**16th Scientific-Professional Colloquium on Geometry and Graphics**
**September 9 – 13, Baška, Krk, 2012**

# Porisms

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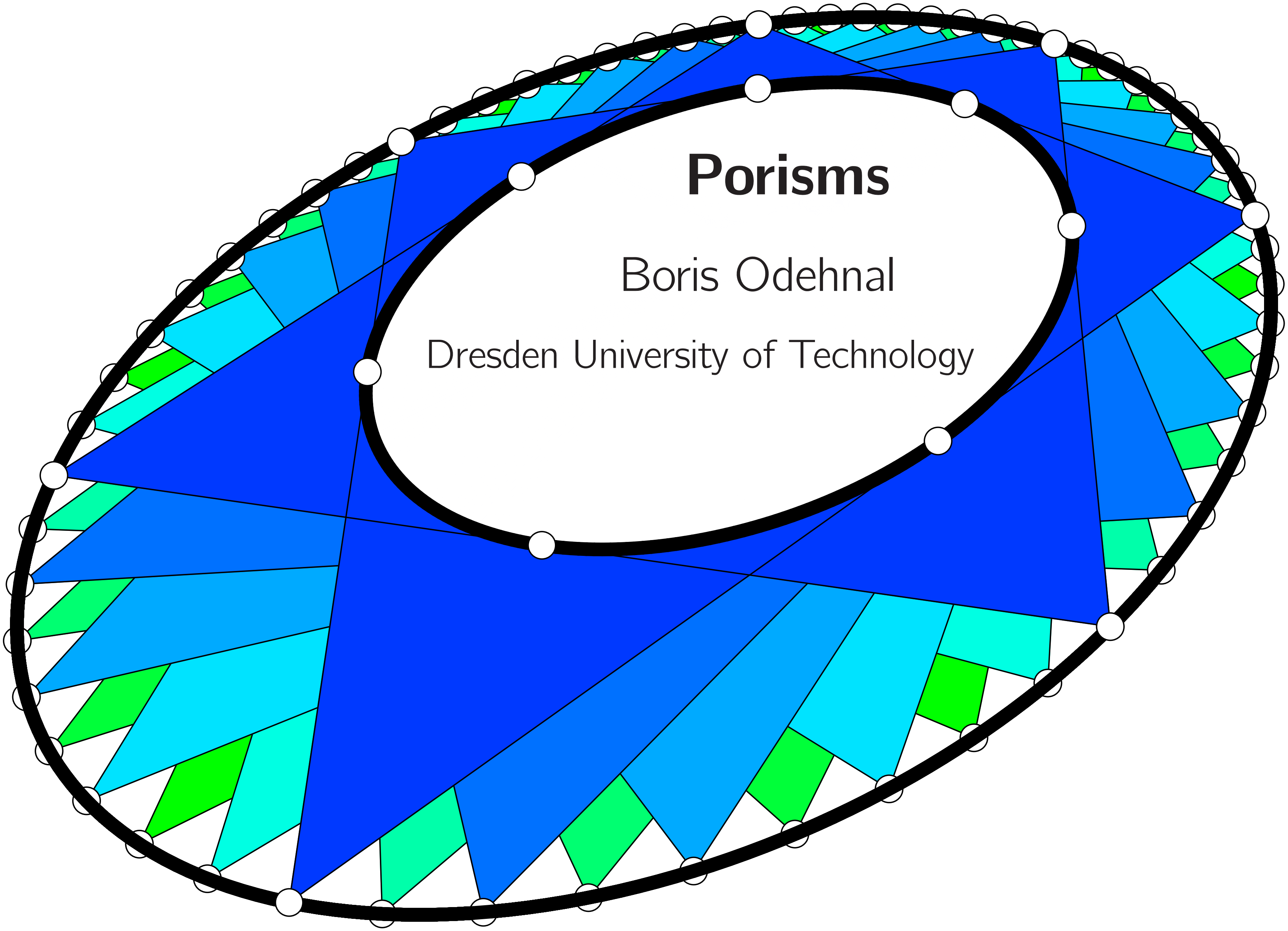


## What is a porism?

- something in between a theorem and a problem (Euklid)
- indetermined or unsolvable problem
- geometric locus
- theorems from projective geometry

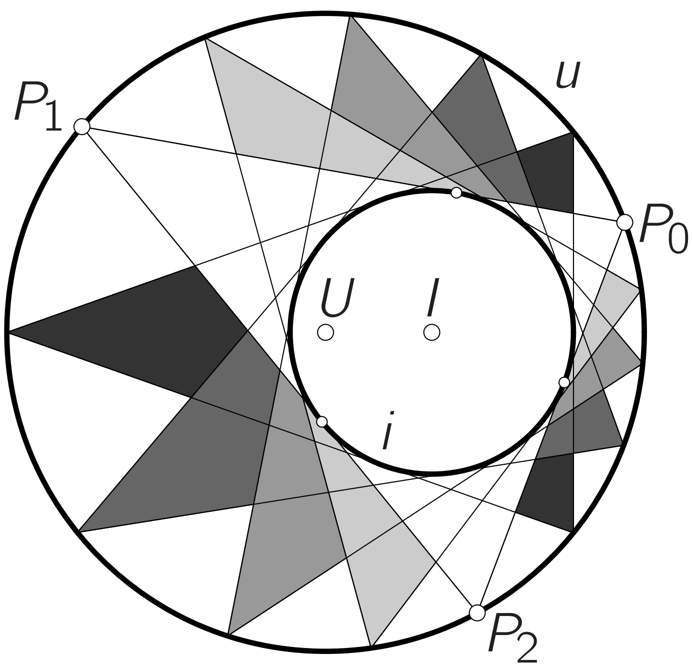
The meaning of the word has changed (more than once)!

Nowadays: A porism is a closure theorem, or closure property,
or a geometric figure/construction that closes somehow.

## Today's menu

| | |
|---|---|
| Apéritif | some examples from triangle geometry |
| Starter | porisms of/with circles and spheres |
| Soup | Poncelet porisms and others |
| Main course | how to prove Poncelet's theorem |
| Desert | Cayley's theorem |
| Digestif | algebraic correspondences, multiple binary forms |

## examples




$(U, u)$ circumcenter, circumcircle

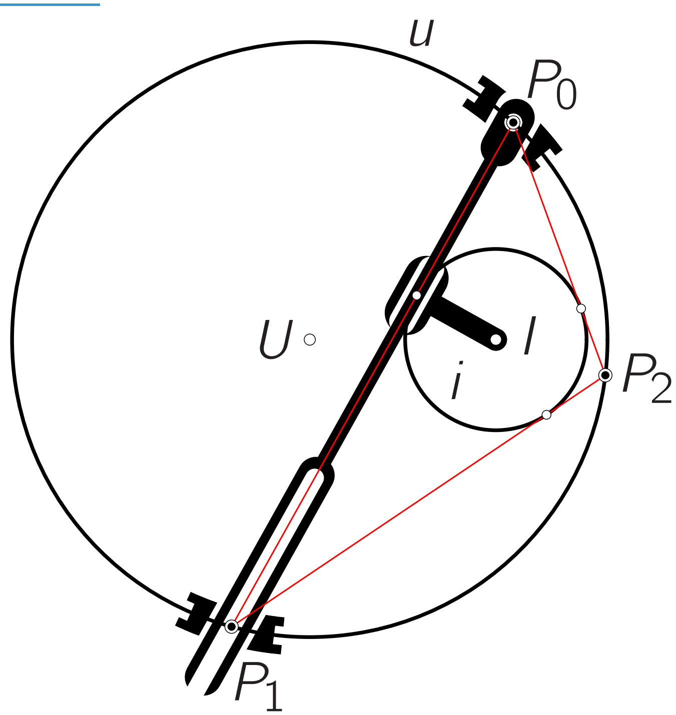
$(I, i)$ incenter, incircle

of a triangle $\Delta = (P_0, P_1, P_2)$ (white)

$u$ and $i$ are circum- an incircle of a smooth one-parameter family of triangles.

Porism: **If** the polygon $(P_0, P_1, P_2)$ (s.t. $P_k \in u$ and $[P_k, P_{k+1}] \in i^\star$, $k$ mod 3) is closed **for one** choice of $P_0 \in u$, **then** it is closed **for any** choice of $P_0 \in u$.

## examples




The poristic motion of $(P_0, P_1, P_2)$ is **not a rigid body** motion.

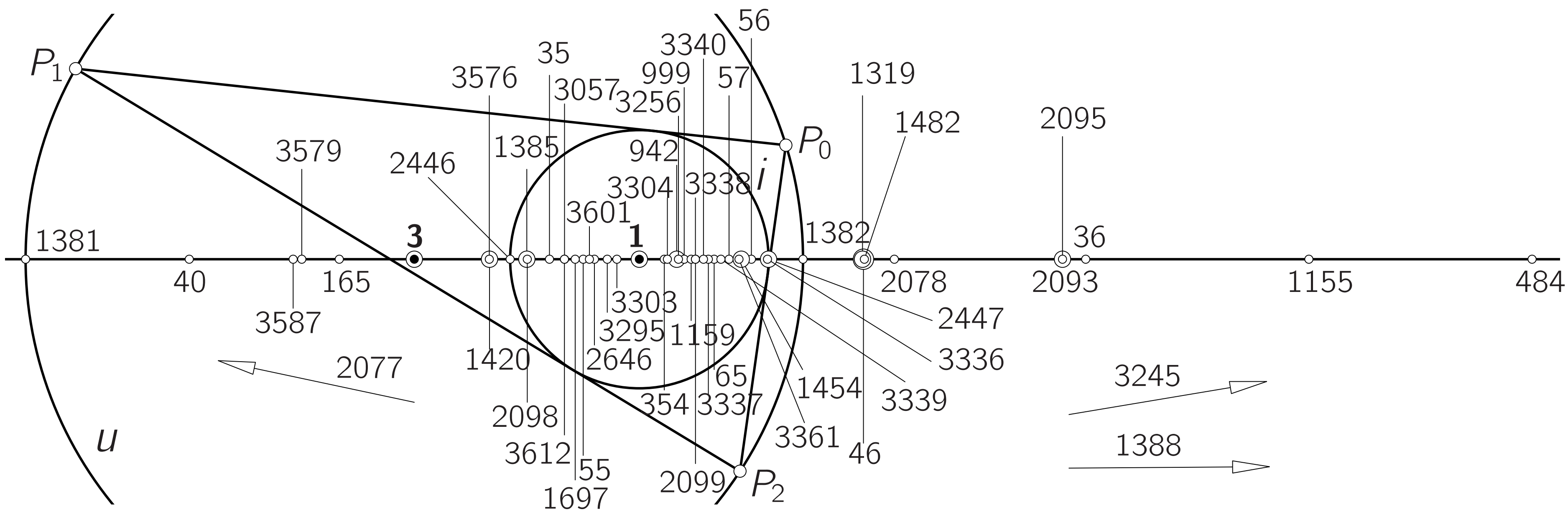
The motion of any **single** side (line) $[P_i, P_{i+1}]$ can be realized by such a $(\longleftarrow)$ mechanism with four moving systems: one rotating about $I$, two gliding along $u$, one $([P_0, P_1] \longleftarrow)$ rotating about $P_0$ and gliding through $P_1$.

Any side of $\Delta$ attains the position of $[P_0, P_1]$ once while $\Delta$ traces the poristic family.

$$\Longrightarrow$$

The poristic path of any point associated with $\Delta$ is traced at least three times.

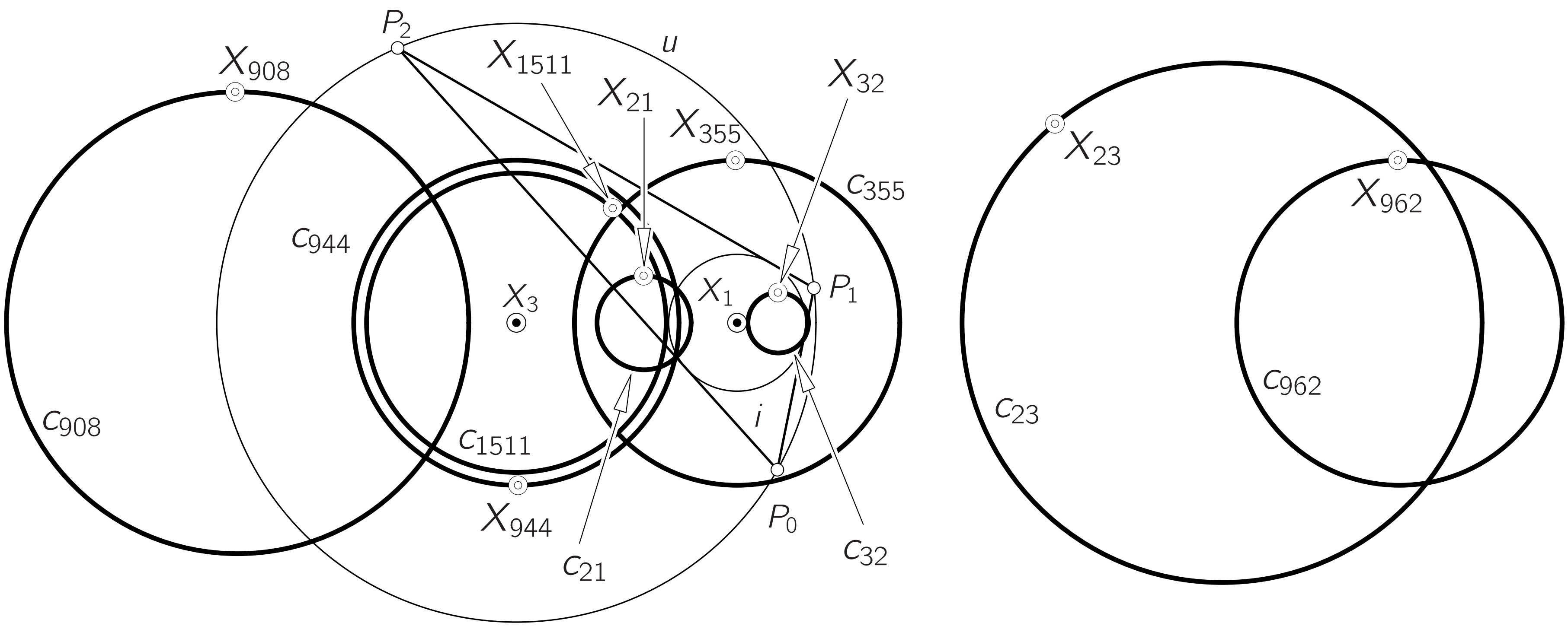
## examples




During the “poristic motion” of $(P_0, P_1, P_2)$ these (↑) triangle centers are fixed.

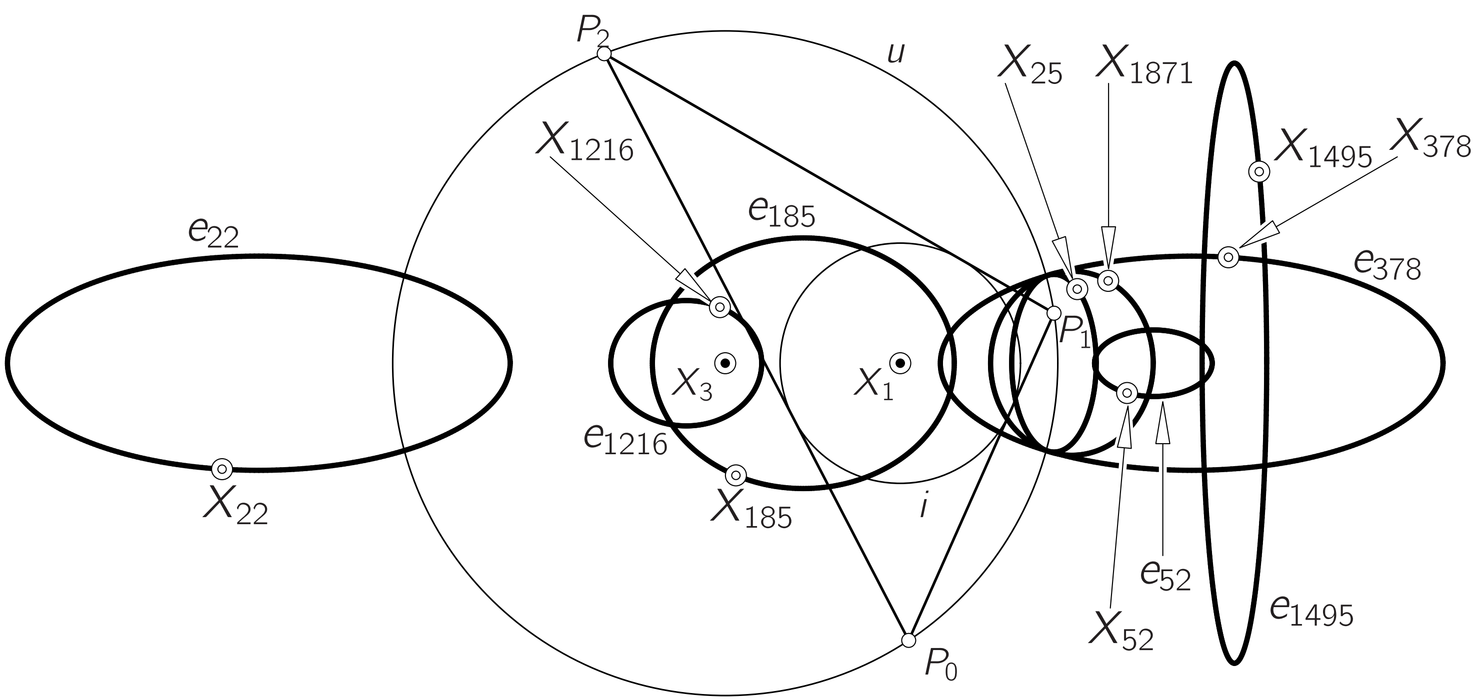
More precise: They are the same for all triangles in the poristic family.

## examples



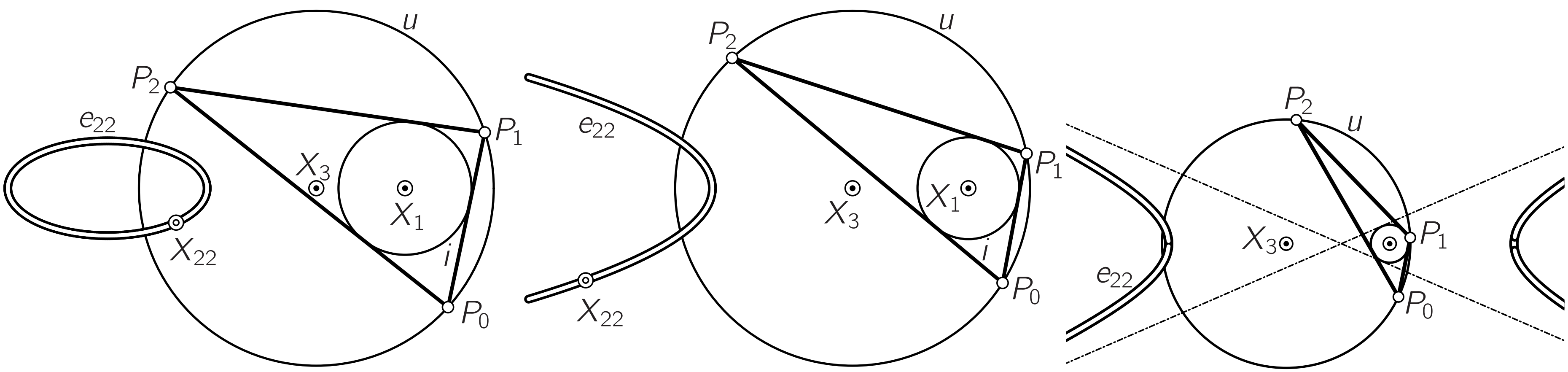

During the "poristic motion" of $(P_0, P_1, P_2)$ these (↑) triangle centers **move on circles**. The circles are traced three times.

## examples




Whereas these (↑) triangle centers move on ellipses. The ellipses are traced three times.

## examples




The poristic trace of the Exeter point $X_{22}$ can be an ellipse, or parabola, or hyperbola.

$X_{22}$ is the perspector of the circummedial triangle and $\Delta_t$.

## examples

Circles $u$ and $i$ are circum-/incircle of one (and than $\infty^1$) triangle(s), if and only if

$$R^2 - 2Rr = d^2$$

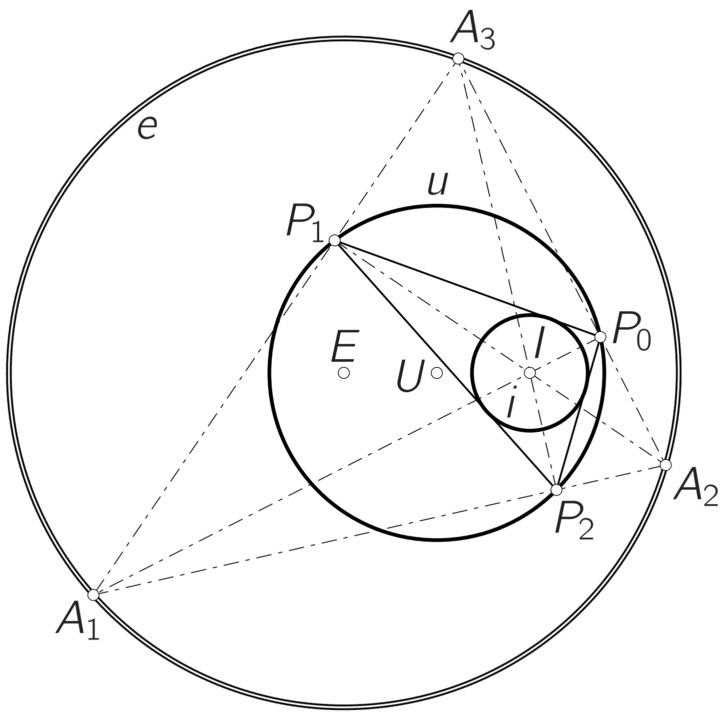
holds, where $d = \overline{UI}$, $r$ and $R$ are in-/circumradius (one of many Euler formulas).

Usually: A family of triangles with a common circumcircle and incircle is called a **poristic family**.

There are many other porisms related to triangles.

## examples

$\Delta = (P_0, P_1, P_2)$ ... base triangle, $\Delta_e = (A_1, A_2, A_3)$ ... excentral triangle of $\Delta$, $\Delta_o(\Delta_e) = \Delta$ ... orthic triangle of $\Delta_e$



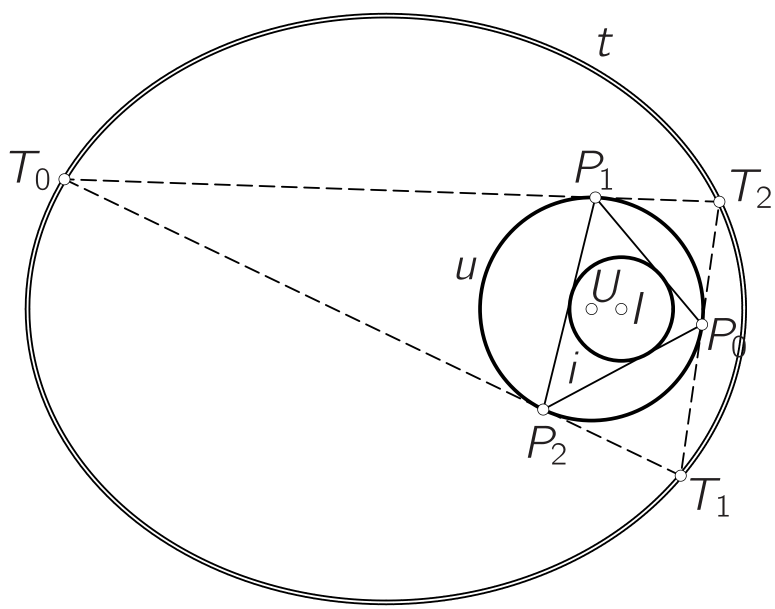

The excenters of triangles from a poristic family trace a circle $e$.

$u$ = ninepoint circle of $\Delta_e \Longrightarrow$ poristic family with common circumcircle $e$ and ninepoint circle $u$

$\Longrightarrow$ poristic family with common circumcircle $e$ and common incircle $i$ of the orthic triangle

## examples

$\Delta = (P_0, P_1, P_2)$ … base triangle, $\Delta_t = (T_0, T_1, T_2)$ … tangent triangle of $\Delta$,
$\Delta_i(\Delta_t) = \Delta$ … intouch triangle of $\Delta_t$



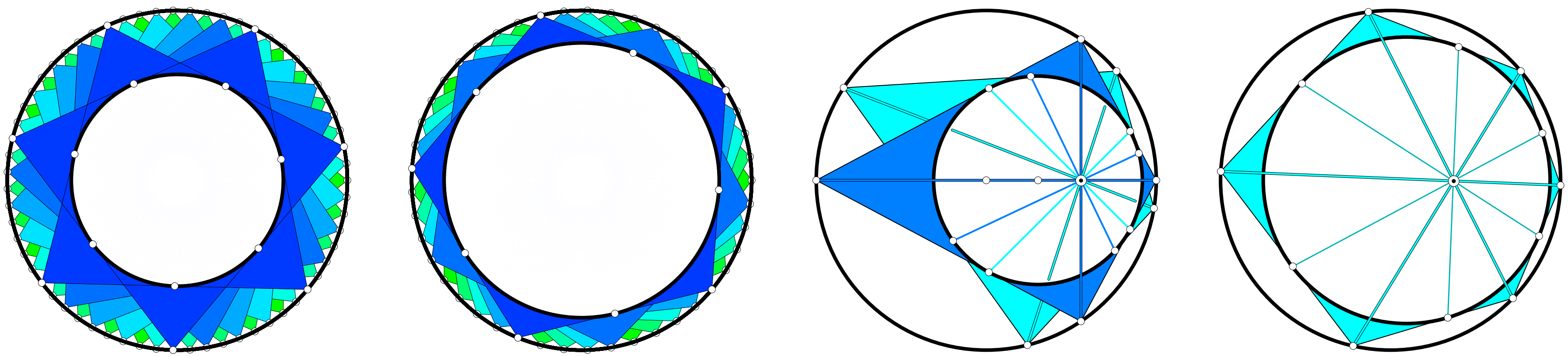

Vertices $T_i$ of $\Delta_t$ of triangles of a poristic family trace an ellipse $t$.

$\Longrightarrow$ poristic family of triangles $\Delta_t$ with a fixed circumellipse $t$ and common incircle $u$

$\Longrightarrow$ poristic family of triangles $\Delta_t$ with a fixed circumellipse $t$ and common incircle $i$ of the intouch triangle $\Delta_i$

## examples

$n$-gons with incircle $i$ and circumcircle $u$ are called bicentric (not necessarily regular, convex). For even $n$ the diagonals (joining opposite vertices) are concurrent.



| star shaped regular | convex regular | $n$ even $\neg$ regular | $n$ even $\neg$ regular |
|---|---|---|---|

## examples

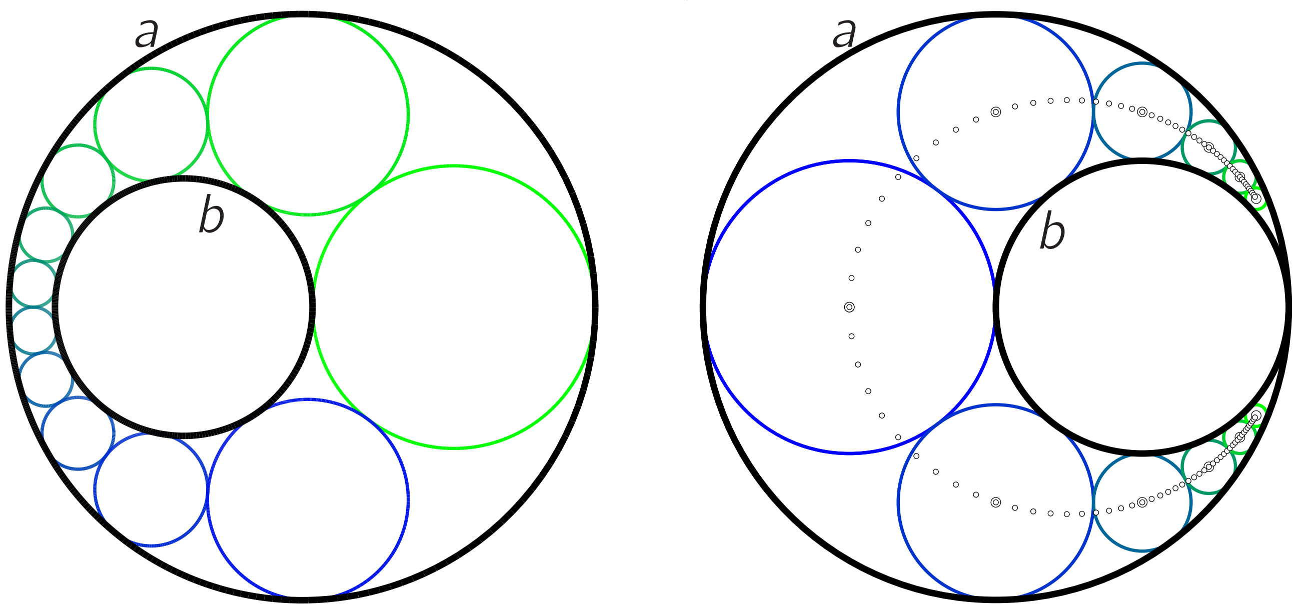
The relations between $R$, $r$, and $d$ are algebraic (indeed polynomial) for any $n$:

| $n$ | relation | |
|---|---|---|
| 3 | $d^2 = R^2 - 2Rr$ | E (S, R, K, A-C) |
| 4 | $(R^2 - d^2)^2 = 2r^2(R^2 + d^2)$ | C, J, K, ... |
| 5 | $r(R-d)=(R+d)\left(\sqrt{(R-r)^2-d^2}+\sqrt{2R(R-r-d)}\right)$ | S, K |
| 6 | $3(R^2-d^2)^4=4r^2(R^2+d^2)(R^2-d^2)^2+16r^4d^2R^2$ | $S^?$, R, K |
| 7 | ... | J, K |
| 8 | ... | J, $S^?$, R |
| 9 | also 10, 12, 14 | R |

A-C=Altshiller-Court (1952), C=Casey (1888), E=Euler (?), J=Jacobi (1823), K=Kerawala (1947), R=Richelot (1830)[257], S=Steiner (1827)

## examples

Steiner chains are sequences $(c) = (c_0, c_1, \ldots, c_n)$ of circles touching two circles $a$, $b$ s.t. $c_i$ is also in contact with $c_{i-1}$ and $c_{i+1}$.




If $c_0 = c_n$ for some $n \in \mathbb{N} \setminus \{0, 1, 2\}$, then $(c)$ is called a poristic chain.

## examples

**Steiner porism: If** the chain $(c)$ is **closed for one** initial circle $c_0$, **then** it is closed **for any** choice of admissable initial circle.

The ring-shaped chain can be "rotated" in between $a$ and $b$.
rotation $=$ equiform rotation

What about the needle-shaped chain?
There is exactly one circle of radius 0 and the equiform transformation becomes singular.

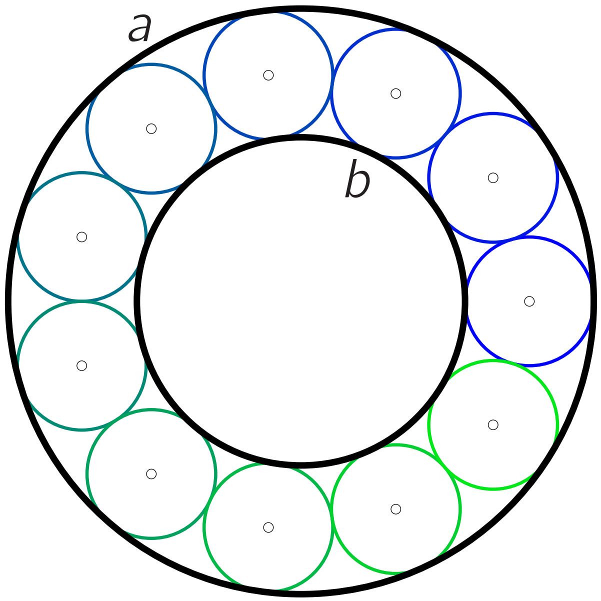
## examples

Apply an inversion and map $a$ and $b$ to concentric circles with $r_a > r_b$. The chain $(c)$ is closed, if

$$\frac{r_a - r_b}{r_a + r_b} = \sin \frac{\pi}{n}.$$

Rational porisms exist [P. Yiu, FG 11/27].

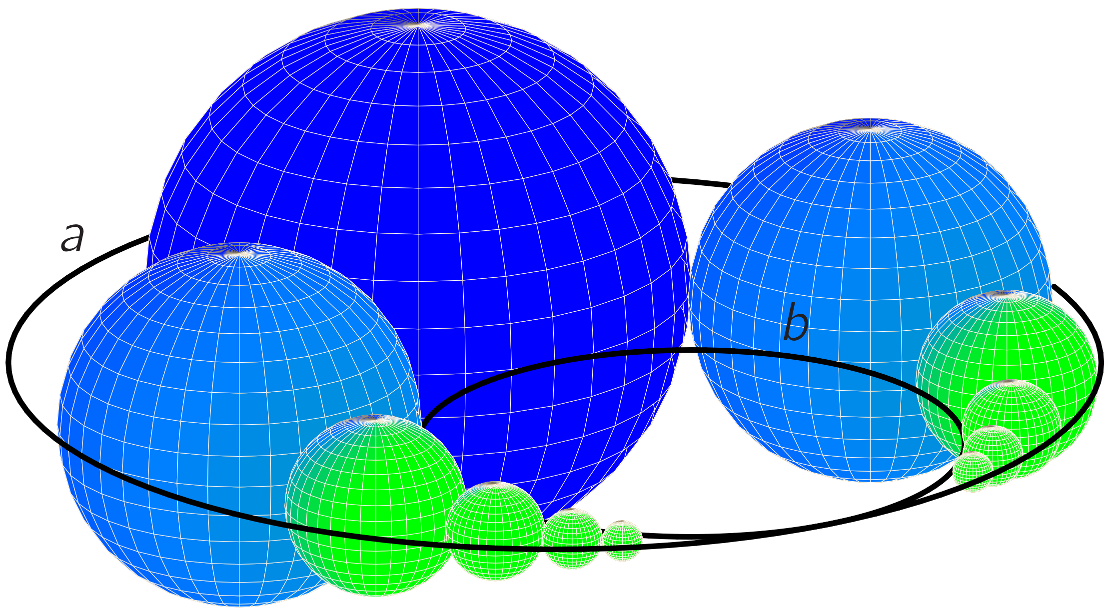
Centers of circles lie on an ellipse if $a$ and $b$ are not concentric.

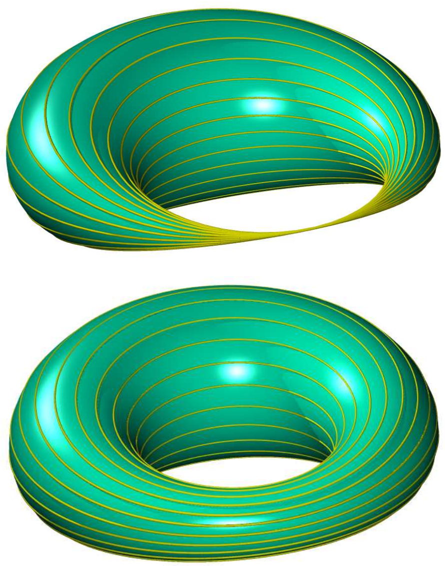



The last three figures can be interpreted as cross sections of Dupin cyclides.

## examples

The envelopes of spheres from Steiner chains are Dupin cyclides.

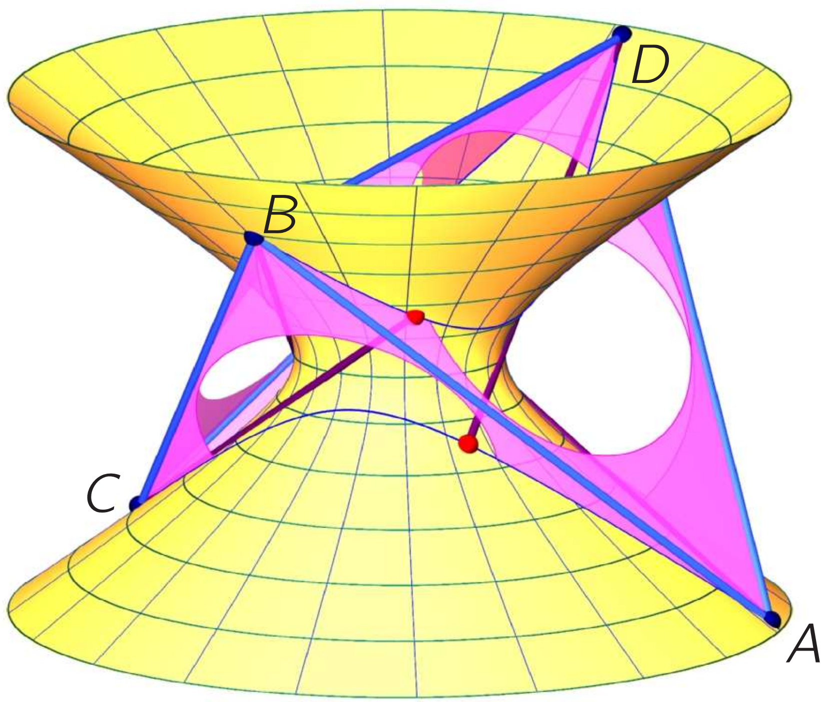


The chain of spheres can be rotated freely in the cyclide (equiform motion). According to P. Yiu there exist rational sphere porisms. $n = 6 \Longrightarrow$ Soddy hexlet

## example

The altitudes of any non-orthocentric tetrahedron $\mathcal{T} = (A, B, C, D)$ lie in a ruled quadric $Q$ centered in $\mathcal{T}$'s Monge point.




If $Q$ is a quadric of revolution, then there are infinitely many such tetrahedra.

The rotation of $\mathcal{T}$ is a rigid body motion.

Porism: **If** there is **one** such tetrahedron $\mathcal{T}$ with altitudes in $Q$, **then** there are **infinitely many** congruent copies of $\mathcal{T}$ with the same properties.

## Poncelet

So far most of the examples deal with porisms in circles. Poncelet's porism is a more general and specific version (no spheres, no tetrahedra) and contains many of the examples:

Assumption:

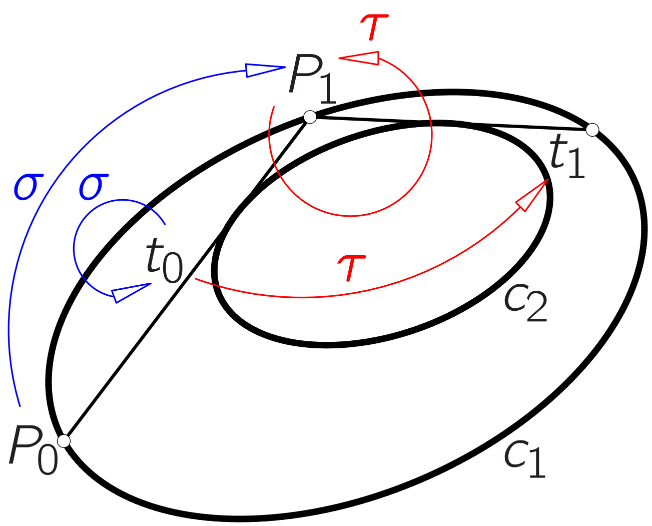
Given two conic sections $c_1$, $c_2$ (in general position).

Theorem:

If there exists an $n$-gon $(P_0, P_1, \dots P_{n-1})$ s.t. $P_i \in c_1$ and $t_i := [P_i, P_{i+1}] \in c_2^\star$, then there exist infinitely many such polygons.

$c_2^\star$ is the set of tangents of $c_2$.

## Poncelet - the proof




**Proof - sketch of a proof.**

Assumption: $c_1$, $c_2$ in general position (in GP), *i.e.*, $\#(c_1 \cap c_2) = 4$ (not algebraically counted).

$\Longrightarrow$ $c_1^\star$ and $c_2^\star$ in GP (four common tangents)

Poncelet correspondence $\mathcal{P} := \{(P, t) : P \in c_1, t \in c_2^\star, P \in t\}$

two mappings on $\mathcal{P}$: $\sigma(P_0, t_0) := (P_1, t_0)$ and $\tau(P_1, t_0) := (P_1, t_1)$

Both are involutive: $\sigma^2 = \mathrm{id}_{\mathcal{P}}$, $\tau^2 = \mathrm{id}_{\mathcal{P}}$.

The Poncelet map $\eta := \tau \circ \sigma$ acts bijective on $\mathcal{P}$.

## Poncelet - the proof

$\mathcal{P}$ is a two-sheeted cover of $c_1$ with branch points $(P, t)$, where $P \in c_1 \cap c_2$

$\mathcal{P}$ is a two-sheeted cover of $c_2^\star$ with branch points $(P, t)$, where $t \in c_1^\star \cap c_2^\star$

$\sigma$ and $\tau$ interchange sheets and leave branches fixed.

The sets of branch points ...

$\mathcal{B}_{c_1} := \{(P, t) \in \mathcal{P} : P \in c_1 \cap c_2\}$, $\mathcal{B}_{c_2^\star} := \{(P, t) \in \mathcal{P} : t \in c_1^\star \cap c_2^\star\}$

... unite to

$\mathcal{F} = \{(P, t) : P \in c_1 \cap c_2, t \in c_1^\star \cap c_2^\star\}$

with

$\mathcal{F} = \emptyset \iff c_1, c_2$ in GP $\iff c_1^\star, c_2^\star$ in GP

Important fact: $c_i, c_i^\star \cong \overline{\mathbb{C}} = \mathbb{C} \cup \infty \Longrightarrow c_i, c_i^\star \cong \mathbb{P}^1(\mathbb{C})$

## Poncelet - the proof

We derive an algebraic equation for $\mathcal{P}$.

$c_0 : \mathbb{C} \to \mathbb{P}^2(\mathbb{C})$ ... affine conic section with $c_0(u) = (1, u, u^2)$, $u \in \mathbb{C}$

define $c_0(\infty) := (0 : 0 : 1) \Longrightarrow c_0 : \overline{\mathbb{C}} \to \mathbb{P}^2(\mathbb{C})$

We start with parametrizations of $c_1$ and $c_2^\star$ over $\overline{\mathbb{C}}$:

$$\begin{array}{rcl} P(u) & = & C_1 \cdot c_0(u) \\ t(v) & = & {C_2^\star}^T \cdot c_0(v) \end{array} \quad u, v \in \overline{\mathbb{C}} \quad \begin{array}{rcl} P : \overline{\mathbb{C}} & \to & c_1 \\ t : \overline{\mathbb{C}} & \to & c_2^\star \end{array} \quad \text{one-to-one \& onto,}$$

where $C_1, C_2^\star \in \mathrm{GL}(\mathbb{C}, 3)$.

$P(u)$ is parametrization of the points on $c_1$, $t(v)$ is a parametrization of the tangents of $c_2$ (or the points of $c_2^\star$).

## Poncelet - the proof

A pair $(P, t)$ is a "point" of $\mathcal{P} \iff P \in t \iff \langle P, t \rangle = 0$. This gives the equation of $\mathcal{P}$ in the $(u, v)$-plane $\overline{\mathbb{C}}^2$:

$$M(u, v) = c_0(v)^{\mathsf{T}} \cdot C_2^{\star\mathsf{T}} \cdot C_1 \cdot c_0(u) = 0.$$

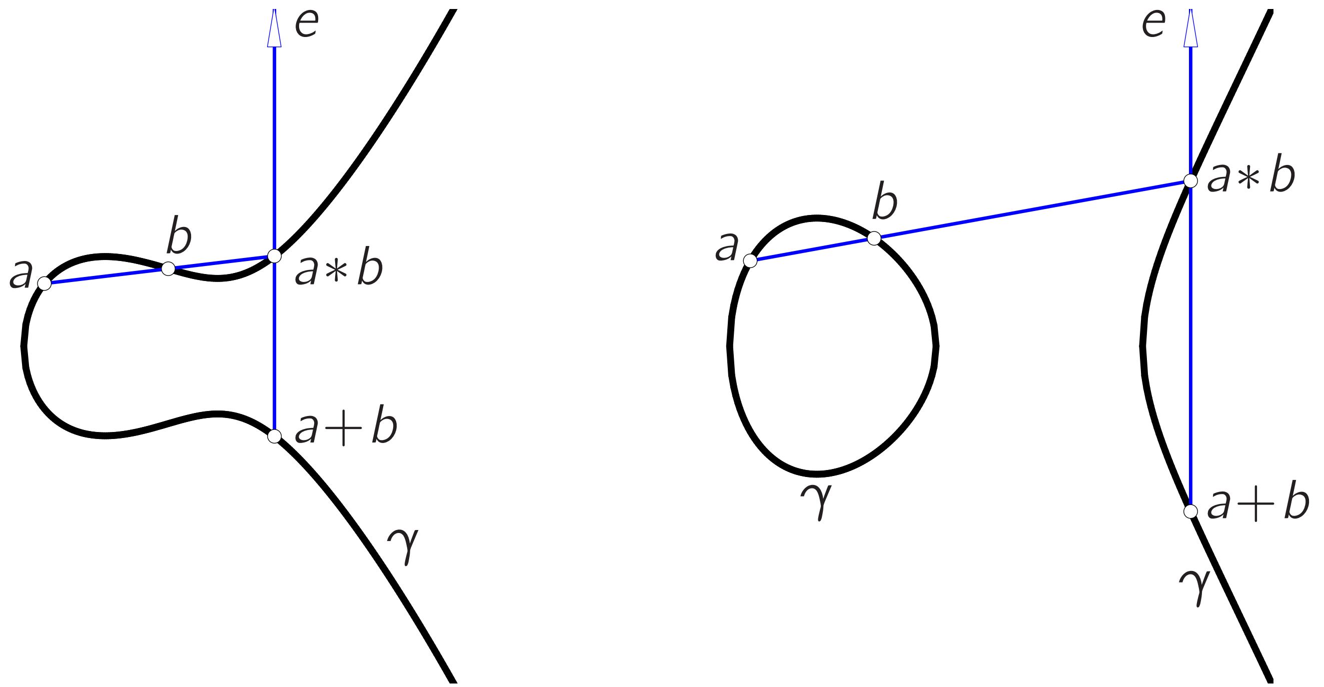
$M(u, v) = 0$ is the equation of an algebraic curve $\gamma \subset \overline{\mathbb{C}}^2$ with $\deg \gamma \leq 4$.

$\gamma$ is elliptic and of genus 1 and as a Riemann surface isomorphic to a torus.

Elliptic curves can be endowed with a unique analytic group structure. They can be parametrized with the Weierstraß-function. There exists an isomorphism $\phi : \mathbb{C}/\Lambda \to \gamma(\mathcal{P})$ for some lattice $\Lambda$.

## Poncelet - the proof

Geometric realization of the group structure on elliptic curves:




These two cases look different but only from the real point of view.

## Poncelet - the proof

The mapping $m(P(u), t(v)) : \overline{\mathbb{C}}^2 \rightarrow c_1 \times c_2^\star$ is one-to-one and onto, also its restriction $m|_\gamma : \gamma \rightarrow \mathcal{P}$.

$m(Q, u) \in \mathcal{F} \Longleftrightarrow (Q, u)$ is a singular point of $\gamma$

For $c_1$, $c_2$ in GP $\Longrightarrow \mathcal{F} = \emptyset \Longrightarrow \gamma$ is smooth in $\overline{\mathbb{C}}^2$.

$\Longrightarrow$ If $c_1$, $c_2$ in GP, then $\gamma$ is smooth in $\overline{\mathbb{C}}^2$ with complex structure inherited by $m$.

$\gamma$ is an elliptic curve: verification rather technical, solving $M = 0$ w.r.t. $u$ (or $v$) involves a cubic discriminant with no multiple roots (since $c_1$, $c_2$ in GP).

## Poncelet - the proof

The mappings $\sigma$, $\tau$, and $\eta = \tau \circ \sigma$ induce automorphisms of $\mathcal{P}$:

$\sigma^\star = m^{-1} \circ \sigma \circ m$ . . . involution on $\gamma$, interchanges points with same $v$-coordinate

$\tau^\star = m^{-1} \circ \tau \circ m$ . . . involution on $\gamma$, interchanges points with same $u$-coordinate

$\gamma$ is a two-sheeted branched cover of the $u$-/$v$-spheres $\Longrightarrow$ $\sigma^\star/\tau^\star$ interchange the $v$-/$u$-sheets. $\Longrightarrow$

$\sigma^\star$, $\tau^\star$ . . . automorphisms of $\gamma$ and $m$ lifts them to automorphisms of $\mathcal{P}$. $\Longleftrightarrow$

$$\eta^\star = \tau^\star \circ \sigma^\star = m^{-1} \circ \tau \circ m \circ m^{-1} \circ \sigma \circ m = m^{-1} \circ \tau \circ \sigma \circ m = m^{-1} \circ \eta \circ m$$

is an automorphism of $\mathcal{P}$.

## Poncelet - the proof

Finally we show that $\eta^\star: \ \mathcal{P} \to \mathcal{P}$ is a translation on this elliptic curve:

$\phi: \ \mathbb{C}/\Lambda \to \mathcal{P}$ ... isomorphism with proper $\Lambda \Longrightarrow \phi$ carries the additive action on $\mathbb{C}/\Lambda$ to that on $\mathcal{P}$.

$\tilde{\sigma} := \phi^{-1} \circ \sigma \circ \phi$, $\tilde{\tau} := \phi^{-1} \circ \tau \circ \phi$ ... conformal involutions with four fixed points (fundamental parallelogram of $\Lambda$) $\Longrightarrow$

$\tilde{\eta} := \tilde{\tau} \circ \tilde{\sigma}$ is a translation of $\mathbb{C}/\Lambda$, *i.e.*, $\tilde{\eta}(z) = z + a$ for all $z$ and some $a$ in $\mathbb{C}/\Lambda$.

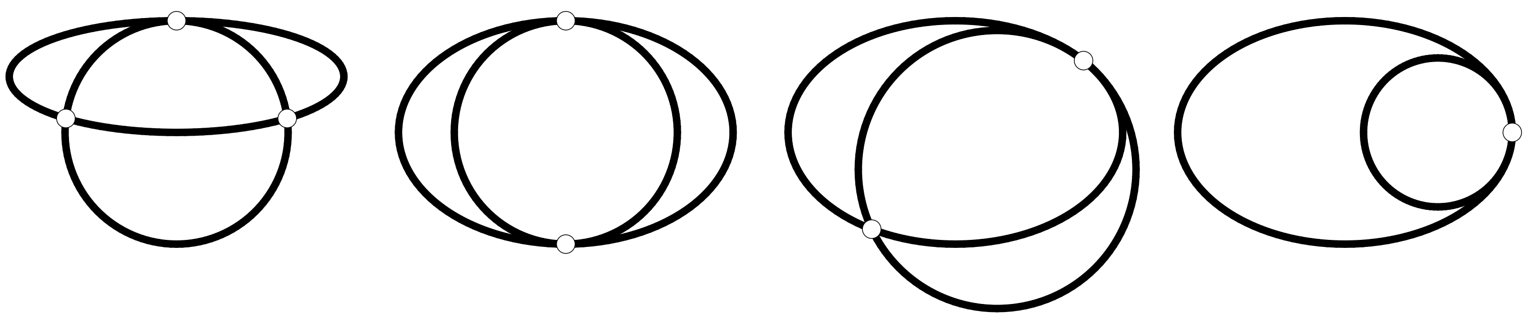
For $z \in \mathcal{P}$ and $b = \phi(a)$ the isomorphism $\phi$ yields

$\eta(y) = \phi \circ \tilde{\eta} \circ \phi^{-1}(y) = \phi(\phi^{-1}(z) + a) = \phi \circ \phi^{-1}(z) + \phi(a) = z + b.$

$\Longrightarrow$ **If** $\eta^n(p_0) = p_0$ **for one** $p_0 \in \mathcal{P}$ and some $n \in \mathbb{N} \setminus \{0, 1\}$, **then** we have $\eta^n(p) = p + nb$. Let now $p = p_0 \Longrightarrow nb = 0 \Longrightarrow \eta^n(p) = p$ **for all** $p \in \mathcal{P}$.

## Poncelet - remarks

The proof for $c_1$, $c_2$ not in GP is similar, but needs separate discussion of the four further configurations of $c_1$, $c_2$:



For ellipses and circles there are many results obtained in an elementary way.

## Cayley's theorem

$c_i: \ x^{\mathsf{T}} \cdot C_i \cdot x = 0 \ldots$ two conic sections, $C_i \in \mathrm{GL}(\mathbb{C}, 3)$

There exists an $n$-sided polygon inscribed in $c_1$ and circumscribed to $c_2$, if and only if, the coefficients $a_i$ in the power series

$$\sqrt{\det(t \cdot C_1 + C_2)} = a_0 + a_1 t + a_2 t^2 + \ldots$$

fulfil

$$\begin{vmatrix} a_2 & \ldots & a_{m+1} \\ \vdots & & \vdots \\ a_{m+1} & \ldots & a_{2m} \end{vmatrix} = 0, \text{ if } n = 2m+1 m \geq 1$$

$$\begin{vmatrix} a_3 & \ldots & a_{m+1} \\ \vdots & & \vdots \\ a_{m+1} & \ldots & a_{2m+1} \end{vmatrix} = 0, \quad \text{if } n = 2m, m \geq 2.$$

The proof use the same techniques!

## multiple binary forms - example

The equation $M(u, v) = 0$ of the elliptic curve $\gamma$ is a special case of a multiple binary form:

$$f(X_0 : X_1; Y_0 : Y_1) = (X_0^k, X_0^{k-1}X_1, \dots, X_1^k) \cdot \begin{pmatrix} f_{00} & \dots & f_{0\kappa} \\ \vdots & & \vdots \\ f_{k0} & \dots & f_{k\kappa} \end{pmatrix} \cdot \begin{pmatrix} Y_0^\kappa \\ Y_0^{\kappa 1}Y_1 \\ \vdots \\ Y_1^\kappa \end{pmatrix}$$

let $X_0 = 1$, $X_1 = x$; $Y_0 = 1$, $Y_1 = y \Longrightarrow$ inhomogeneous equation

The relation to incidence conditions of points on normal curves of order $k$ and normal varieties of order $\kappa$ is obvious.

To any pair $(x_0, y_0)$ there exist $\kappa - 1$ further $y_1$, $y_2$, $\dots (\neq y_0)$ and $k - 1$ further $x_1$, $x_2$, $\dots (\neq x_0)$.

## multiple binary forms - example

If now $x_0 = x_n$ and $y_0 = y_\nu$ for positive integers $n$ and $\nu$, then $f$ shows a closure and there is a geometric realization in form of a configuration

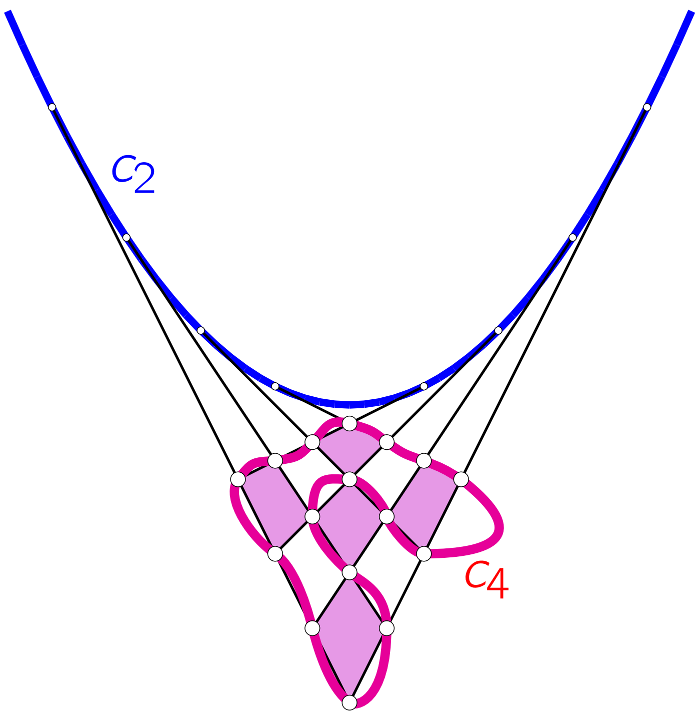
$$\Delta^{k,\kappa}_{n,\nu}(x_0, x_1, \dots, x_{n-1}; y_0, y_1, \dots, y_{\nu-1})$$

with $n\kappa = k\nu$ pairs of elements.

If $f$ defines infinitely many such configurations, then $f$ shows a porism.

For conditions on multiple binary forms to be poristic, see: A.B. Coble: Multiple binary forms with closure property. American J. 43 (1921), 1–19.

## multiple binary forms - the closing example




A special form $f$ describes the configuration $\Delta$:

If a $c_4$ passes through the 16 points of checker board grid (projective version) formed by $2 \cdot 4$ tangents of a $c_2$, then there are infintely many such diagrams with points on $c_4$ and lines tangent to $c_2$.

Thank You For Your Attention!