Variations on Frégier's Theorem

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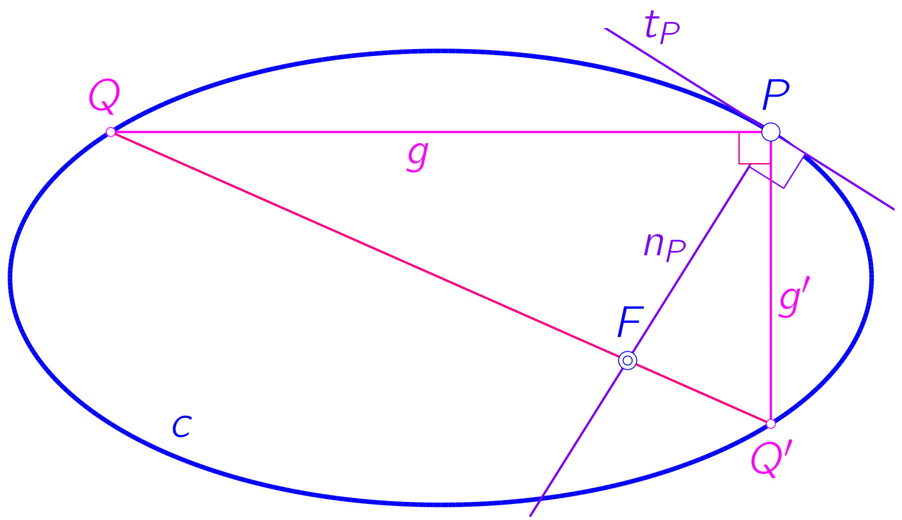
overview and contents

Frégier's theorem	original form
Frégier conics	
Generalized offsets	conic shaped
arbitrary angles	many Frégier conics
related porisms	just by-catch
arbitrary projective mappings	still pencils of Frégier conics

Frégier's theorem



c ... arbitrary (regular) conic

$P \in$ ... arbitrary point on c

$g, g' \ni P$ ... $g \perp g'$

$Q = c \cap g \setminus \{P\}$, $Q' = \dots$

Frégier:

$[Q, Q'] \ni F$ for all pairs (g, g').

F ... Frégier point of P w.r.t. c

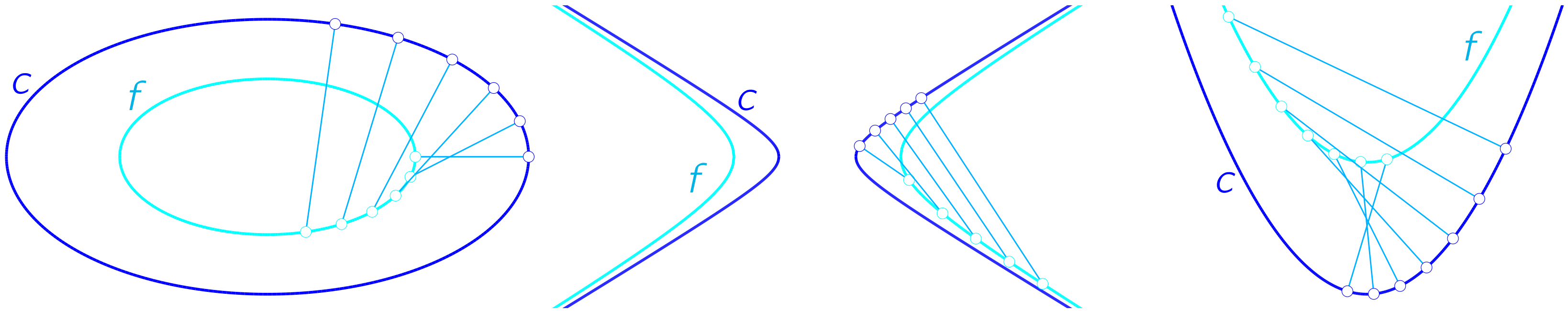
obviously $F \in n_P$

F = center of the involution of right angles lifted to $c \Longrightarrow F$ exists for all $P \in c$

$\Longrightarrow$ Affine type of c does not matter.

Frégier conics

The set f of all Frégier points F of a conic c is called the Frégier conic of c.



Conics with center have Frégier conics (with center) homothetic to the initial conics.

similarity factor $\dfrac{a^2 - b^2}{a^2 + b^2}$ similarity factor $\dfrac{a^2 + b^2}{a^2 - b^2}$

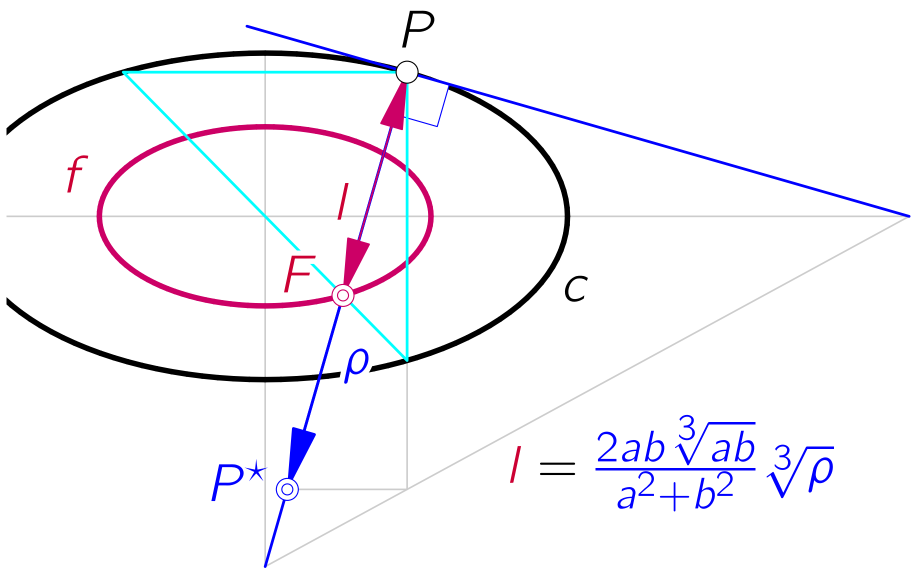
Parabolas are congruent to their Frégier conics.

generalized offsets of conics

ordinary offset p of a curve c = set of all points at equal (orthogonal) distance to c

Offset of conics ($\neq$ circles) are not conics!

generalized offset g of a curve c = offset distance not constant, "offset function"



$l = \frac{2ab\sqrt[3]{ab}}{a^2+b^2}\sqrt[3]{\rho}$

$\rho := \overline{PP^\star}$ ... curvature radius of c at P

$l := \overline{PF}$

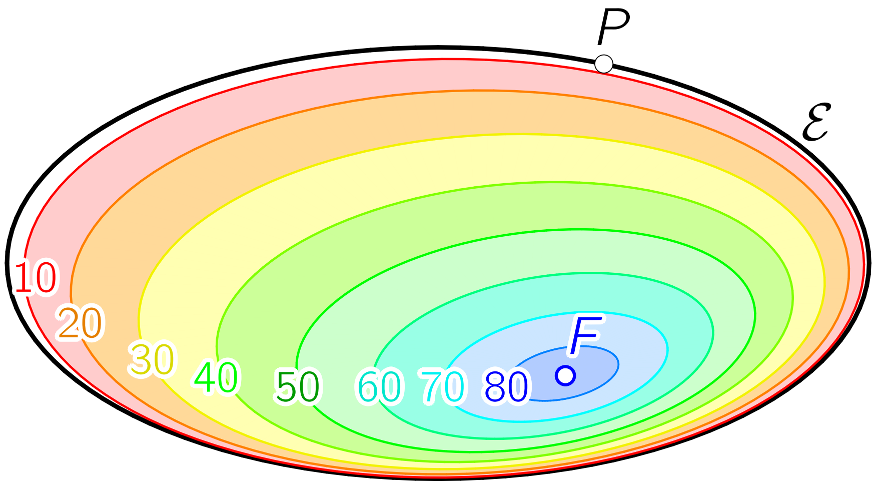
Thm.:

The Frégier conic f is a generalized offset of the conic c.

ellipse	$8a^4b^4\rho = (a^2+b^2)^3 l^3$
hyerpbola	$8a^4b^4\rho = (a^2-b^2)^3 l^3$
parabola	$8q^2\rho = l^3$ (parameter q)

pencils of Frégier conics

We replace the right angle between g and g' by an arbitrary constant angle $0 < \varphi < \frac{\pi}{2}$:



Thm.:
The chords $[Q, Q']$ cut out by the legs enclosing constant angles φ (with vertices at P) envelop a single conic f_φ.

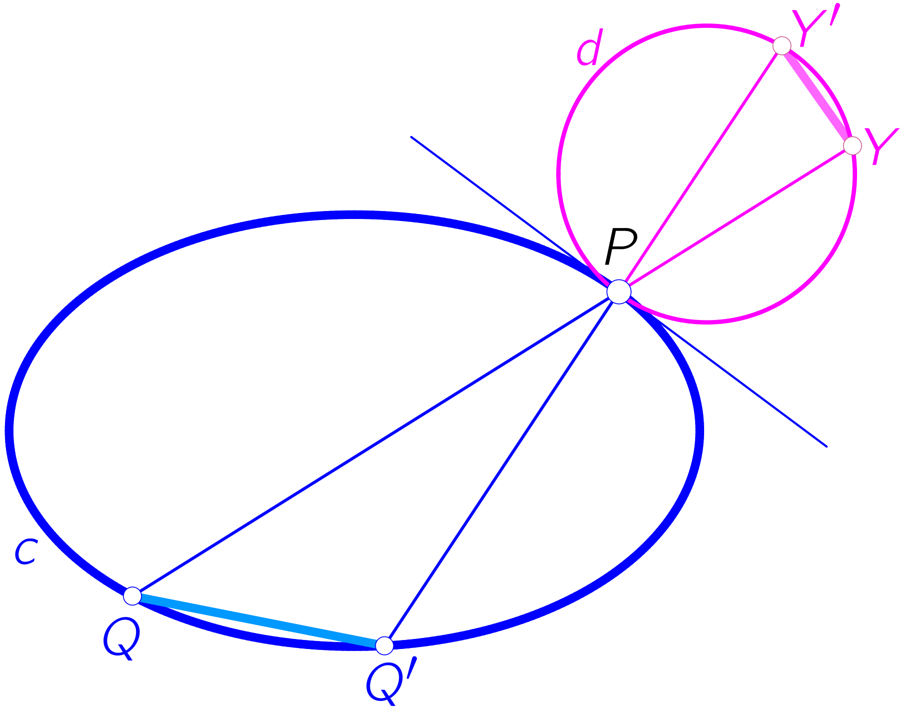
The conic f_φ is called generalized Frégier conic.

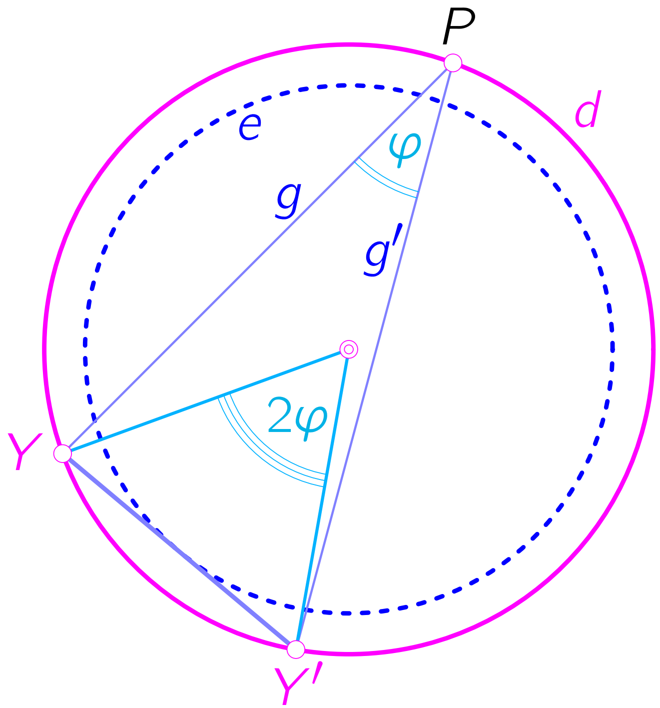
Thm.: The generalized Frégier conics constitute a pencil of the third kind.

Note: $f_\varphi = f_{-\varphi}$

pencils of Frégier conics

Proof: Either brute force computation or synthetic:





$g \mapsto g'$ s.t. $\measuredangle(g, g') = \varphi$ induces a projectivity on c: $\gamma: \ Q \mapsto \gamma(X)$

perspective collineation $c \to d$ (circle d, tangent to c at d),

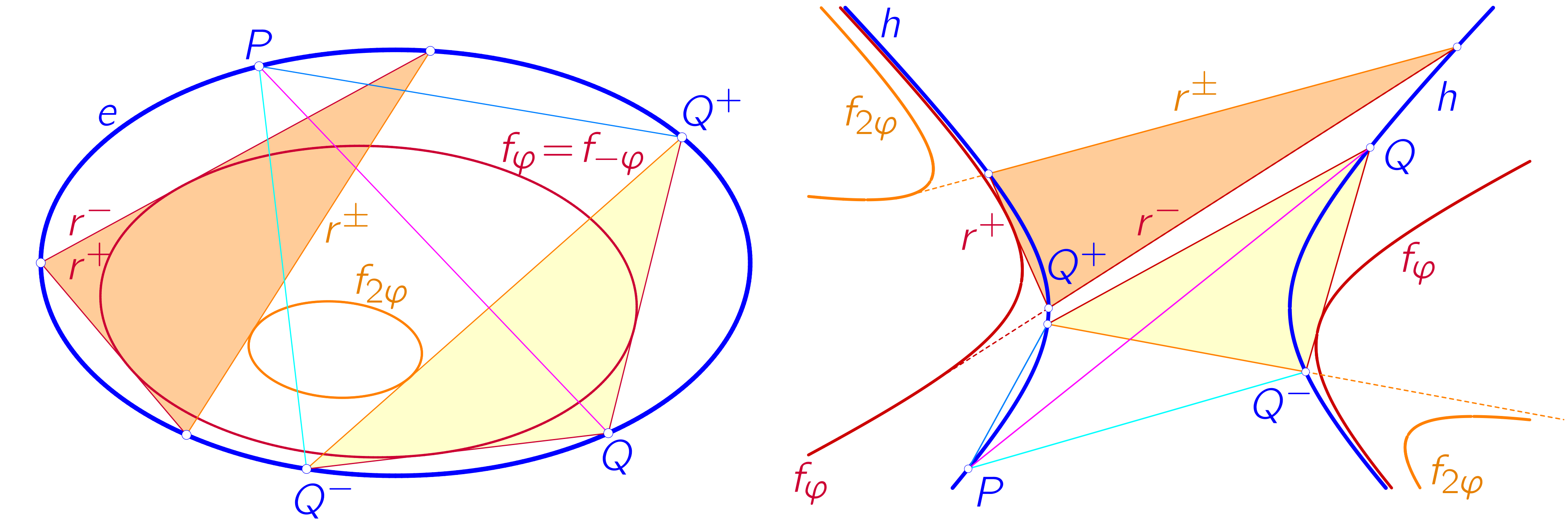
axis = common tangent, center = P

on d: chords envelop circles e concentric with d $\Longrightarrow$ pencil of the 3$^{\text{rd}}$ kind

related porisms

The computational proof yields some by-catch:

$\sphericalangle\,(g, g^+) = +\varphi$, $\sphericalangle\,(g, g^-) = -\varphi$, $Q^+ := c \cap g^+ \setminus \{P\}$, ...



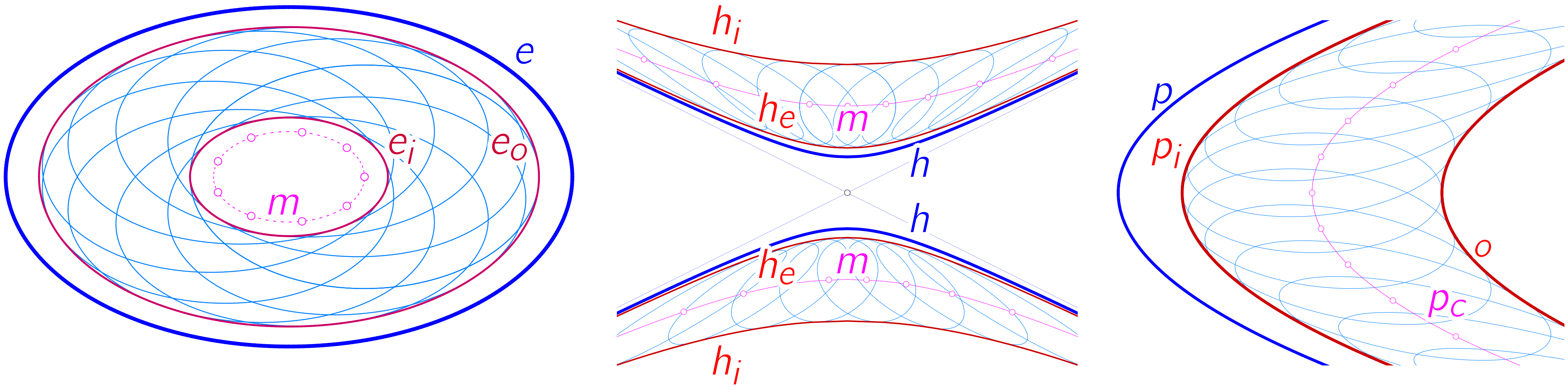
Computation of envelopes f_φ / $f_{-\varphi} = f_\varphi$, $f_{2\varphi}$ of $[Q, Q^+]$ / $[Q, Q^-]$, $[Q^-, Q^+]$ yields additional tangents r^+ / r^-, $r^\pm$ of f_φ and $f_{2\varphi}$ $\Longrightarrow$ Since c, f_φ, $f_{2\varphi}$ lie in a pencil of the third kind, there exists a Poncelet porism (billiard with three caustics).

Again: The affine type of the conic does not matter!

envelopes of generalized Frégier conics

Thm.:

The generalized Fregier conics of c to a fixed angle φ envelop pairs of conics homothetic to c if the pivot P traces c.



polarities and other involutive mappings

Any involutive mapping in the pencil about P has a center F.
$\Longrightarrow$ Any center of an involution can be viewed as a Frégier point.

Involutive mappings can be induced by polarities π.

This leads to non-Euclidean Frégier points.
$\Longrightarrow$ Frégier points/conics in elliptic/hyperbolic geometry if π is elliptic/hyperbolic (regular in any case).

Singular polarities (projective mappings on lines or in pencils):
$\Longrightarrow$ pseudo-Euclidean and Euclidean versions of Frégier

arbitrary projective mappings . . .

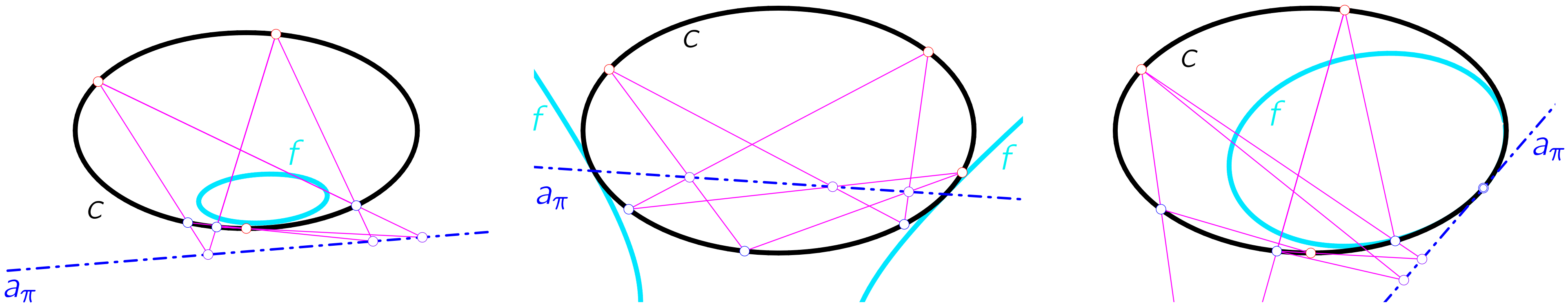
. . . generate conics.

$\pi: \; A \mapsto A', B \mapsto B', C \mapsto C'$ generic projective mapping on a conic c.

Thm.:

The chords $[X, \varphi(X)]$ envelope a conic f.

c and f span a pencil of the 3rd kind (or, in the limit, of the 5th kind).



The conic f is not assigned to a unique point on c, it is assigned to the projective mapping on c.

arbitrary projective mappings . . .

Proof (unfortunately only analytic):

W.l.o.g.: $c: \ x_1^2 = x_0 x_2$, $A = 1:0:0$, $\quad B = 1:1:1$, $\quad C = 0:0:1$

and $A' = 1:u:u^2$, $B' = 1:v:v^2$, $C = 1:w:w^2$

with $u, v, w \neq 0, 1, \infty$, $u \neq v, w$, $v \neq w$, $w = v/(1-u+v)$

For $X = 1:q:q^2$ ($q \in \mathbb{R} \neq 0, 1, \infty, u, v, w$) compute $X' = \pi(X)$

Then, determine the envelope f of all $[X, X']$:

$$f: \ (x_0, x_1, x_2)^{\mathrm{T}} \begin{pmatrix} u^2 d_2^2 & u d_1 d_4 & -d_1 d_2 d_3 \\ u d_2 d_4 & c_{11} & -d_1 d_4 \\ -d_1 d_2 d_3 & -d_1 d_4 & d_1^2 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = 0.$$

singular conic in $[c, f]$: $(u d_2 x_0 + d_4 x_1 - d_1 x_2)^2 = 0$

$u - v = d_1, v - w = d_2, 2w - u = d_3, uw - vw - v + w = d_4$

$c_{11} := d_1^4 + 2(d_2 + d_3) d_1^3 + (d_2 + d_3)^2 d_1^2 - 2(3d_1 + 3d_2 + d_3) d_1 d_2 + d_2^2$

some remarks

Frégier's theorem	where does it belong to?
	elementary geometry
	projective geometry
	differential geometry
projective generalization	allows for non-Euclidean interpretation
three-dimensional versions	work in progress

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Thank You For Your Attention!