Convergent Triangle Tunnels

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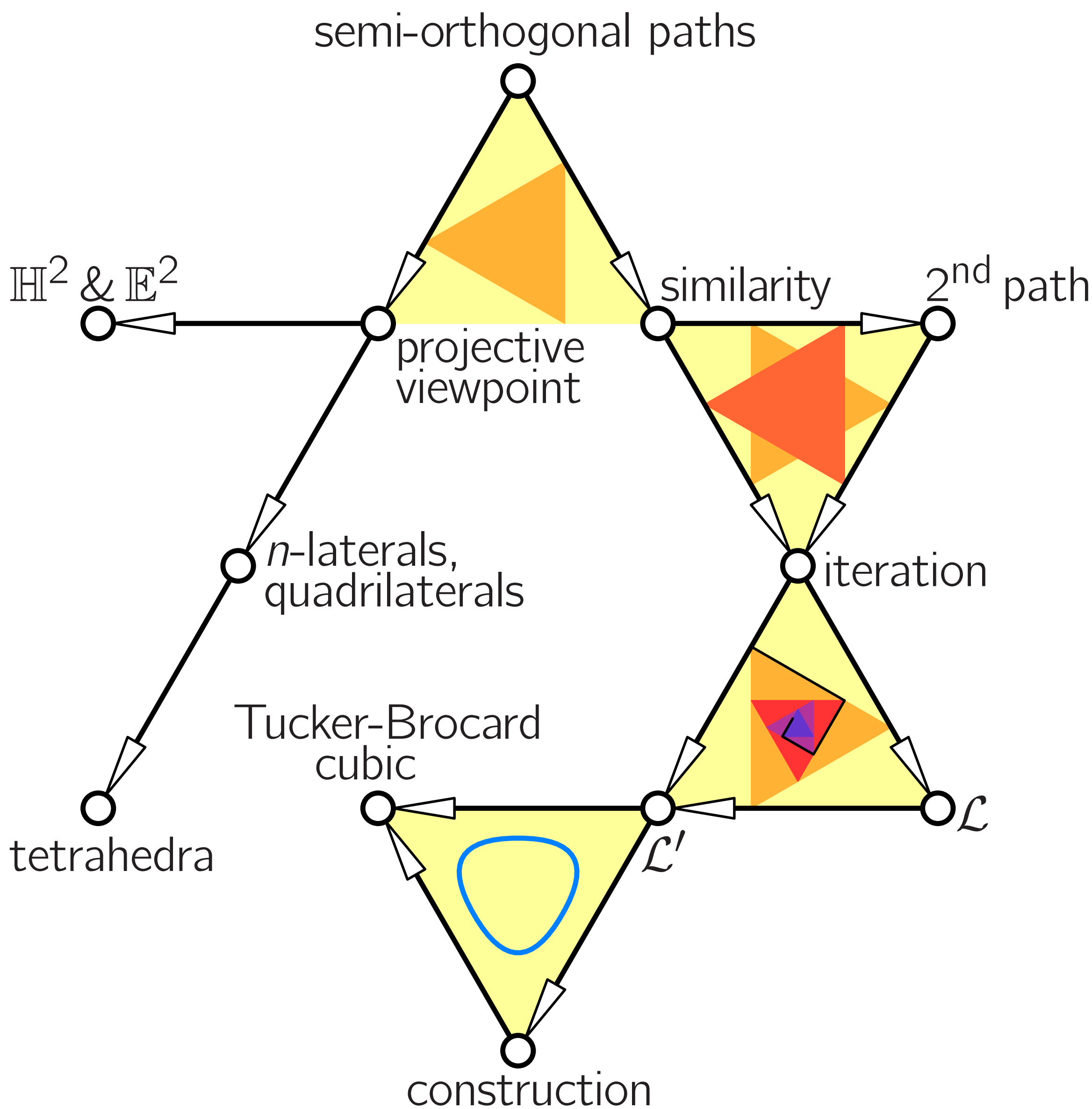
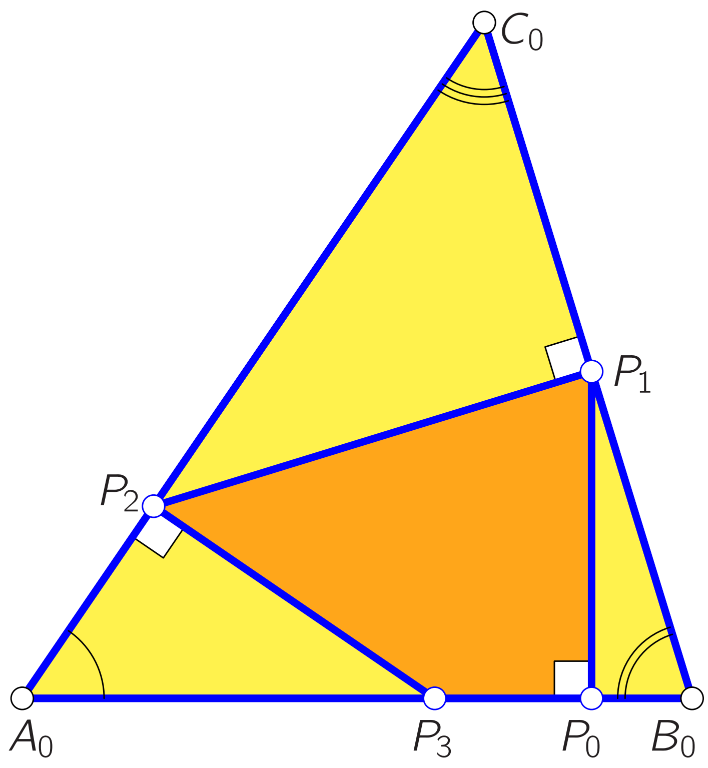
today we serve



semi-orthogonal paths



A semi-orthogonal path $P_0P_1P_2P_3$

in a triangle $\Delta_0 = A_0B_0C_0$:

$P_0 \in [A_0, B_0]$,

$[P_0, P_1] \perp [A_0, B_0]$,

$P_1 \in [B_0, C_0]$

and cyclic reordering with:

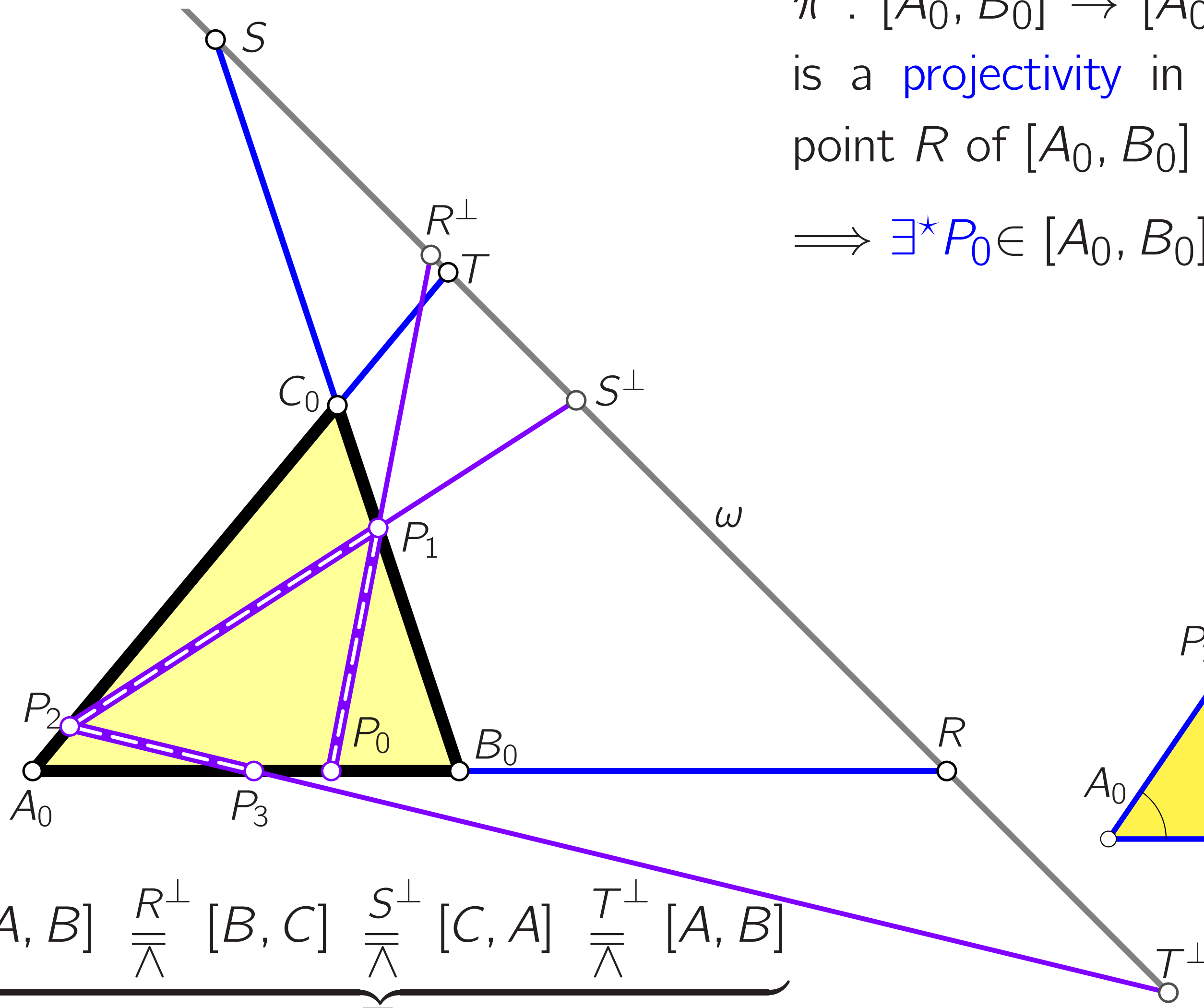
$0 \to 1 \to 2 \to 3 \to 0$, $A \to B \to C \to A$

The path is closed if $P_0 = P_3$.

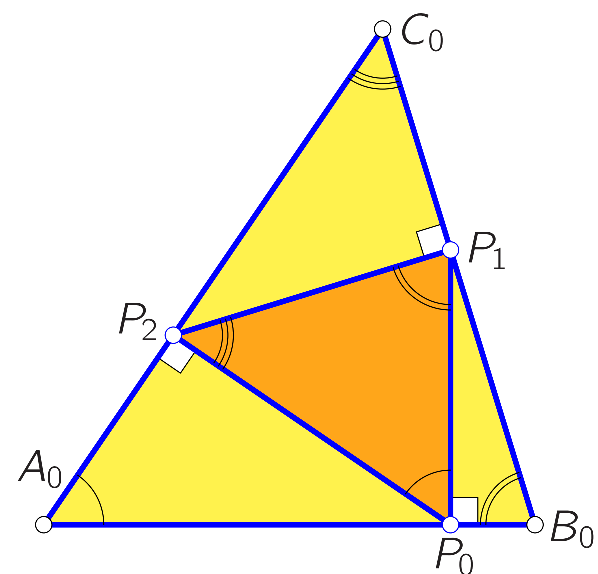
projective point of view

$\pi : [A_0, B_0] \to [A_0, B_0],\ P_0 \mapsto P_3$ is a projectivity in which the ideal point R of $[A_0, B_0]$ is self-assigned.

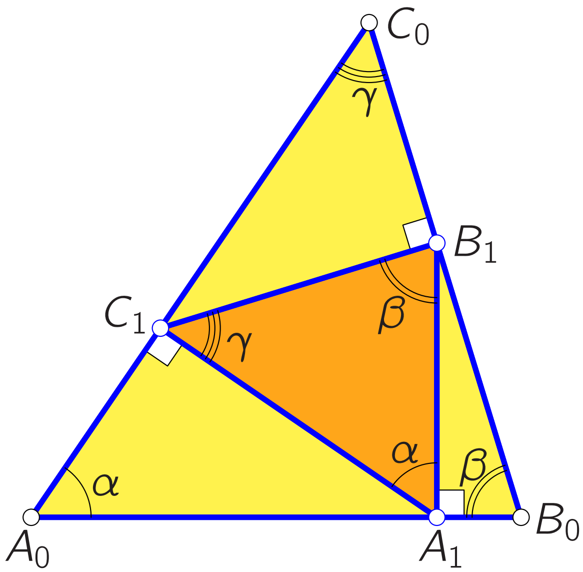
$\Longrightarrow \exists^{\star} P_0 \in [A_0, B_0]$ with $P_0 = P_3$



$$\underbrace{[A, B] \overset{R^{\perp}}{\overline{\wedge}} [B, C] \overset{S^{\perp}}{\overline{\wedge}} [C, A] \overset{T^{\perp}}{\overline{\wedge}} [A, B]}_{\overline{\wedge}}$$



important property: similarity



If $P_0 = P_3$, then let $P_0 = A_1$, $P_1 = B_1$, $P_2 = C_1$.

$\Delta_0 = A_0B_0C_0$ is similar to $\Delta_1 = A_1B_1C_1$.

$\sphericalangle\ C_1A_1B_1 = \pi - (\frac{\pi}{2} - \alpha) - \frac{\pi}{2} = \alpha$,

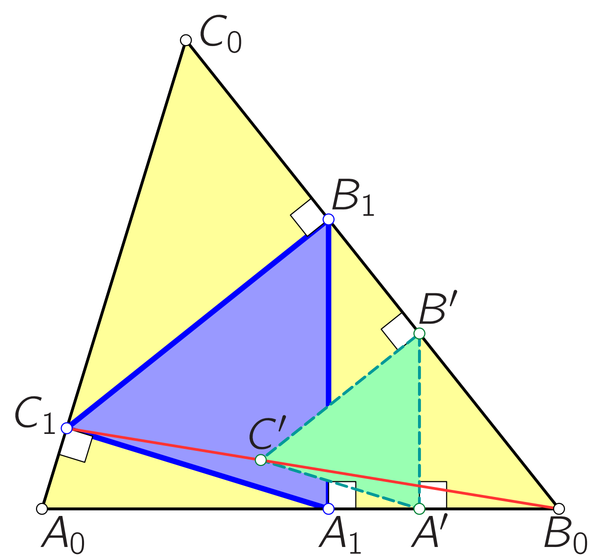
and cyclic . . . .

construction: central similarity $\Delta_0 \sim \Delta_1$

$\Longrightarrow$ There exists a further triangle

$$\nabla^1 \sim \Delta_0 \sim \Delta_1$$

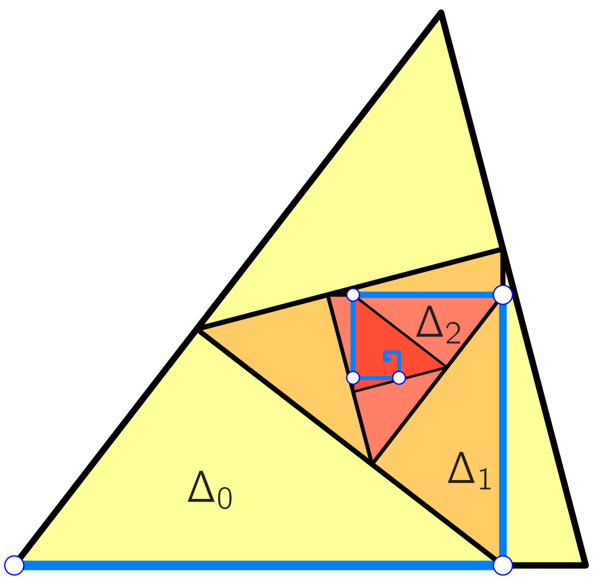
whose edges form a semi-orthogonal path.

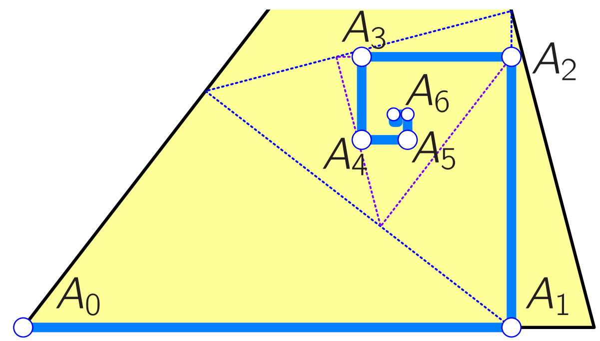


iteration . . .

. . . produces a sequence of nested and similar triangles $\Delta_0, \Delta_1, \Delta_2, \dots$.

The orbit of $A_0 A_1 A_2 \dots$ of A_0 is a discrete logarithmic spiral.





The limit position $\mathcal{L} = \lim_{i \to \infty} A_i$ equals the asymptotic point of the spiral.

$[A_i, A_{i+1}] \parallel_a [A_{i+2}, A_{i+3}]$

$\overline{A_i A_{i+1}} = \lambda \cdot \overline{A_{i+1} A_{i+2}}, \; i = 0, 1, \dots$

$\Longrightarrow$ The position of $\mathcal{L}$ is obtained by computing the sums of infinite geometric series, since $\lambda < 1$.

$\mathcal{L}$

Elementary computations yield: $\lambda = 4F\sigma^{-1}$ with $\sigma = a^2 + b^2 + c^2$, F ... area of Δ_0

$(\Longrightarrow \lambda < 1)$ and $\overline{A_0A_1} = 2b^2c\sigma^{-1}$.

coordinates of $\mathcal{L}$ (w.r.t. the Cartesian frame $A_0 = (0,0)$, $B_0 = (c,0)$):

$x_{\mathcal{L}} = \overline{A_0A_1} - \overline{A_2A_3} + \overline{A_4A_5} - \overline{A_6A_7} \pm \ldots = \overline{A_0A_1} \cdot (1 - \lambda^2 + \lambda^4 \mp \ldots)$,

$y_{\mathcal{L}} = \overline{A_1A_2} - \overline{A_3A_4} + \overline{A_5A_6} - \overline{A_7A_8} \pm \ldots = \overline{A_0A_1} \cdot \lambda \cdot (1 - \lambda^2 + \lambda^4 \mp \ldots)$.

$\lambda < 1 \Longrightarrow 1 - \lambda^2 + \lambda^4 \mp \ldots = \frac{1}{1+\lambda^2}$

$$\Longrightarrow \mathcal{L} = \frac{b^2c}{2\tau}(\sigma, 4F)$$

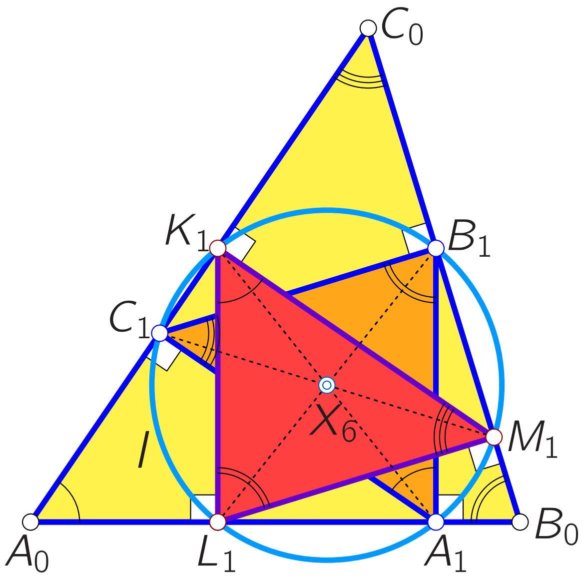
with $\tau = a^2b^2 + b^2c^2 + c^2a^2$

$y_{\mathcal{L}} = \overline{\mathcal{L}[A_0, B_0]} \Longrightarrow$

homogeneous trilinear coordinates w.r.t. Δ_0 $\quad \mathcal{L} = (ac^2 : ba^2 : cb^2)$

$\Longrightarrow \mathcal{L}$ is the first Brocard point Δ_0.

the second semi-orthogonal path and $\mathcal{L}'$

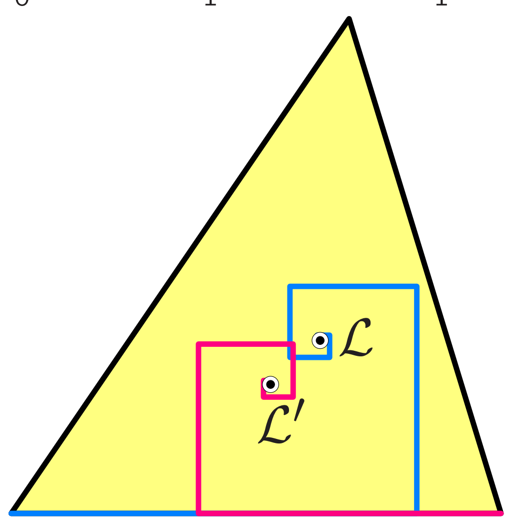


$[L_1, K_1] \perp [A_0, B_0]$, $[K_1, M_1] \perp [C_0, A_0]$, ...

$\nabla_1 = K_1 L_1 M_1 \sim \Delta_0 \Longrightarrow \nabla_1 \sim \Delta_1$

Δ_1 & ∇_1 share the circumcircle I - centered at X_6.

$$\Longrightarrow \Delta^1 \cong \nabla^1$$

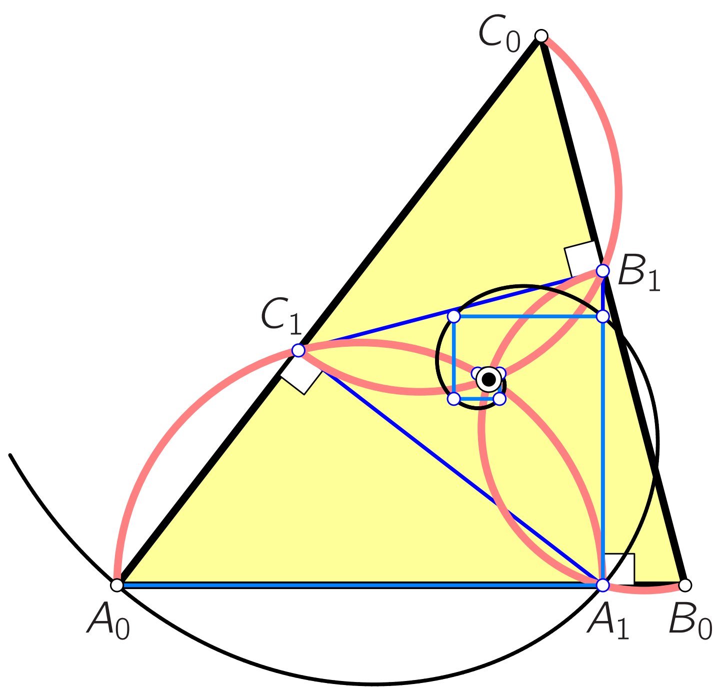


Second family $\Delta_0 \nabla_1 \nabla_2 \ldots$ of triangles also converges to a point.

$$\lim_{i \to \infty} \nabla^i = \mathcal{L}' = (ab^2 : bc^2 : a^2 c)$$

$\Longrightarrow \mathcal{L}'$ is the second Brocard point of Δ_0.

a simple construction of the limits $\mathcal{L}$ and $\mathcal{L}'$



The Thaloids of the three segments A_0A_1, B_0B_1, and B_0B_1 (A_0K_1, B_0L_1, C_0M_1) are concurrent in $\mathcal{L}$ ($\mathcal{L}'$).

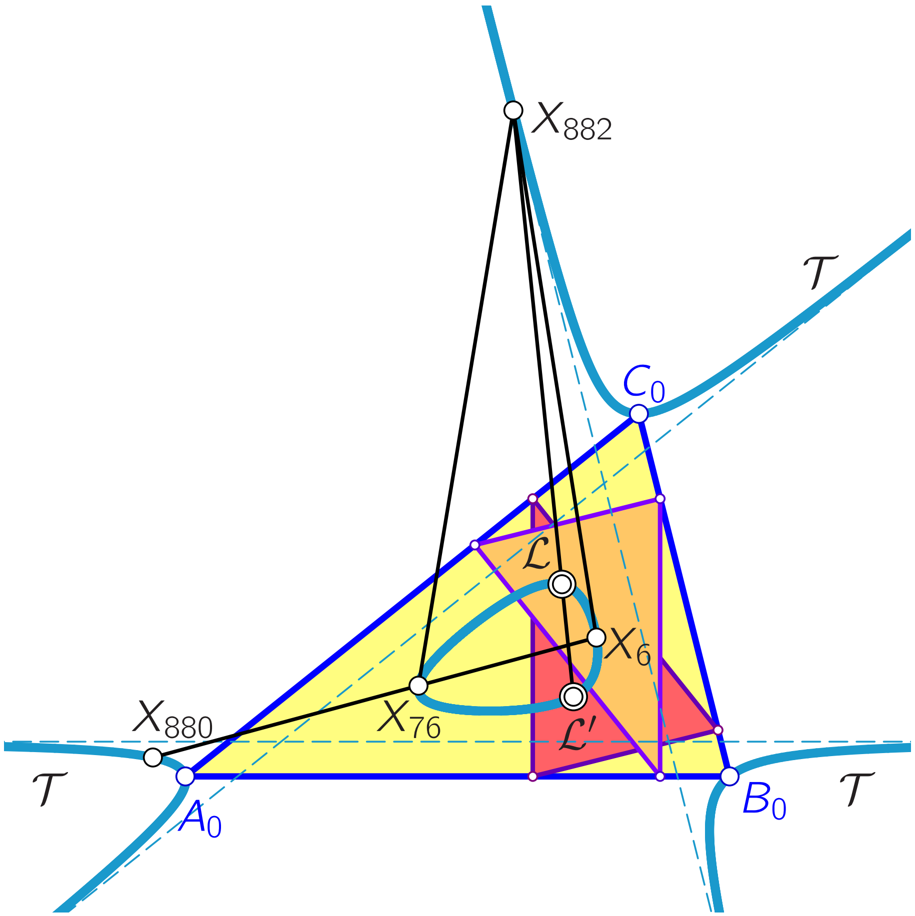
Each segment of the discrete logarithmic spiral corresponds to a 90° turn.

$\Longrightarrow$ Each segment (and so $A_0A_1, \ldots$) is seen at a right angle from the asymptotic point ($\mathcal{L}$ or $\mathcal{L}'$).

The proof can also be done by computation.

This is a new and elementary construction of the two Brocard points of a triangle.

Tucker-Brocard cubic - K012

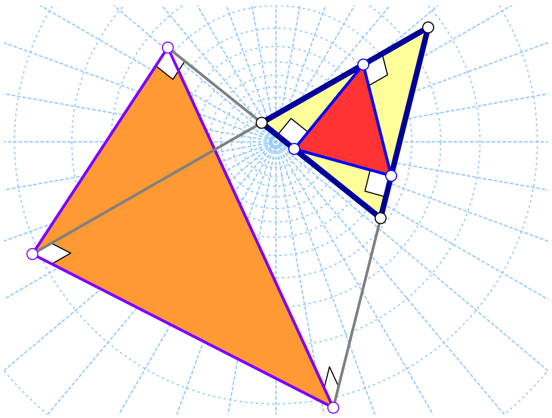


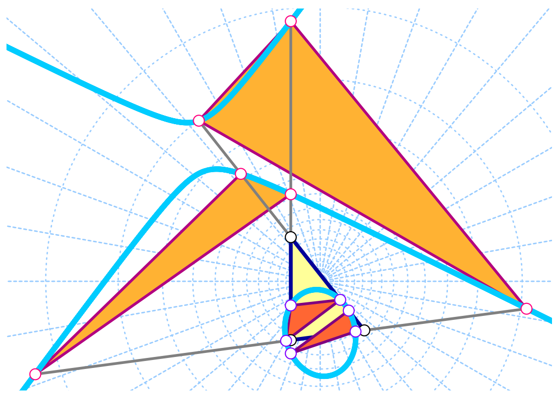
Tucker-Brocard cubic $\mathcal{T}$:
locus of points X such that their Cevian triangles have the same area as the Cevian triangle of the symmedian point X_6
(Lemoine point, Grebe point)

$\mathcal{T}$ = self-isotomic pivotal cubic, pivot = X_6

and contains further:
X_{76} … 3rd Brocard point, $\alpha_3^{[\mathrm{tril}]} = a^{-3}$
X_{880}, X_{882}
and (more or less) surprisingly
the tunnel limits $\mathcal{L}$ and $\mathcal{L}'$
(1. & 2. Brocard point)

non-Euclidean versions

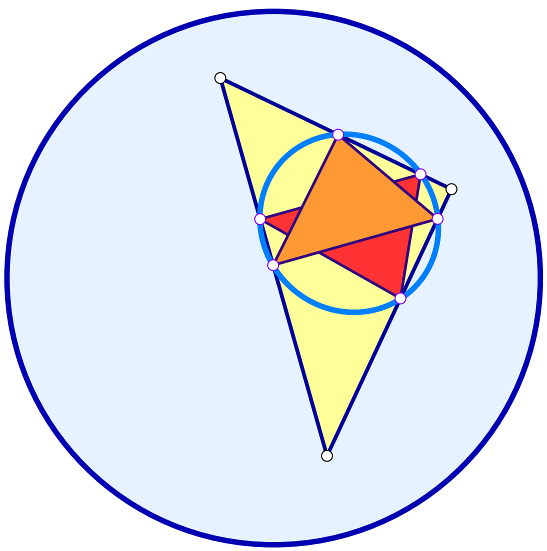




interesting only in non-degenerate CK-geometries ($\mathbb{E}^2$ & $\mathbb{H}^2$):

projective mapping $\pi: \; P_0 \mapsto P_3$ without self-assigned point

$\Longrightarrow$ two different fixed points



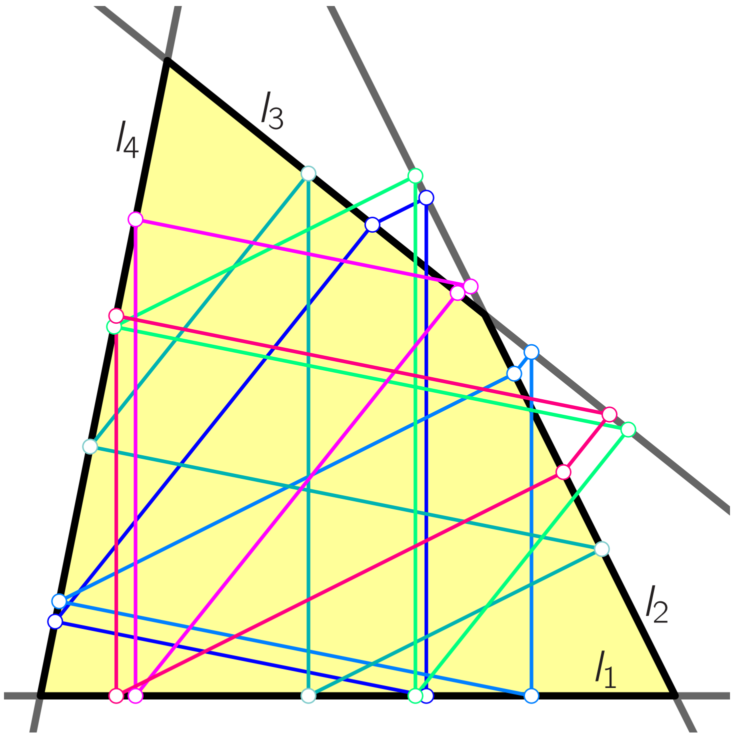
$\mathbb{H}^2$: Only one solution is proper.

Absolute polar triangle of the non-proper solution lies con-conical with the proper one.

$\mathbb{H}^2$ & $\mathbb{E}^2$:

One solution for one orientation of the path finds a companion from the other orientation such that the two triangles lie on one conic.

n-gons



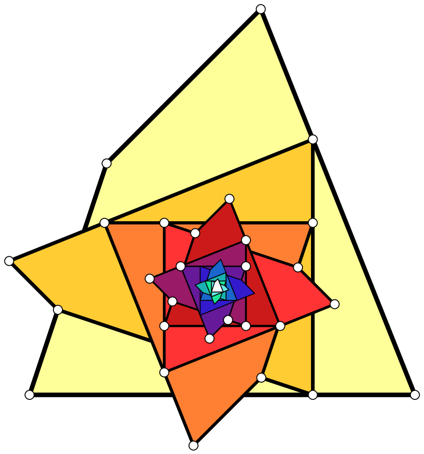
In a generic n-lateral in the Euclidean plane there exist up to $(n-1)!$ different closed semi-orthogonal paths.

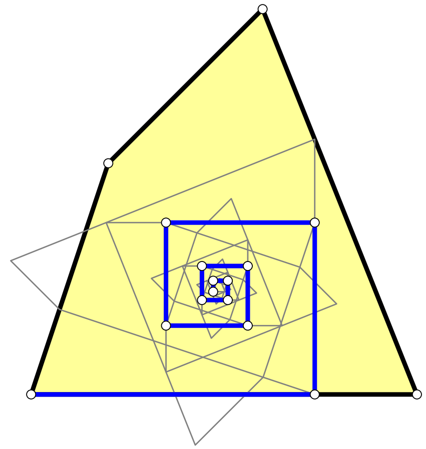
generic n-lateral . . . n different straight lines in the (Euclidean) plane, no right angle enclosed

closed semi-orthogonal path . . . projective mapping $\pi: \ l_i \to l_i$ with self-assigned ideal point

$\Longrightarrow$ one proper fixed point for any ordering of the lines l_i

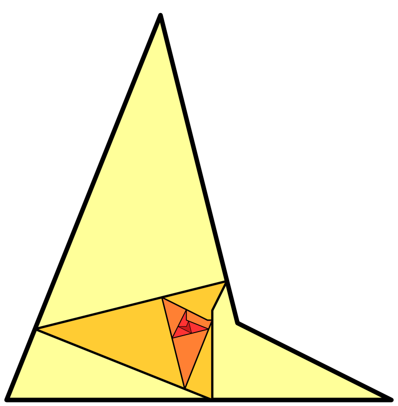
n-gons, quadrilaterals

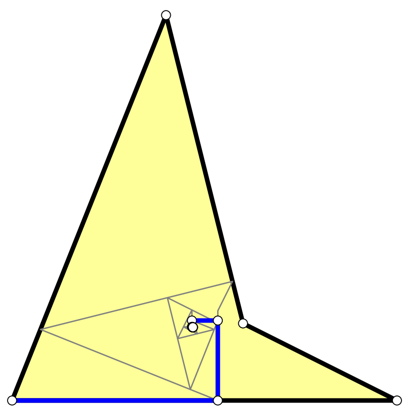




The sequence $Q_0, Q_1, Q_2, \ldots$ of quadrilaterals does in general not contain similar quadrilaterals.

Measures of interior angles of Q_0 can be found in any Q_i $(i > 0)$.



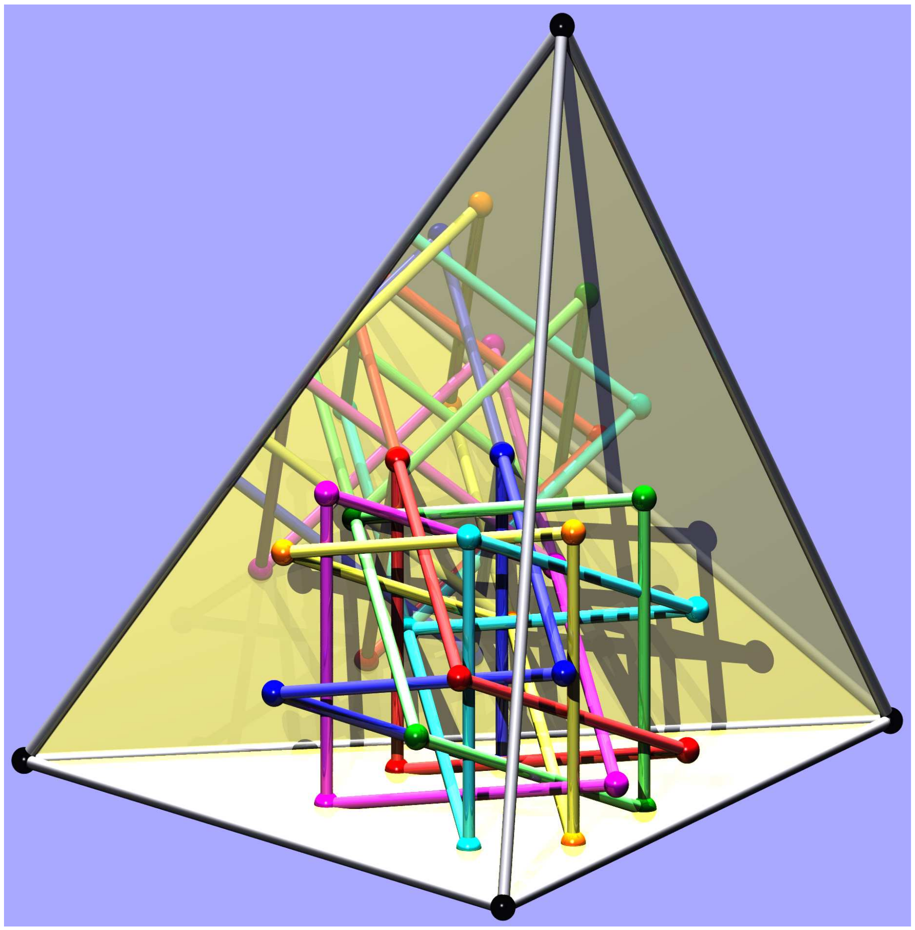


Convexity is not necessary.

The sequences of vertices converge.

The orbits differ from logarithmic spirals.

in three-space



given: a tetrahedron $\mathcal{T} = ABCD$

start at $P_0 \in [A, B, C]$ and find

$P_1 \in [B, C, D]$ with $[P_0, P_1] \perp [A, B, C]$,

$P_2 \in [C, D, A]$ with $[P_1, P_2] \perp [B, C, D]$,

$P_3 \in [D, A, B]$ with $[P_2, P_3] \perp [C, D, A]$,

$P_4 \in [A, B, C]$ with $[P_3, P_4] \perp [D, A, B]$.

$\pi : \; P_0 \mapsto P_4$ is a projective mapping, finite sequence of perspectivities with four coplanar perspectors in the ideal plane

only one real fixed point $P_0 = P_4$ which is the intersection of a pair conjugate complex lines in $[A, B, C]$

in fact: six different paths

related work

[1] B. Gibert: *Tucker-Brocard cubic.* Available at:
`http://bernard.gibert.pagesperso-orange.fr/Exemples/k012.html`

[2] M. Hajja: *On nested sequences of triangles.* Result. Math. **54** (3), 289–299.

[3] C. Kimberling: *Triangle Centers and Central Triangles.* (Congressus Numerantium Vol. 129) Utilitas Mathematica Publishing, Winnipeg, 1998.

[4] C. Kimberling: *Encyclopedia of Triangle Centers*. Available at:
`http://faculty.evansville.edu/ck6/encyclopedia`

[5] C. Kimberling: *Bicentric Pairs of Points and Related Triangle Centers.* Forum Geometricorum **3** (2003), 35–47.

[6] E.L. Wachspress: *A rational basis for function approximation.* Lecture Notes in Mathematics. Vol. 228, Springer, New York, 1971.

Thank You For Your Attention!