Frégier’s Theorem in 3 Dimensions

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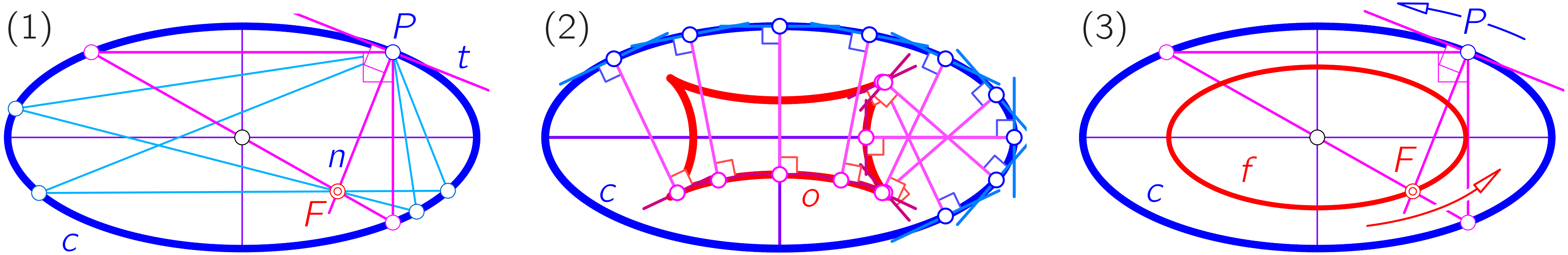
rough sketch of the talk

Frégier's theorem	in the plane, the original version
generalizations	arbitrary angles, polarities, . . .
3D version	2 proofs: synthetic and analytic
Frégier transformation	and its consequences
various types of quadrics	including singular ones
alternative use	of the 2D version
generalized offsets	in arbitrary dimensions

Frégier's theorem – the original version [1 – 6,9,11,14 – 16]

Frégier's theorem: The hypotenuses of right triangles rotating around the vertex P of the right angle placed on a conic c pass through a single point F (the Frégier point).

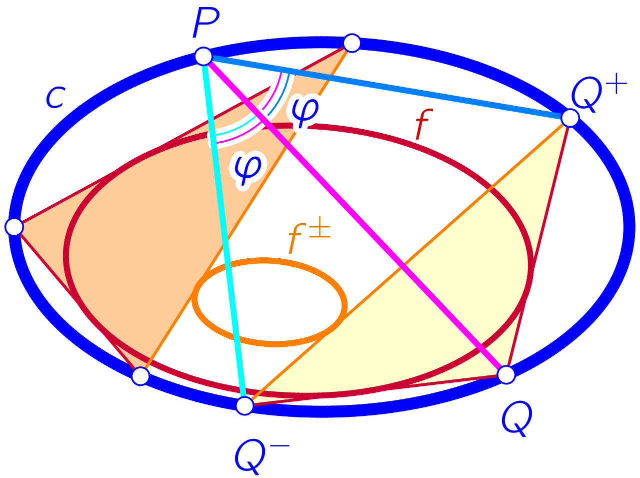


Frégier point F of $P \in c$ — offset o of a conic c — Frégier conic f of c

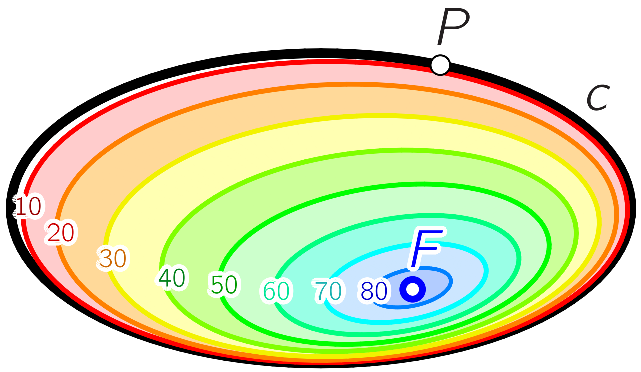
obvious: $F \in n_P$

less obvious: $\overline{PF} = \frac{2(ab)^{\frac{4}{3}}}{a^2+b^2} \cdot \sqrt[3]{\rho}$

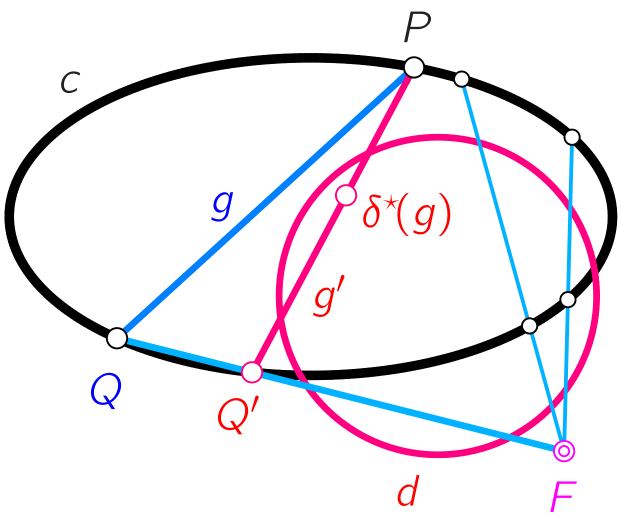
$\Longrightarrow$ Frégier conic = conic shaped generalized offset of a conic

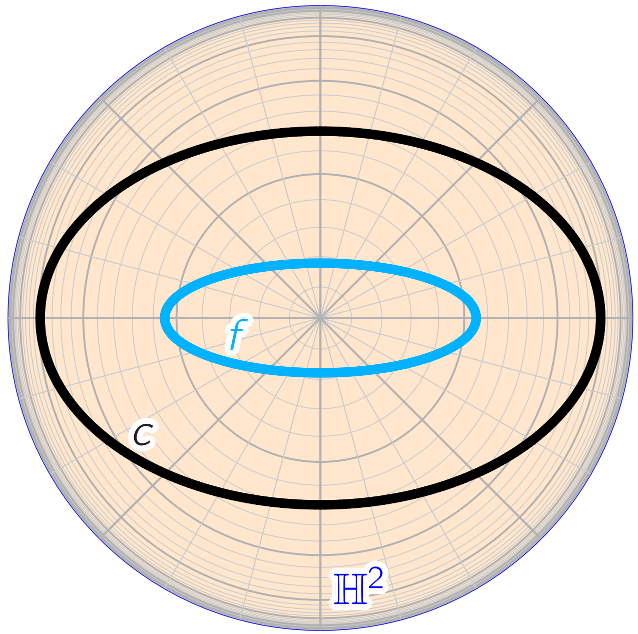


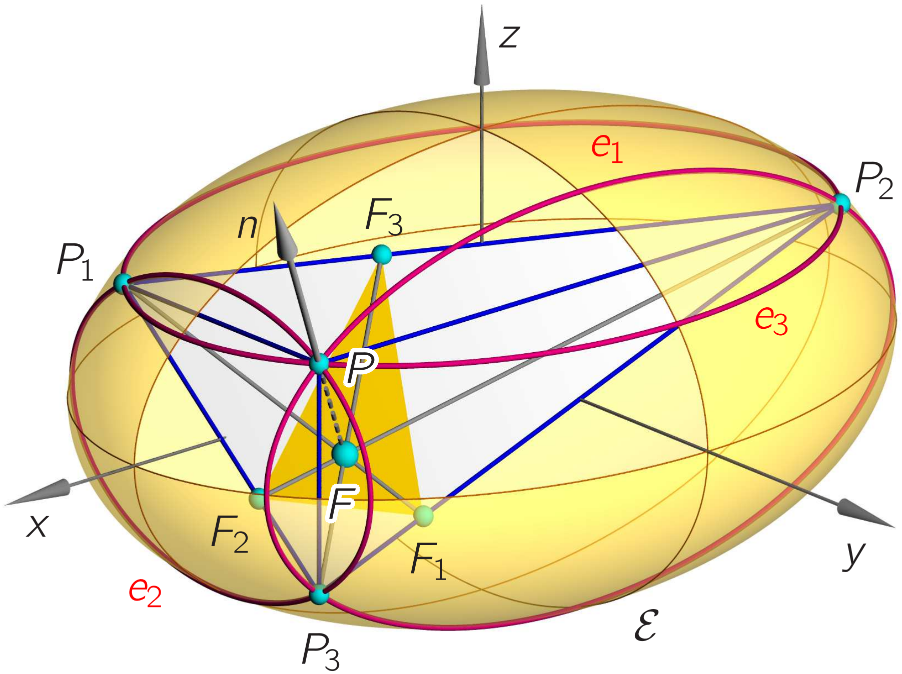
rotating fixed non-right angles: Chords envelop conics in a pencil and give rise to poristic triangle families.



fixed angle rotation ⟶ some polarity/projectivity: definition of non-Euclidean Frégier conics







In analogy to the planar case:

Theorem: The planes $\pi = [P_1, P_2, P_3]$ spanned by the endpoints P_i of a rotating tripod inscribed into the ellipsoid $\mathcal{E}$ and centered at $P \in \mathcal{E}$ pass through a fixed point F on $\mathcal{E}$'s normal n at P.

F = Frégier point of P w.r.t. $\mathcal{E}$

Proof: (1) Start with some tripod (ev. parallel to the axes of $\mathcal{E}$). (2) rotate tripod about a fixed a leg, say, $[P, P_3]$. (3) $\Longrightarrow F_3 \in [P, P_1, P_2]$ is fixed. (4) $[F_3, P_3]$ intersects n, since F_3 is on e_3's normal $e_3 = [P, P_1, P_2] \cap \mathcal{E}$, $[P, P_3] \perp [P, P_1, P_2]$, and n is orthogonal to e_3's tangent at P. (5) This holds true for any index combination. (6) Since any rotation of the tripod (about P) can be decomposed into such elementary rotations, the three points $n \cap [P_i, F_i]$ are identic.

Frégier's theorem for ellipsoids – analytic approach [13]

(1) Center tripod at $P=(u,v,w)\in\mathcal{Q}$: $\mathbf{x}^\top\,\mathbf{Q}\mathbf{x}=1$ with $\mathbf{Q}=\mathbf{Q}^\top$.

(2) Symmetry: endpoints of tripod's legs are $P_1=(-u,v,w),\ \dots$.

(3) Hence, $\mathbf{p}_k=\mathbf{p}+t\cdot\mathbf{b}_k,\ t\in\mathbb{R},\ k\in\{1,2,3\}$, $\mathbf{b}_k$ canonical basis of $\mathbb{R}^3$.

(4) Plane $\varepsilon=[P_1,P_2,P_3]$: given by $xu^{-1}+yv^{-1}+zw^{-1}=1$.

(5) $\mathcal{Q}$'s normal P: $\mathbf{x}(\lambda)=\mathbf{p}+\lambda\mathbf{n},\ \lambda\in\mathbb{R}$ and $\mathbf{n}=\frac{1}{2}\operatorname{grad}\mathcal{Q}(\mathbf{p})=\mathbf{Q}\mathbf{p}$.

(6) $F=\varepsilon\cap n$: $\boxed{\mathbf{f}=\mathbf{p}+\lambda_F\mathbf{Q}\mathbf{p}\quad\text{with}\quad\lambda_F=-2a^2b^2c^2(a^2b^2+b^2c^2+c^2a^2)^{-1}.}$ **($\star$)**

(7) Move tripod around P: Let $\mathbf{D}\in SO_3$, *i.e.*, $\mathbf{D}\mathbf{D}^\top=\mathbf{I}_3$ with $\det\mathbf{D}=+1$.

(8) New legs: $\mathbf{p}'_k=\mathbf{p}+\lambda_k\mathbf{D}\mathbf{b}_k$, where λ_k such that $\mathbf{p}'_k\in\mathcal{Q}$.

(9) $\Longrightarrow\ \lambda_k=-2q(\mathbf{p},\mathbf{D}\mathbf{b}_k)/q(\mathbf{D}\mathbf{b}_k,\mathbf{D}\mathbf{b}_k)$ **($\star\star$)**

(10) Plane $\varepsilon'=[P'_1,P'_2,P'_3]$: given by $\sum\frac{1}{\lambda_k}\langle\mathbf{D}\mathbf{b}_k,\mathbf{x}-\mathbf{p}\rangle=1$.

(11) Show that F **($\star$)** lies in ε', *i.e.*, show $\sum\frac{1}{\lambda_k}\langle\mathbf{D}\mathbf{b}_k,\lambda_F\mathbf{Q}\mathbf{p}\rangle=1$.

(12) First: $\langle\mathbf{D}\mathbf{b}_k,\mathbf{Q}\mathbf{p}\rangle=\mathbf{p}^\top\,\mathbf{Q}^\top\,\mathbf{D}\mathbf{b}_k=q(\mathbf{p},\mathbf{D}\mathbf{b}_k)$.

(13) With **($\star\star$)**, this yields $\sum\frac{1}{\lambda_k}\langle\mathbf{D}\mathbf{b}_k,\lambda_F\mathbf{Q}\mathbf{p}\rangle=-\frac{\lambda_F}{2}\sum q(\mathbf{D}\mathbf{b}_k,\mathbf{D}\mathbf{b}_k)$.

(14) Second: $q(\mathbf{D}\mathbf{b}_k,\mathbf{D}\mathbf{b}_k)=\mathbf{b}_k{}^\top\,\mathbf{D}^\top\,\mathbf{Q}\mathbf{D}\mathbf{b}_k=d_{1k}^2a^{-2}+d_{2k}^2b^{-2}+d_{3k}^2c^{-2}$.

(15) $\mathbf{D}\mathbf{D}^\top=\mathbf{I}_3$ & $\det\mathbf{D}=+1\Longrightarrow$ columns & rows are unit vectors.

(16) $\Longrightarrow\ \sum q(\mathbf{D}\mathbf{b}_k,\mathbf{D}\mathbf{b}_k)=a^{-2}+b^{-2}+c^{-2}$.

(17) With **($\star$)**: $\sum\frac{1}{\lambda_k}\langle\mathbf{D}\mathbf{b}_k,\lambda_F\mathbf{Q}\mathbf{p}\rangle=1$.

Frégier's transform [13]

$\varphi : \mathbf{p} \mapsto \mathbf{p} + \lambda_F(a, b, c)\mathbf{Q}\mathbf{p} = \underbrace{(\mathbf{I}_3 + \lambda\mathbf{Q})}_{=\mathbf{F}}\mathbf{p}$ shall be called Frégier transform.

φ = centro affine mapping, can be singular $\Longrightarrow$

(1) The Frégier set $\mathcal{F}$ of a quadric is a quadric, the Frégier quadric.

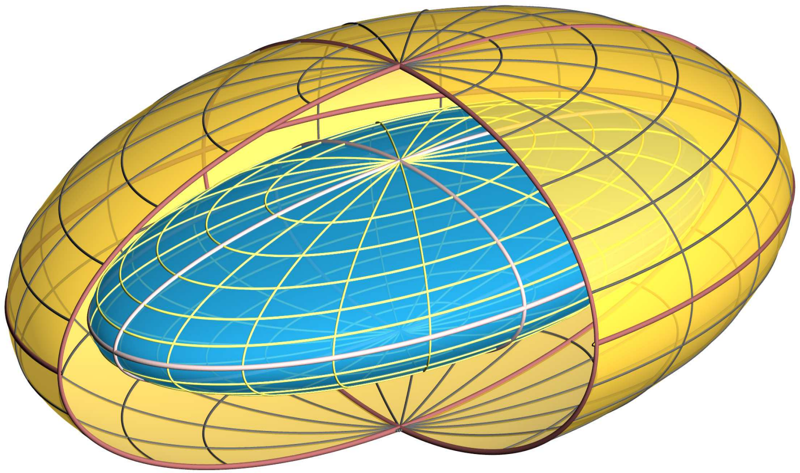
(2) If φ is regular, then $\mathcal{F}$ is a quadric of the affine type of $\mathcal{Q}$.

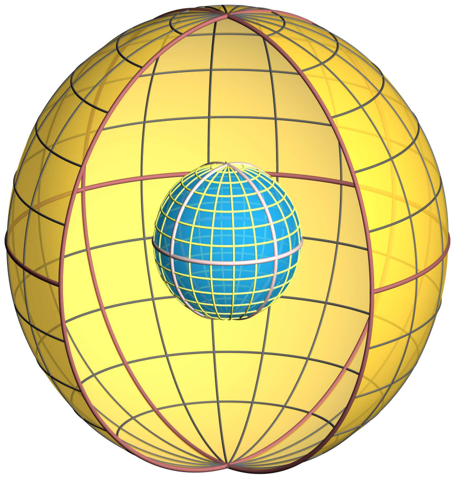
(3) Since $\mathbf{F}$ is diagonal, $\mathcal{F}$ and $\mathcal{Q}$ share the symmetry planes.

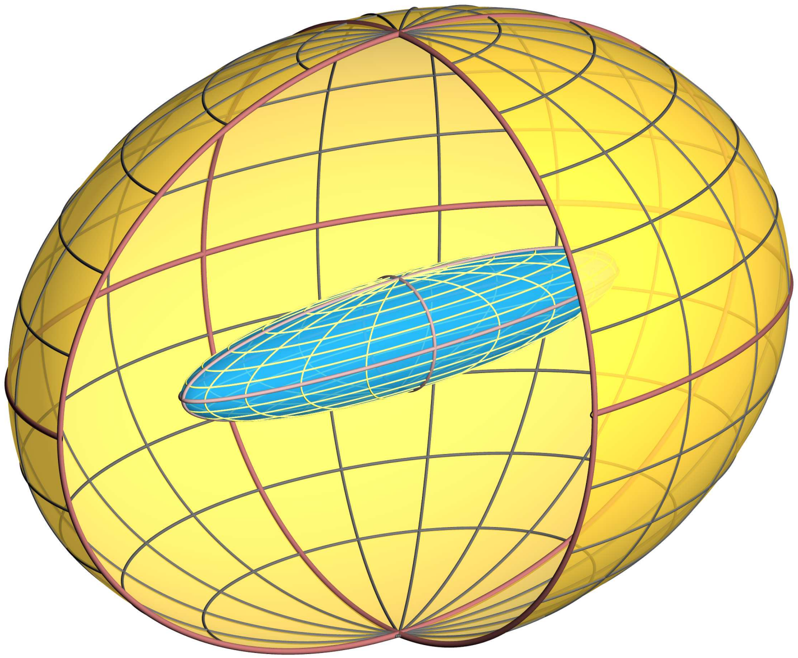
(4) The Frégier quadric $\mathcal{F}$ is a ruled quadric if $\mathcal{Q}$ is ruled.

(5) $\mathcal{F}$ can be singular even if $\mathcal{Q}$ is regular.

Frégier quadrics - ellipsoids [17,18]





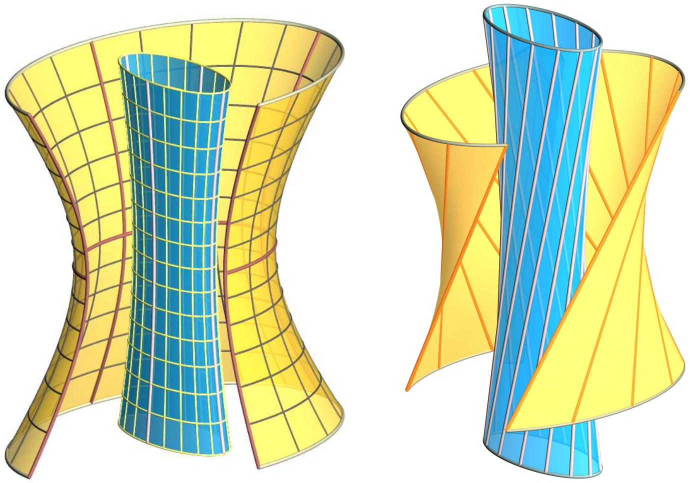


$\uparrow$ $\mathcal{Q}$ = sphere, radius $R \Longrightarrow$
$\mathcal{F}$ = sphere, radius $\frac{1}{3}R$

$\nwarrow$ $\mathcal{Q}$ = triaxial $\Longrightarrow$ $\mathcal{F}$ = triaxial

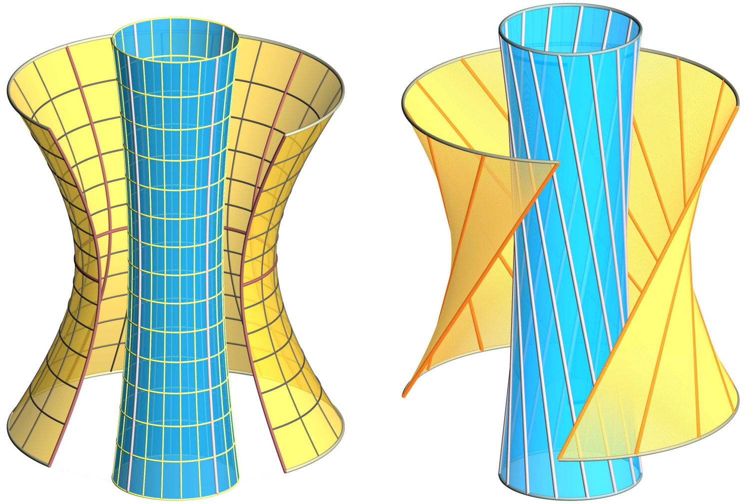
$\leftarrow$ $\mathcal{Q}$ = ellipsoid of revolution $\Longrightarrow$
$\mathcal{F}$ = ellipsoid of revolution

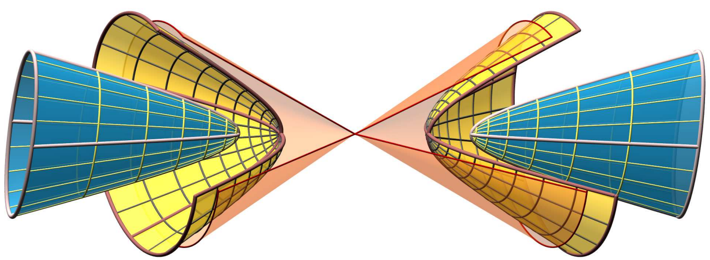
Frégier quadrics - hyperboloids



← Ruled quadrics have ruled Frégier quadrics.

Rotational symmetry is preserved. →

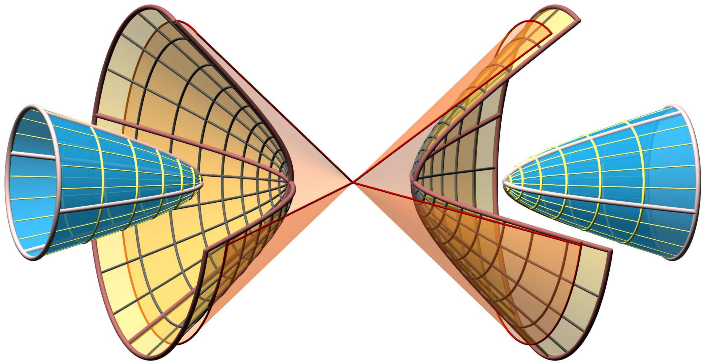




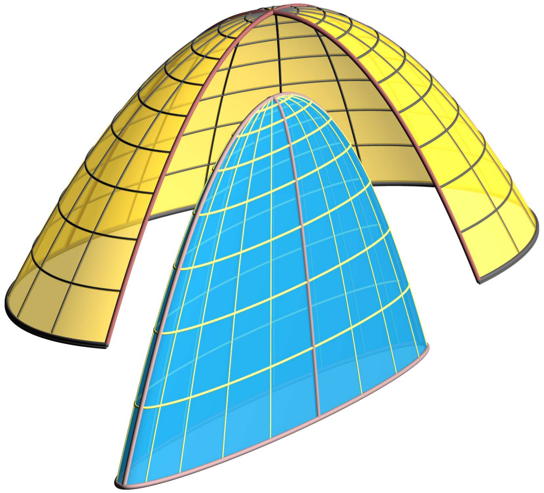
←

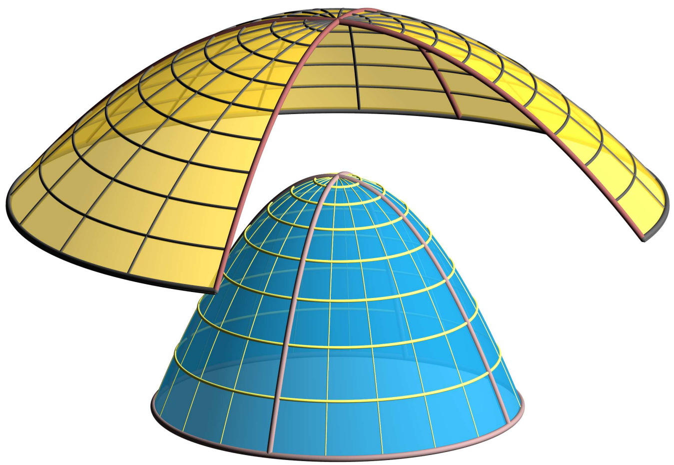
The affine type remains the same.

→

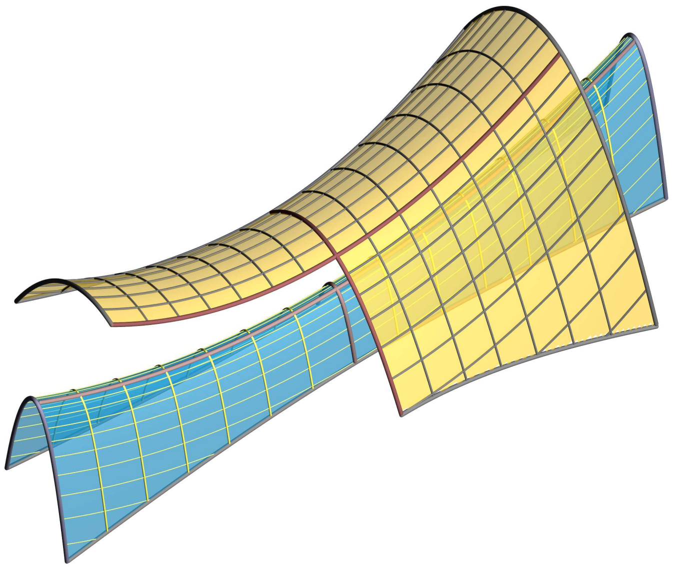


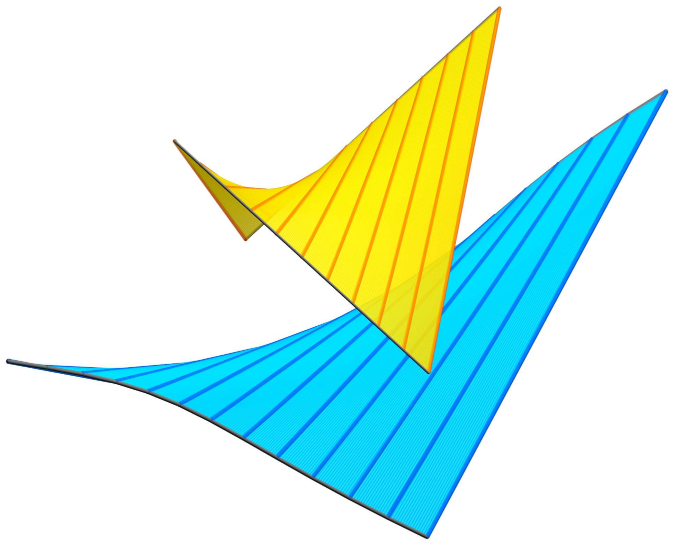
Frégier quadrics - paraboloids, cones

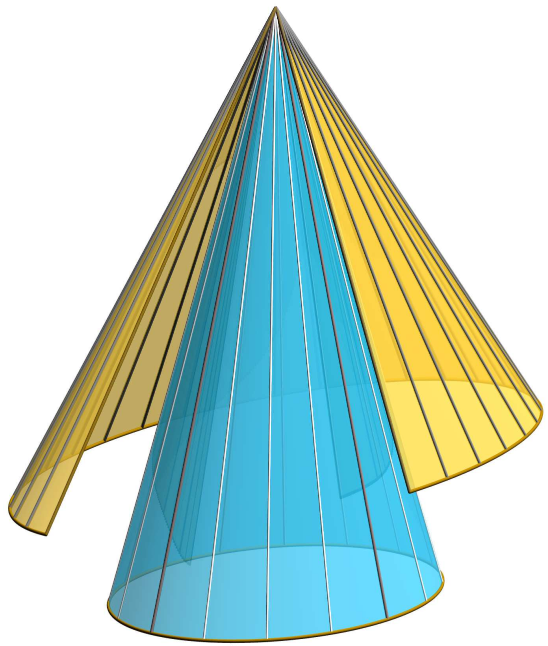




The same is true for paraboloids (← oval, ↙ ruled) and for quadratic cones (↓).







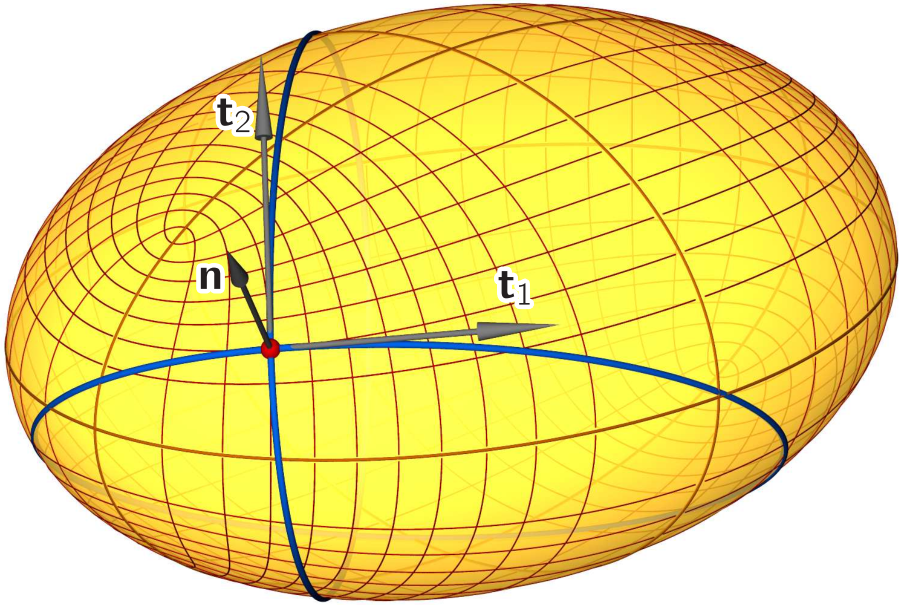
2D vs. 3D [7,8]

conic	Frégier set	quadric	Frégier set
circle	center	sphere	sphere
equilateral hyperbola	line at infinity	orthogonal hyperboloid	plane at infinity
no singular Frégier conic		$\exists$ singular Frégier quadrics	

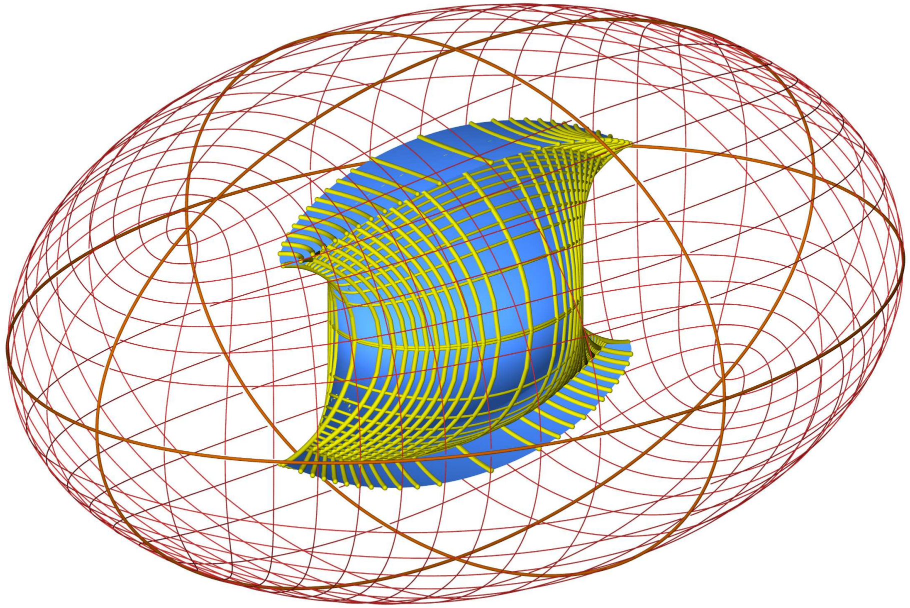
orthogonal hyperbola/hyperboloid/cone/hyperbolic cylinder ... $\operatorname{tr} \mathbf{Q} = 0$

orthogonal hyperboloid/cone = quadric of altitudes in certain tetrahedra

Frégier's theorem applied to planar sections I



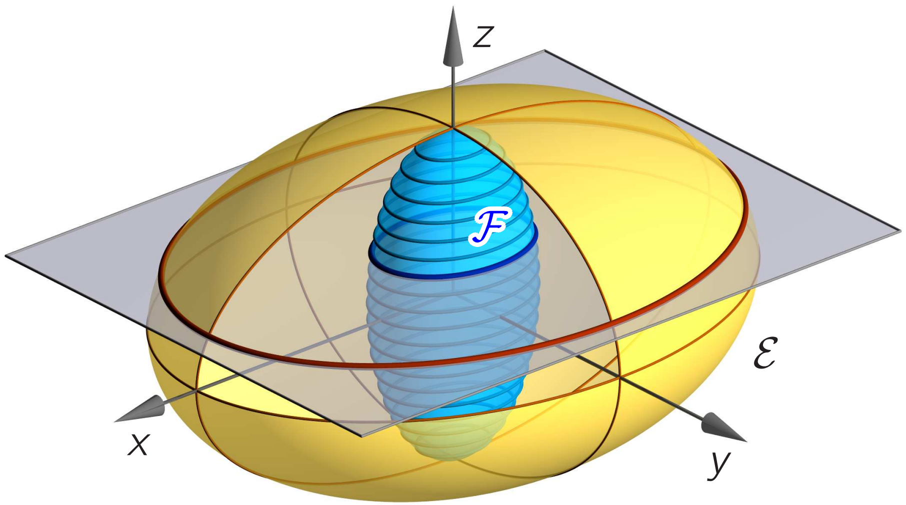
normal sections along principal directions

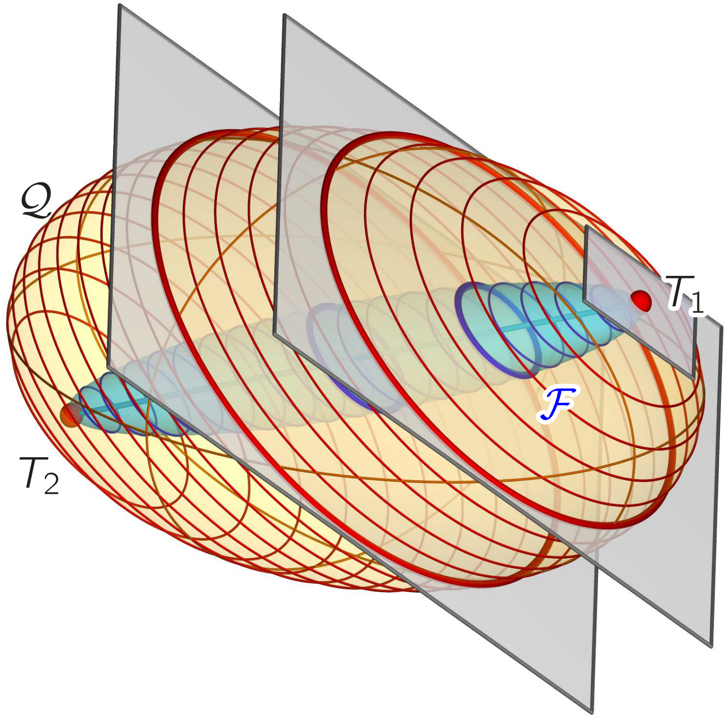


Theorem:
The Frégier points of a quadric's planar intersection along all principal tangents form a single surface of degree 12.

Proof:
Use a curvature line parametrization.

Frégier's theorem applied to planar sections II





intersection with pencil of parallel planes

Proof:
Use a shear transformation.

Theorem:
Ordinary Frégier conics of the intersection with planes from a pencil of parallel planes form a quadric $\mathcal{F}$ that touches $\mathcal{Q}$ at the end points of the pencil carrier's conjugate diameter.

Frégier quadrics as generalized offsets (in n dimensions) [10,13,19]

$\mathcal{Q}$ regular non-parabolic hyperquadric $\mathcal{Q} \subset \mathbb{R}^n$: $\sum_{k=1}^{n} \frac{x_k^2}{a_k^2} = \mathbf{x}^\top \mathbf{Q}\mathbf{x} = 1$ all $a_k \neq 0$,

but not necessarily all $a_k^2 > 0$, $\mathbf{Q} := \operatorname{diag}(a_1^{-2}, \ldots, a_n^{-2})$

Gaussian curvature $K = (-d)^{n+1} \det \mathbf{Q}$ with support function $d = -\frac{1}{\sqrt{\mathbf{p}^\top \mathbf{Q}^2 \mathbf{p}}}$

Frégier point F on $N_P = \mathbf{p} + \lambda_F \mathbf{n}$ with $\mathbf{n} = -d\,\mathbf{Q}\mathbf{p}$ in $\varepsilon := [P_1, \ldots, P_n] : \sum_{k=1}^{n} \frac{x_k}{u_k} = n-2$

with P_i being P's reflection in $x_i = 0$ yields $\lambda_F = 2/d/\operatorname{tr}\mathbf{Q} = \overline{PF}$

Theorem: F is independent of the Cart. n-pod's pose and $\lambda_F = -2\frac{\sqrt[n+1]{\det \mathbf{Q}}}{\operatorname{tr}\mathbf{Q}} \cdot \frac{1}{\sqrt[n+1]{K}}$.

Theorem:

(1) Frégier quadrics of quadrics are generalized offsets in the shape of quadrics, same affine type, offset function $\lambda = \kappa(\mathbf{Q}) / \sqrt[n+1]{K}$.

(2) Frégier Quadrics of quadrics with center are coaxial and concentric with their pivot quadrics.

and finally

1. Results for quadrics with center also hold for parabolic quadrics.
2. Computational proof (existence of the Frégier point, Frégier quadric and transformation) works in any dimension in the same way.
3. Generalized offsets in higher dimensions for parabolic quadrics need special treatment.
4. The results from slides 11, 12 are only two examples taken out of a huge variety.

Thank You For Your Attention!

some references

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