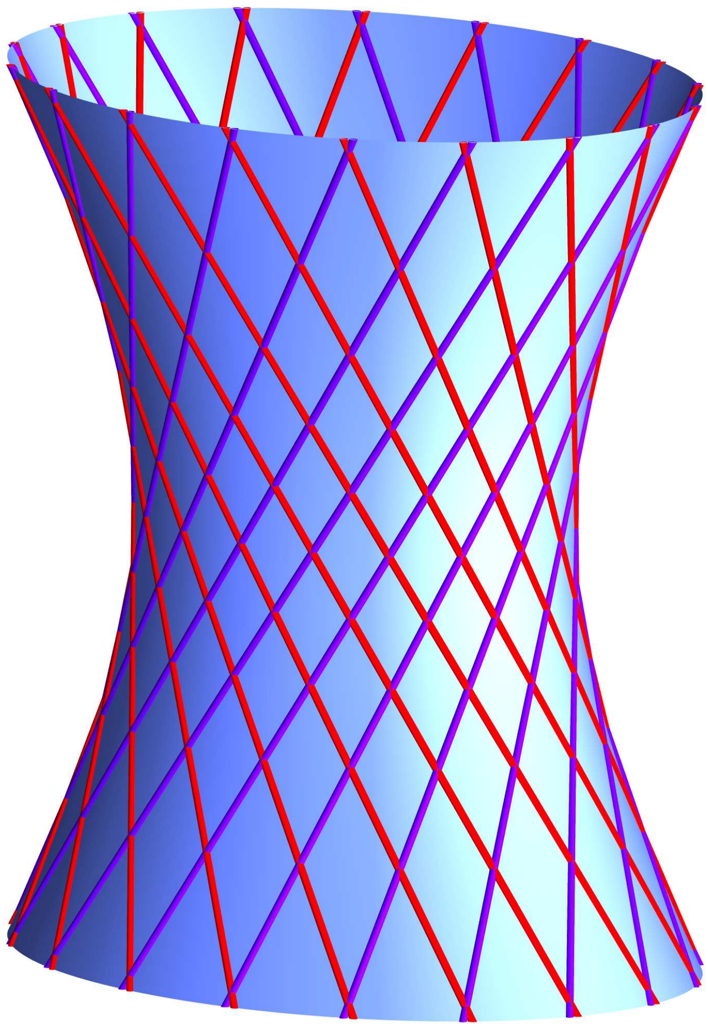
carries two ruled surfaces (reguli)

$$\mathcal{R}_{1,2} = \begin{pmatrix} a\cos u \\ b\sin u \\ 0 \end{pmatrix} + v \cdot \begin{pmatrix} -a\sin u \\ b\cos u \\ \pm c \end{pmatrix}$$

with $u \in [0, 2\pi[$ and $v \in \mathbb{R}$

assumption: $a < b$,

excluded: surfaces of revolution



Gaussian curvature and support function

usual formulas apply to parametrizations: $(x, y, z(x, y))^{\mathsf{T}}$ "upper half"

$$\Longrightarrow \quad K = -\frac{1}{a^2b^2c^2} \cdot \frac{1}{\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)^2}$$

(valid on upper and lower half)

support function d of a surface = oriented distance of tangent planes to the origin

$$\operatorname{grad} \mathcal{S} = 2\left(\frac{x}{a^2}, \frac{y}{b^2}, -\frac{z}{c^2}\right)^{\mathsf{T}} \Longrightarrow d = \frac{1}{\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}}$$

$$\Longrightarrow K = -\frac{d^4}{a^2b^2c^2}$$

in case of an ellipsoid or a two-sheeted hyperboloid: $K = +\frac{d^4}{a^2b^2c^2}$

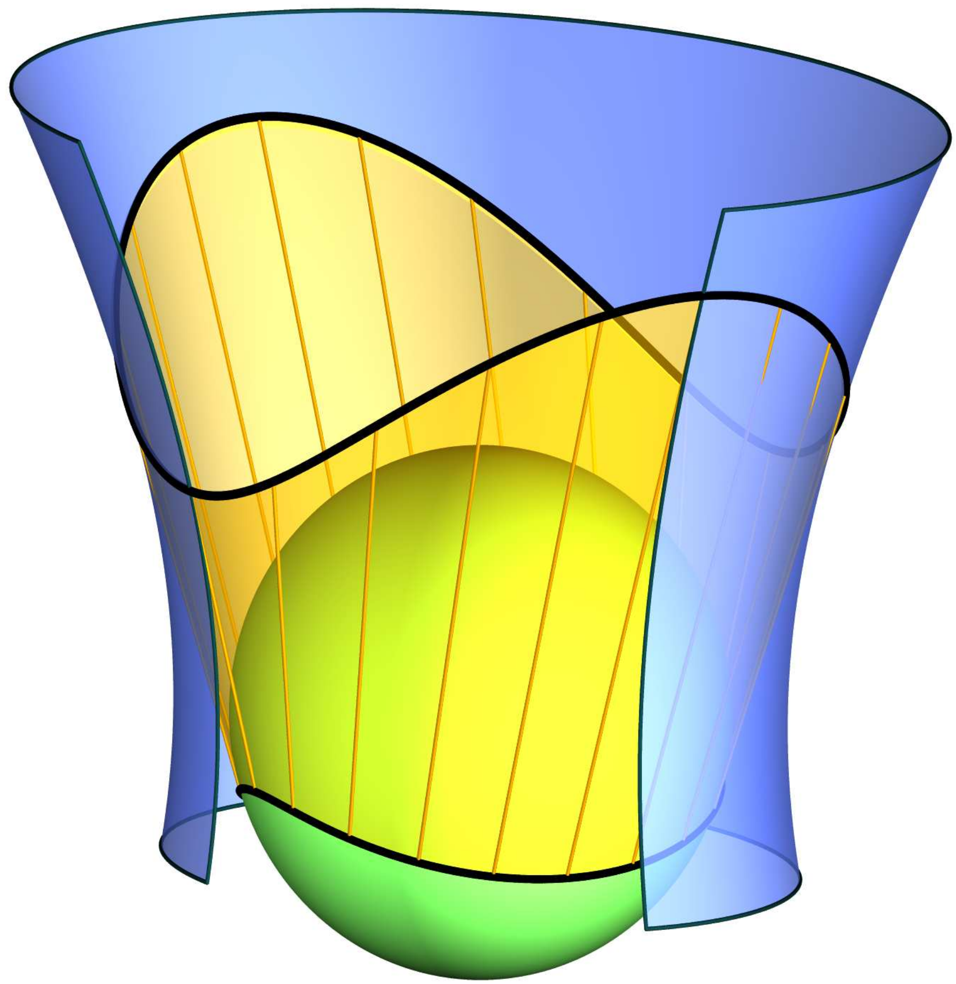
Gaussian curvature and support function

Theorem 1:

The tangent planes of $\mathcal{S}$ along a curve of constant Gaussian curvature K_0 are at fixed distance $d_0 = \sqrt{abc}\sqrt[4]{-K_0}$ from the origin, and thus, they envelope a concentric sphere with radius d_0.

Theorem 2:

The curves of constant Gaussian curvature $K_0 < 0$ on $\mathcal{S}$ are the contact curves of a developable ruled surface tangent to $\mathcal{S}$ and a concentric sphere of radius d_0.

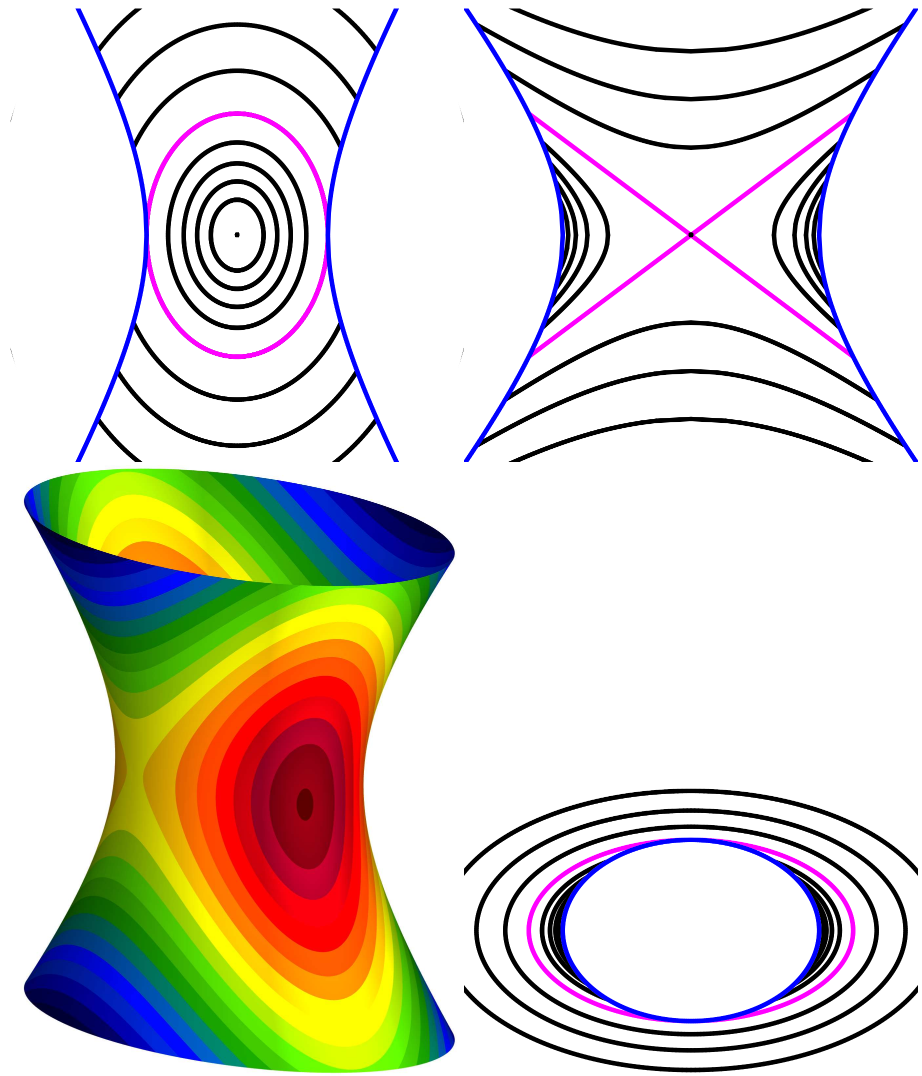


Curves of constant Gaussian curvature

Theorem 3:

The curves of constant Gaussian curvature on $\mathcal{S}$ are the quartic curves of intersection of the hyperboloid with concentric and coaxial ellipsoids

$$\mathcal{E}: \ \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} = \frac{1}{abc\sqrt{-K}} = \frac{1}{d^2}.$$



front view . . . . hyperbolae

top/side view . . . . ellipses

degenerate curve = pair of ellipses if

$$K_0 = -a^2(bc)^{-2}$$

minimal G. curvature at $(0, \pm b, 0)^{\mathrm{T}}$

$$K_{\min} = -b^2(ac)^{-2}$$

Curves of constant Gauss curvature and the central curve

choose one ruled surface $\mathcal{R}_{1,2} \subset \mathcal{S}$

compute the distribution parameter δ and use Lamarle's formula (cf. [Ho,Mu])

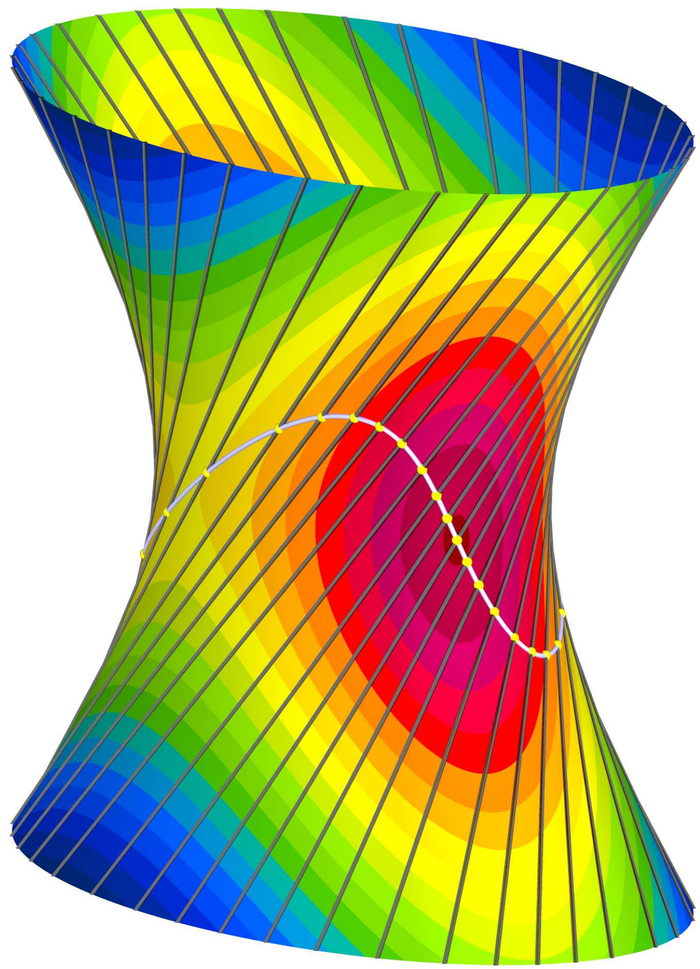
$$K = -\frac{\delta^2}{(\delta^2 + v^2)^2}$$

where v is the surface point's distance to the central point (measured on a ruling)

$\Longrightarrow$ K is minimal $\Longleftrightarrow$ $v = 0$

$\Longrightarrow$ K is minimal at the central point.

$\Longrightarrow$ The iso-lines of K touch the rulings at the central points.



Curves of constant Mean curvature

With $(x, y, z(x, y))^{\mathrm{T}}$ and the support function d we have

$$M = \frac{d^3}{2a^2b^2c^2}L$$

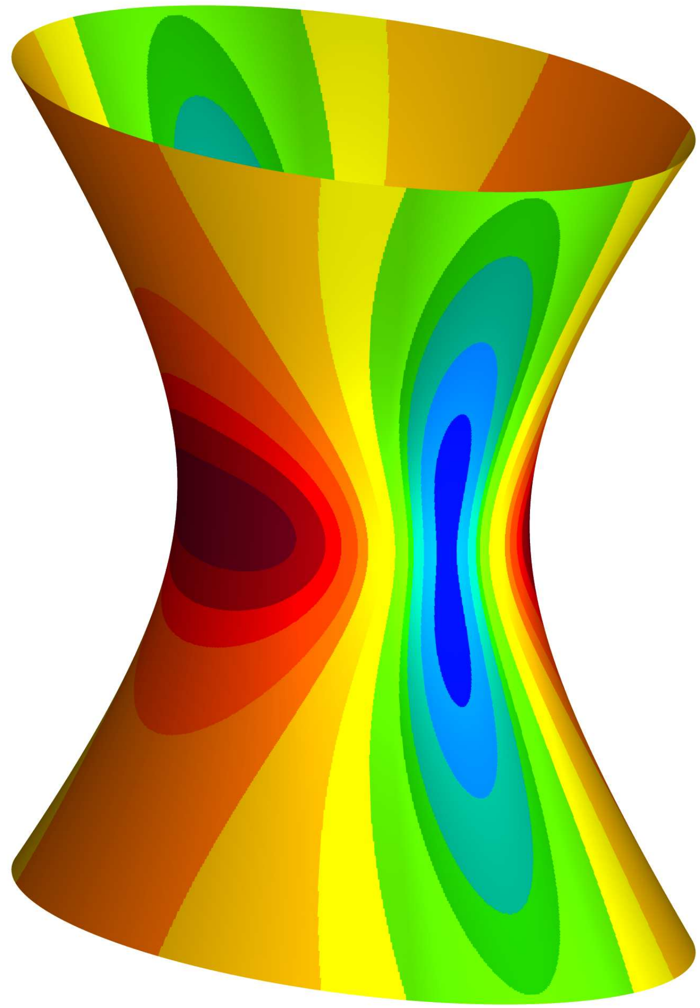
where

$$L = \left(b^2 - c^2\right)\frac{x^2}{a^2} + \left(a^2 - c^2\right)\frac{y^2}{b^2} - \left(a^2 + b^2\right)\frac{z^2}{c^2}.$$

Theorem 4:

Curves of constant Mean curvature on $\mathcal{S}$ are algebraic curves of degree 12.

The principal views are algebraic curves of degree 6 (due to the symmetry of $\mathcal{S}$).



Mean curvature

Theorem 5:

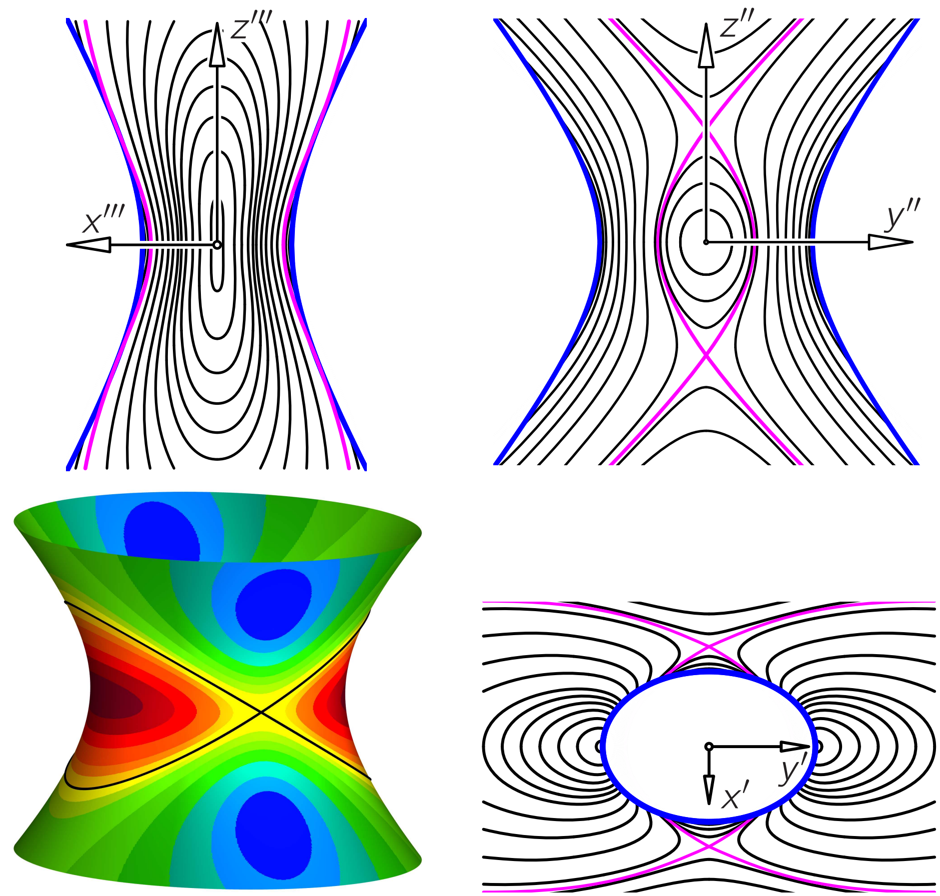
The curve of vanishing Mean curvature on $\mathcal{S}$ is the pair of smallest circles $\iff a = c$.

Proof: Compute the top view and show that the curve with $M = 0$ is an ellipse which is the image of a pair of circles.

Related result (cf. [Kr]):

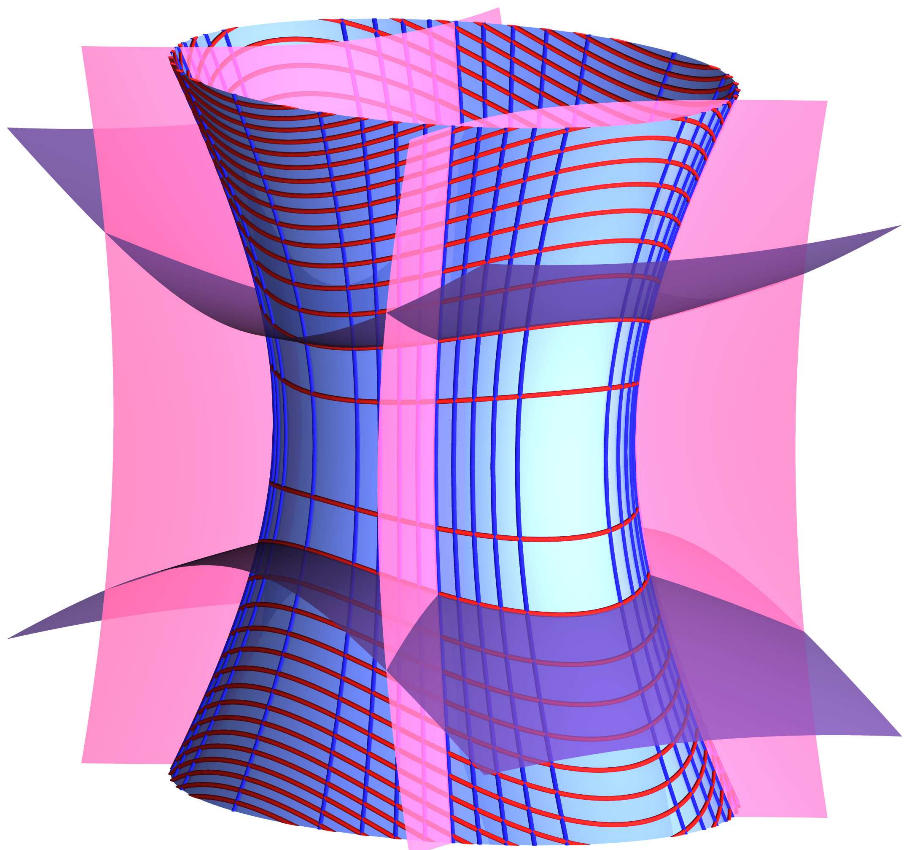
If one point on a circular section s of $\mathcal{S}$ has vanishing Mean curvature, then any point on s shows $M = 0$.

Proof: Search for pairs of orthogonal rulings (through one point).



Curves of constant principal curvatures $\neq$ principal curvature lines

principal (curvature) lines (pcl)= intersection of a quadric with confocal quadrics (not from the same family)



- pcl form an orthogonal system of curves,
- pcl are quartic curves,
- tangents to pcl are principal tangents,
- no curvature function is constant along a pcl

Curves of constant principal curvature

The principal curvatures $\kappa_{1,2}$ are related to the Gauss and Mean curvature via:

$$2M = \kappa_1 + \kappa_2 \quad \text{and} \quad K = \kappa_1 \kappa_2.$$

Alternatively: $K = \det W$ and $2M = \operatorname{trace} W$ with W being the coordinate matrix of the Weingarten map ω (cf. [dC] or [Sp]) $\Longleftrightarrow$ $\kappa_{1,2}$ are eigenvalues of ω.

Solving for either κ (index doesn't matter) means solving

$$\kappa^2 + 2M\kappa + K = 0 \Longleftrightarrow a^2b^2c^2\kappa^2 - d^3L\kappa - d^4 = 0$$

or (without radicals by squaring once)

$$(a^2b^2c^2\kappa^2 - d^4)^2 - d^6\kappa^2L^2 = (a^2b^2c^2\kappa^2 - d^4 - d^3\kappa L)(a^2b^2c^2\kappa^2 - d^4 + d^3\kappa L) = 0.$$

Theorem 6:

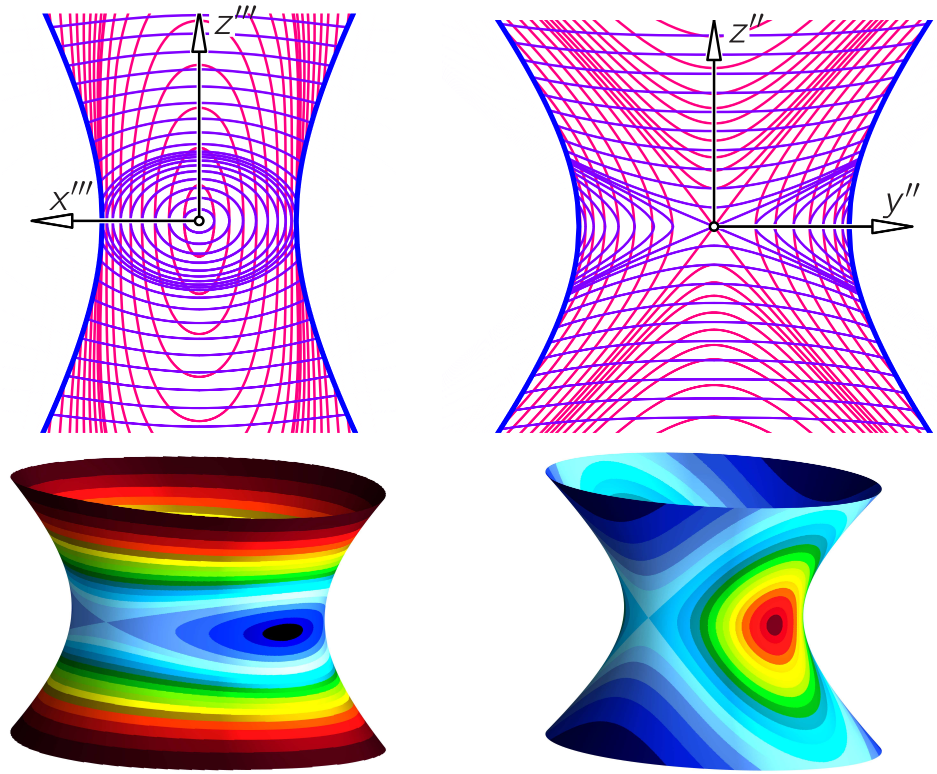
The curves of constant principal curvatures on $\mathcal{S}$ are two families of algebraic curves of degree 16. The principal views are of degree 8 (due to the symmetry of $\mathcal{S}$).

Curves of constant principal curvatures: algebraic parametrization

Curves of constant principal curvatures can be parametrized by the support function d:

$$
\begin{aligned}
x^2 &= \frac{a^4}{\beta\gamma\kappa d^3}(d^3 - b^2c^2\kappa)(a^2\kappa + d),\\
y^2 &= -\frac{b^4}{\alpha\gamma\kappa d^3}(d^3 - a^2c^2\kappa)(b^2\kappa + d),\\
z^2 &= \frac{c^4}{\alpha\beta\kappa d^3}(d^3 + a^2b^2\kappa)(d - c^2\kappa)
\end{aligned}
$$

with $\alpha = b^2 + c^2$, $\beta = c^2 + a^2$, $\gamma = a^2 - b^2$.



Ratio of principal curvatures - shape of Dupin's indicatrix

at some regular surface point P:

$\kappa_1 : \kappa_2 = \;\; 1 \Longrightarrow$ indicatrix at P is a circle,

surface behaves locally like a sphere $\longrightarrow$ cannot happen on $\mathcal{S}$

$\kappa_1 : \kappa_2 = -1 \Longrightarrow$ indicatrix at P is a pair of conjugate equilateral hyperbolae,

surface behaves locally like a minimal surface

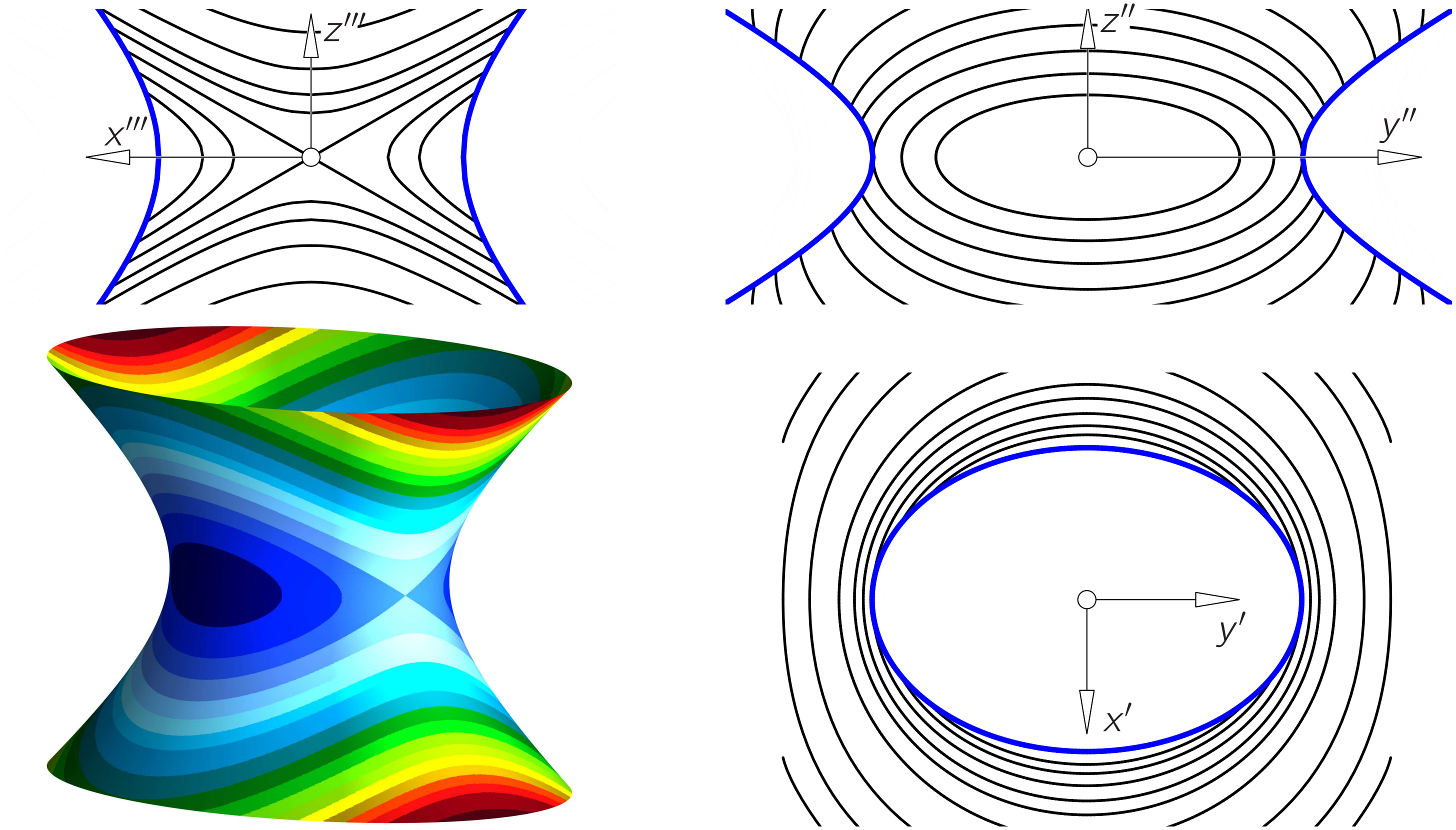
$$\kappa_{1,2} = \frac{d^2}{2a^2b^2c^2}(dL \mp Q) \quad \text{with} \quad Q = \sqrt{d^2L^2 + 4a^2b^2c^2}$$

$$\Longrightarrow R = \kappa_1 : \kappa_2 = (dL - Q) : (dL + Q) \Longrightarrow a^2b^2c^2(1+R)^2 + Rd^2L^2 = 0.$$

Theorem 7:

The curves of constant ratio of principal curvatures on $\mathcal{S}$ are algebraic curves of degree 12. The principal views of the curves of constant ratio of principal curvatures on $\mathcal{S}$ are algebraic curves of degree 6 (due to the symmetry of $\mathcal{S}$).

Curves of constant ratio of principal curvatures



Thank You For Your Attention!

References

[dC] M.P. do Carmo: *Differential Geometry of Curves and Surfaces.* Prentice Hall, 1976.

[Ho] J. Hoschek: *Liniengeometrie.* Bibligraphisches Institut, Zürich, 1971.

[Kr] J. Krames: *Zur mittleren Krümmung einschaliger Hyperboloide.* Anz. Akad. Wiss. Wien, math.-naturw. Klasse Nr. 1, 1971, 1–3.

[Mu] E. Müller: *Vorlesungen über Darstellende Geometrie.* Band III: Konstruktive Behandlung der Regelflächen. Deuticke, Leipzig und Wien, 1931.

[Sp] M. Spivak: *A comprehensive introduction to differential geometry.* Publish or Perish, 1978.

[Wu] W. Wunderlich: *Überblick über die Krümmungsverhältnisse auf dem Ellipsoid.* Festschrift Emil Doležal, 1967.