Geometry Proofs: Really Automated?

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today we serve

proofs	synthetic, analytic, by machine
automatization	calculation, synthetic (algebraic) reasoning
a matter of language	translation, propositional logic, machine language(s)
What do we accept?	
algebraic approach	technique(s), byproducts, side conditions
	interpretation of results
AI guided theorem proving	technique(s), byproducts, new results
comparison, discussion	(dis)advantages, drawbacks

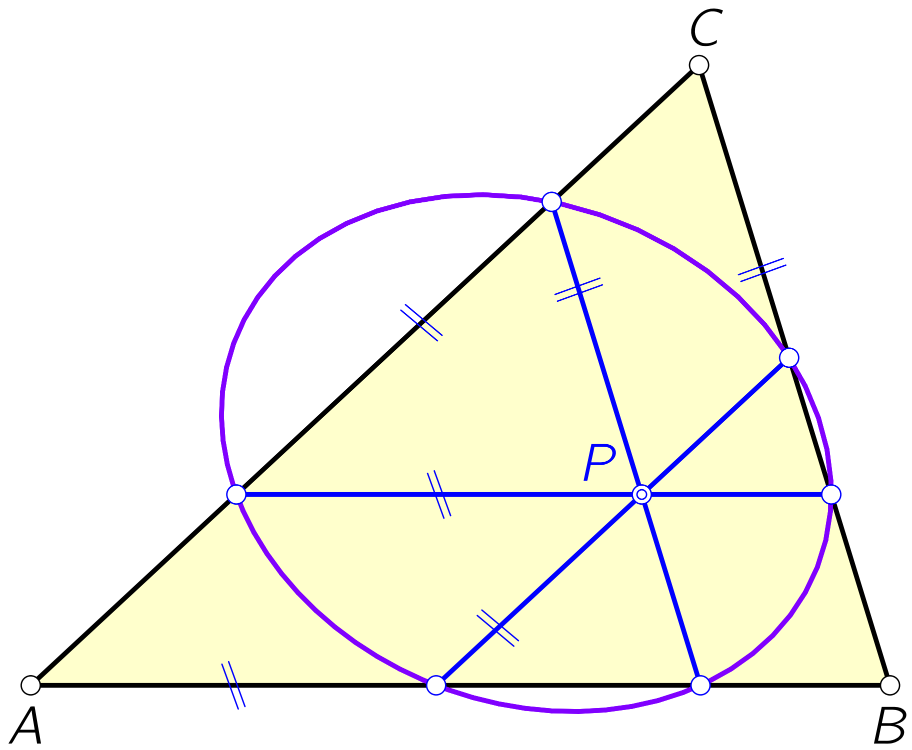
a simple example with different (?) kinds of proofs

Theorem:
Let P be a point in the (affine) plane of a triangle $\Delta = ABC$. P does not lie on any of Δ's sides. The lines parallel to Δ's sides through P intersect Δ's sides in (up to) six conconic points (the parallelians).

Usually, we continue with a proof here.

What do we accept as proof?

- synthetic (algebraic) reasoning
- analytic proof (calculation)



proof - example - analytic

Proof:

1. Introduce (affine) coordinates.
2. Show that the 6 parallelians lie on a single conic

   by either

   a. finding an equation of a conic on 5 of them and checking that the 6^{th} lies on it or

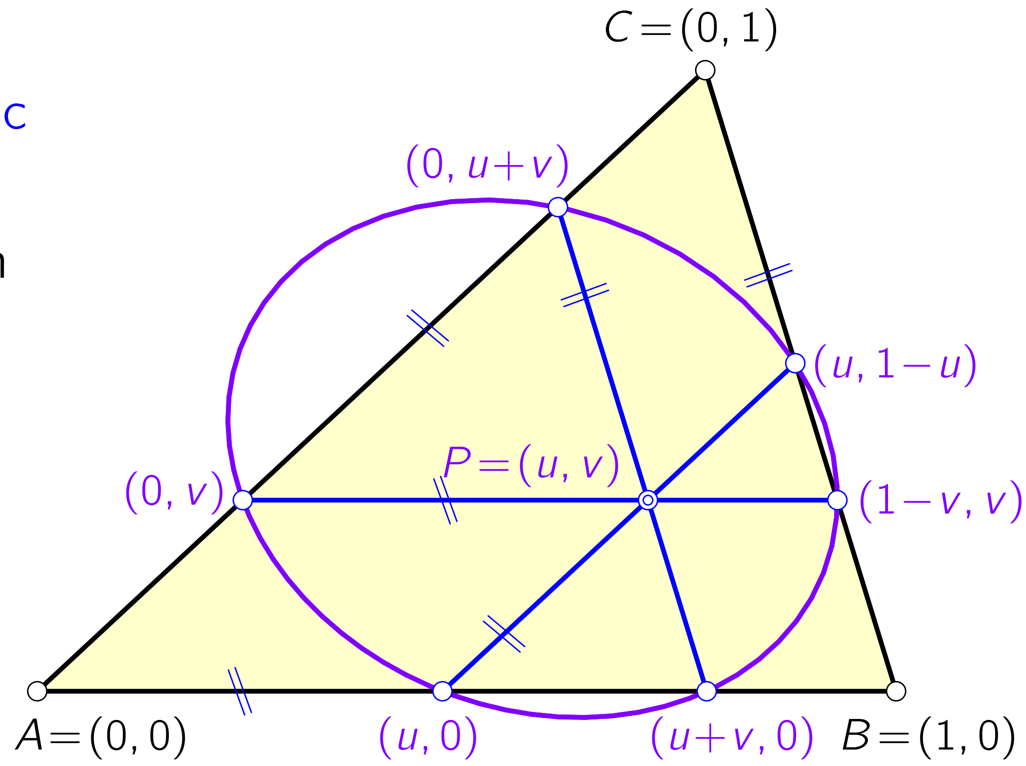
   b. checking without equation or

   c. showing that Pascal's theorem holds.

a. $vx^2+(2u+2v-1)xy+uy^2$
$-v(2u+v)x-u(u+2v)y+uv(u+v)=0$

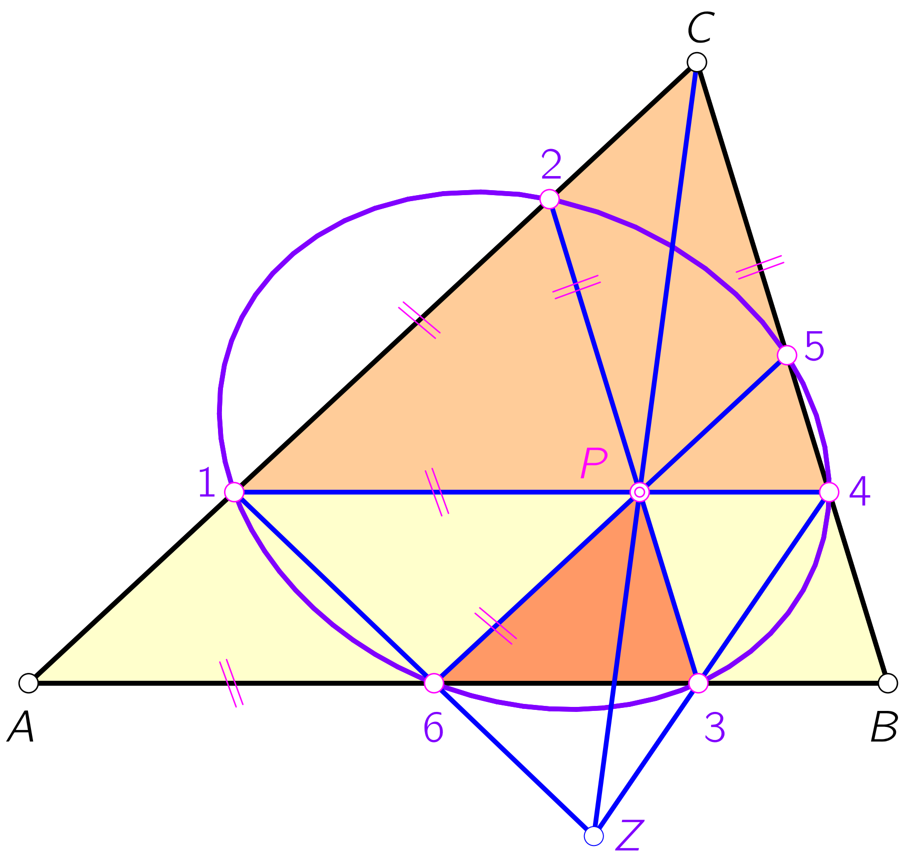
b. Veronese mapping & $\det(M_{6,6})=0$

c. boring, simple undergraduate linear algebra



Proof:

1. The six parallelians $1, \ldots, 6$ have to fulfill Pascal's criterion:
   $[1,2] \cap [4,5]$ (cyclic) are collinear
2. Proper labelling simplifies the verification:
   $[1,2] \cap [4,5] = C$, $[2,3] \cap [5,6] = P$
3. Because of parallel lines:
   $\zeta : 14C \rightarrow 63P$ is a central similarity
   with perspectrix = ideal line $\Longrightarrow$
4. Perspector $Z = [3,4] \cap [6,1] \in [C,P]$.



Requires knowledge of Pascal's and Desargues's theorems (?) and perspective collineations. Automatization? How?

automatized analytic proofs

automatization - analytic/algebraic

The algebraic approach uses Wu's method:

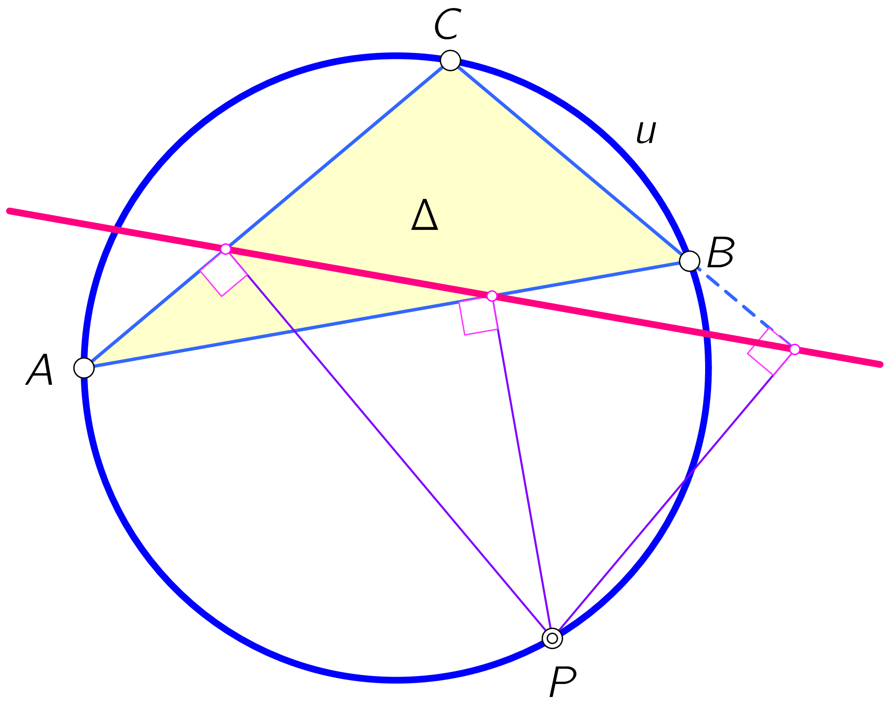
1. Translate the geometric theorem into a system of algebraic hypothesis equations plus a conclusion equation expressing the statement.
2. Transform the system of equations into a triangular form using pseudodivison.
3. Perform pseudodivision of the triangular system and the conclusion equation. If the final remainder equals zero, the conclusion follows from the hypotheses.
4. Examine all non-degenerate conditions found in the triangulation process. Some of them are natural, some give constraints and restrictions necessary for the validity of the hypotheses.

intermission: Wallace-Simson - planar version

$\Delta = ABC$ ... triangle (in the Euclidean plane)

u ... Δ's circumcircle

$P \in u$ ... arbitrary point on u



Theorem:

The feet of the normals from P to Δ's sides are collinear if, and only if, P is chosen on u.

In the plane: The locus of such points P is never degenerate!

Wu's method - example - step 1

Wallace-Simson - spatial version

We are given a skew quadrilateral $ABCD$ and ask for all points P such that the feet

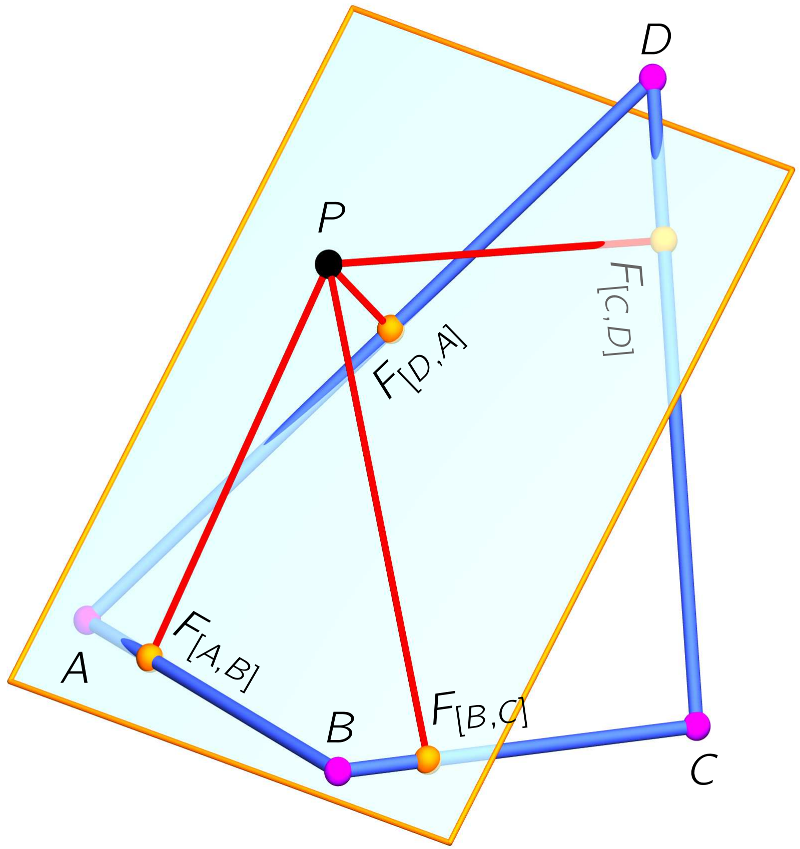
$$F_{[A,B]},\ F_{[B,C]},\ F_{[C,D]},\ F_{[D,A]}$$

of the normals from P to the side lines

$$[A,B],\ [B,C],\ [C,D],\ [D,A]$$

are coplanar.

All points P with four coplanar feet lie on a cubic surface $\mathcal{K}$ passing through the vertices of the quadrilateral.



Are there **conditions on $ABCD$ such that $\mathcal{K}$ is degenerate**, *i.e.*, $\mathcal{K}$ splits into a plane and a quadric?

Wu's method - example - step 1

coordinate vectors of the vertices A, B, C, D

$\mathbf{a} = (0,0,0), \mathbf{b} = (a,0,0), \mathbf{c} = (b,c,0), \mathbf{d} = (d,e,f)$

feet of normals from $P = \mathbf{x}$

$$F_{[A,B]} = \mathbf{b}\alpha,\ F_{[B,C]} = \mathbf{b}(1-\beta) + \mathbf{c}\beta,\ \dots \text{ with}$$

parameters $\alpha = \langle \mathbf{x}, \mathbf{b} \rangle \|\mathbf{b}\|^{-2}, \beta = \langle \mathbf{x} - \mathbf{b}, \mathbf{c} - \mathbf{b} \rangle \|\mathbf{c} - \mathbf{b}\|^{-2}, \dots$

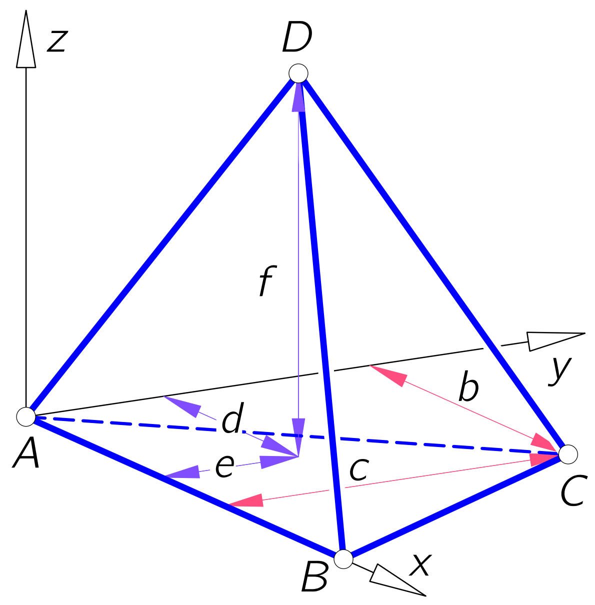
condition on four points $\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}$ to lie in one plane:

$$\det(\mathbf{p},\mathbf{q},\mathbf{s}) + \det(\mathbf{q},\mathbf{r},\mathbf{s}) + \det(\mathbf{r},\mathbf{p},\mathbf{s}) - \det(\mathbf{p},\mathbf{q},\mathbf{r}) = 0$$

equation of $\mathcal{K}$, the locus of all $P = \mathbf{x}$ such that ...

$$\mathcal{K}:\ \varepsilon_0 - \varepsilon_1 + \varepsilon_2 - \varepsilon_3 = 0$$

ε_i ... i-th elementary symmetric function in $\alpha, \beta, \gamma, \delta$



Suitable choice of the coordinate system simplifies the computation and causes two cases to be distinguished.

Wu's method - example - step 1

a cubic surface instead of the circumcircle

Four points A, B, C, D define three different skew quadrilaterals

$$ABCD, ABDC, ACBD.$$

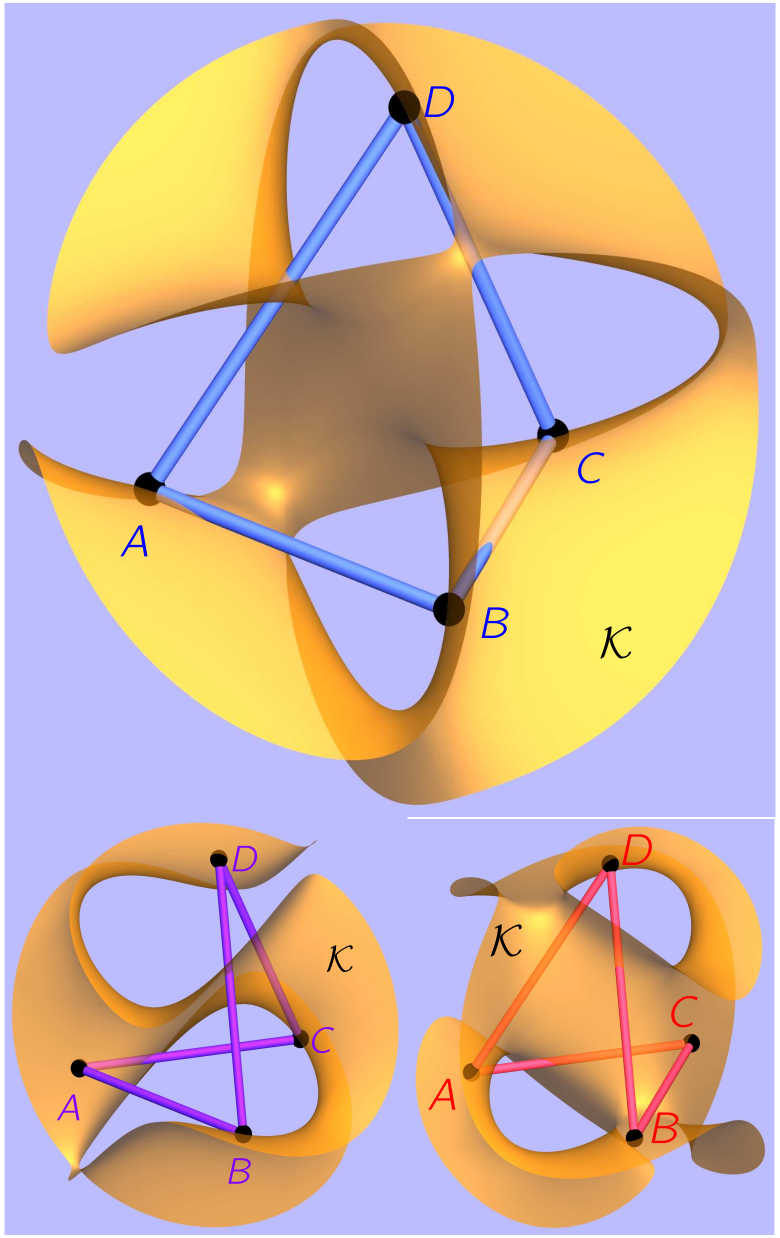
$\Longrightarrow$ There are three different cubic surfaces $\mathcal{K}$.

Degenerate surfaces $\mathcal{K}$ can be found by choosing the vertices of a regular tetrahedron.

Are these the only cases?

Is there a condition on $\mathcal{K}$ such that it degenerates?

How to find conditions on $ABCD$ such that $\mathcal{K}$ degenerates?



not Wu's method: basic algebra

important fact:

Each univariate cubic polynomial with real coefficients has at least one real root.

$\iff$

If a trivariate cubic polynomial (with real coefficients) factors,

then there is at least one real factor of degree 1.

Wu's method - example - step 1

degeneracy conditions for cubic surfaces

$$\mathcal{K}: \sum_{r+s+t\leq 3} k_{r,s,t} x^r y^s z^t = 0 \ldots \text{equation of the cubic surface}$$

assume $\mathcal{K}$ is degenerate $\Longrightarrow$ union of a plane $\mathcal{P}$ and something, say $\mathcal{Q}$, of degree 2

$$\mathcal{P}: \; l_0 + l_1 x + l_2 y + l_3 z = 0, \quad \mathcal{Q}: \sum_{r+s+t\leq 2} q_{r,s,t} x^r y^s z^t = 0$$

$$A \in \mathcal{K} \text{ and } \mathbf{a} = (0,0,0) \Longrightarrow k_{000} = 0$$

two cases to be treated separately (due to the special choice of the coordinate system):

(A) $A \in \mathcal{P} \Longleftrightarrow l_0 = 0$

(B) $A \in \mathcal{Q} \Longleftrightarrow q_{000} = 0$

$$\mathcal{K} = \mathcal{P} \cup \mathcal{Q} \Longleftrightarrow \sum_{r+s+t\leq 3} k_{r,s,t} x^r y^s z^t - (l_0 + l_1 x + l_2 y + l_3 z) \cdot \left(\sum_{r+s+t\leq 2} q_{i,j,k} x^i y^j z^k \right) = 0$$

collect the coefficients of monomials $x^r y^s z^t$, eliminate l_i and $q_{r,s,t}$

and take either case into account! $\Longrightarrow$

Wu's method - example - step 1

degeneracy conditions for cubic surfaces - case (A)

$$k_{010}^3 k_{300} - k_{010}^2 k_{100} k_{210} + k_{010} k_{100}^2 k_{120} - k_{030} k_{100}^3 = 0,$$
$$k_{001}^3 k_{300} - k_{001}^2 k_{100} k_{201} + k_{001} k_{100}^2 k_{102} - k_{003} k_{100}^3 = 0,$$
$$k_{001}^3 k_{030} - k_{001}^2 k_{010} k_{021} + k_{001} k_{010}^2 k_{012} - k_{003} k_{010}^3 = 0,$$
$$k_{010}^2 k_{200} - k_{010} k_{100} k_{110} + k_{020} k_{100}^2 = 0,$$
$$k_{001}^2 k_{200} - k_{001} k_{100} k_{101} + k_{002} k_{100}^2 = 0,$$
$$k_{001}^2 k_{020} - k_{001} k_{010} k_{011} + k_{002} k_{010}^2 = 0,$$
$$-2k_{001} k_{010}^3 k_{300} + k_{00,1} k_{010}^2 k_{100} k_{210} - k_{001} k_{030} k_{100}^3 +$$
$$+k_{010}^3 k_{100} k_{201} - k_{010}^2 k_{100}^2 k_{111} + k_{010} k_{021} k_{100}^3 = 0.$$

7 equations in 19 unknowns $k_{r,s,t}(a, \ldots, f)$ of degree ≤ 5

+143 polynomial side conditions on a, b, c, d, e, f

We skip case (B).

Wu's method - example - step 2, 3

application to skew quadrilaterals

Conjecture:

If the tetrahedron $ABCD$ has no symmetries, shows no right angles between any pair of edges (whether skew or not), and has no pair of equally long edges, then none of the three cubic surfaces $\mathcal{K}$ associated with the three types of skew quadrilaterals ($ABCD$, $ABDC$, $ACBD$) degenerates.

Justification: (no proof, it's not a theorem!)

insert coefficients of $\mathcal{K}$'s equation (for any case) into the degeneracy conditions,

try to solve the emerging systems of equations ...

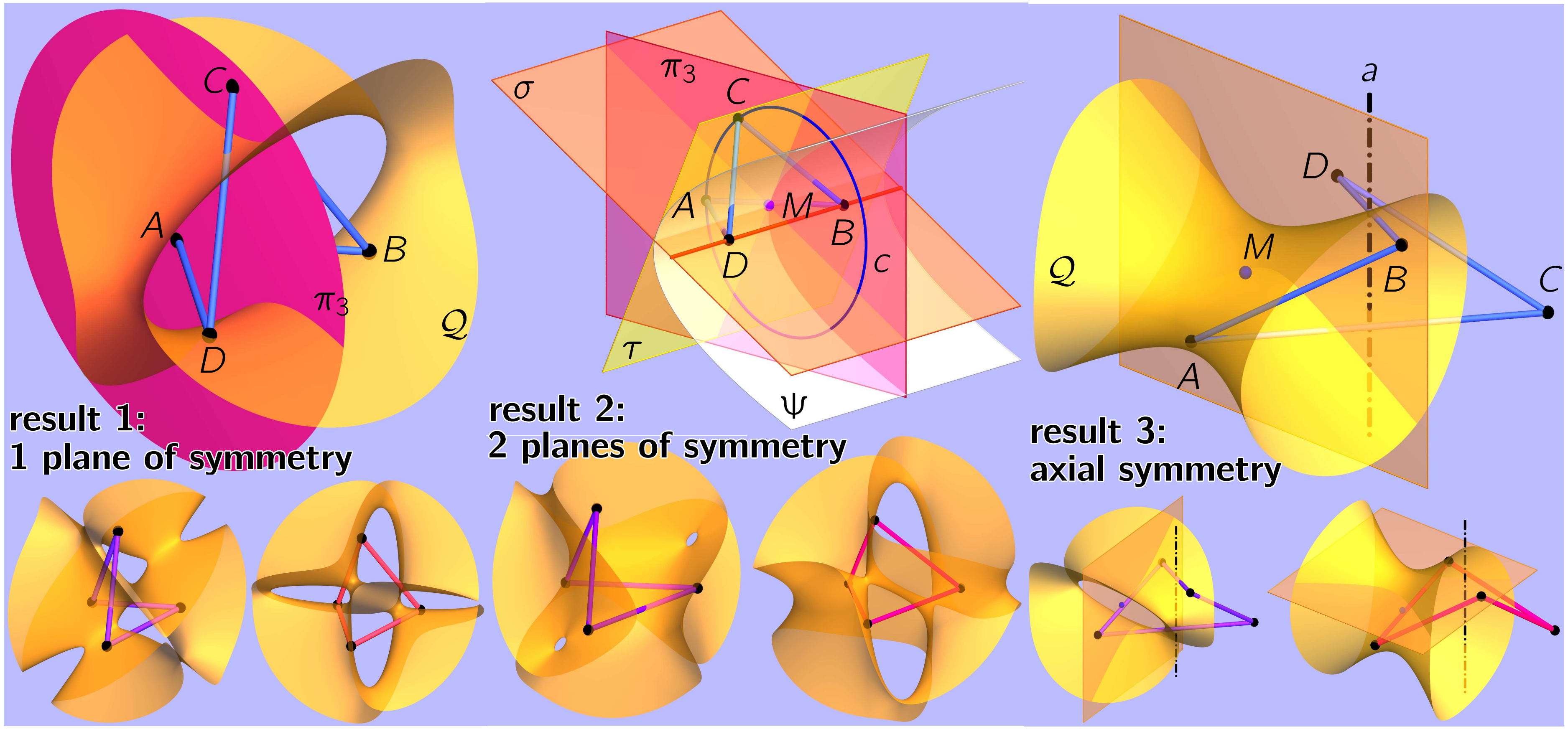
factors that are only vanishing if there are right angles or symmetries can be canceled

...

nothing useful remains ...

but we didn't get through all computations!

Wu's method - example - step 3, 4



results 3, 4, 5: orthoschemes, cuboid corner, regular tetrahedron

Wu's method - aftermath

hypotheses & conjecture equation(s)	writing them down $\longrightarrow$	not automatic
	elimination works well $\longrightarrow$	partly automatic
	depends on processing power & capacity	
reading the results	$\longrightarrow$	definitely not automatic
	works well, if resultants can be built	
interpretation	$\longrightarrow$	definitely not automatic
	sometimes not so easy	
new & further reaching results	$\longrightarrow$	???

Some people believe(d) that automated (analytic) theorem proving yields new results (theorems) expressed in terms of remainders (byproducts).

Seems hopeless, since remainders only give constraint equations, side conditions, and further new polynomials in the considered ideal will not show up.

AI guided theorem proving

proof assistants & AI systems

apply to algebra, number theory, . . .

ACL 2, Agda, Albatross, Coq, $F^{\star}$, HOL Light, HOL4, Idris, Lean, LEGE, Metamath, Mizar, Nqthm, NuPRL, PVS, Twelf

applies only to Euclidean geometry:

AlphaGeometry = AI program, supposed to solve Euclidean geometry problems
developed by DeepMind (Google subsidiary)

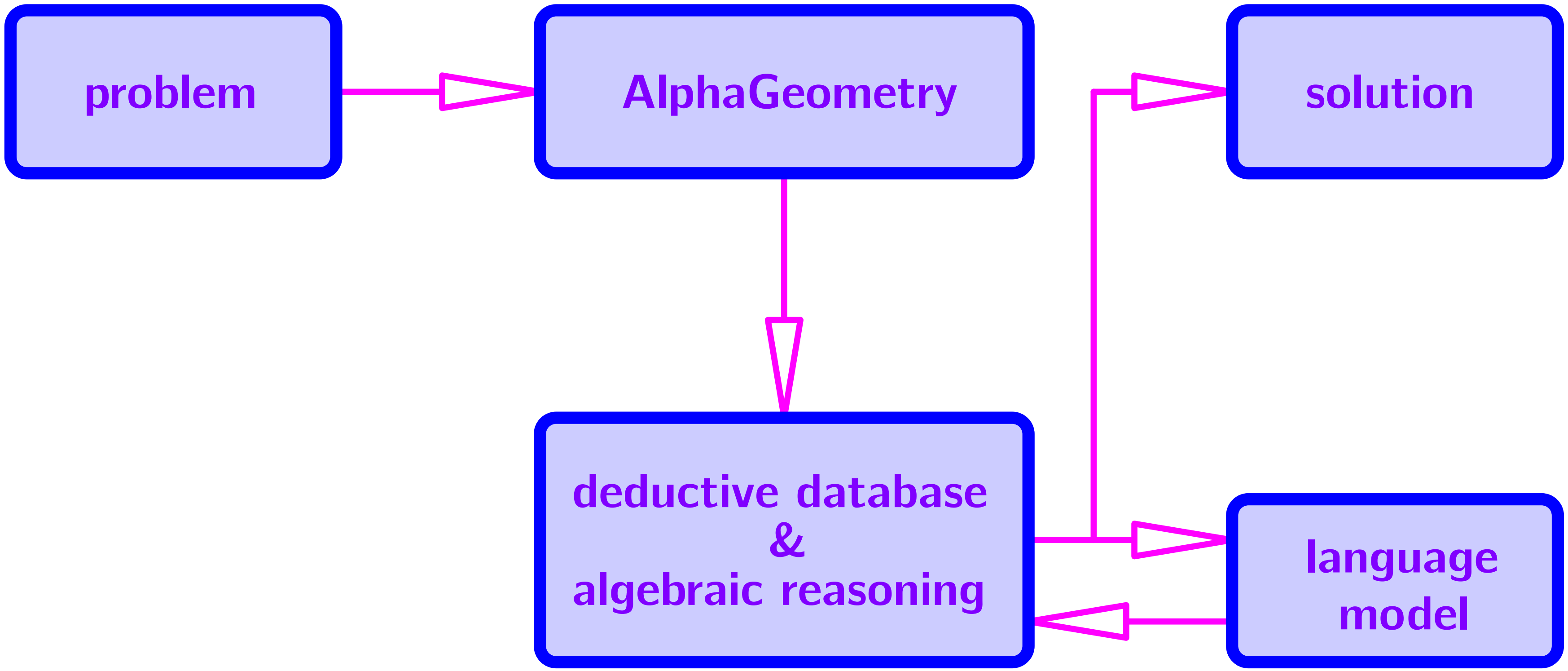
performance: solved 25 of 30 IMO geometry problems (with competition time limits)
compares to the average human gold medallist

previous AI programs using Wu's method solved only 10 of 30

traditionally: symbolic engines, rely exclusively on human-coded rules, lack flexibility

AlphaGeometry . . .

. . . combines symbolic engine & specialized large language model trained on synthetic data of geometrical proofs.
If symbolic engine fails to find a formal & rigorous proof, it solicits the LLM, which suggests a geometrical construct to move forward.



AlphaGeometry - deductive database

premises

A, B, C non-collinear

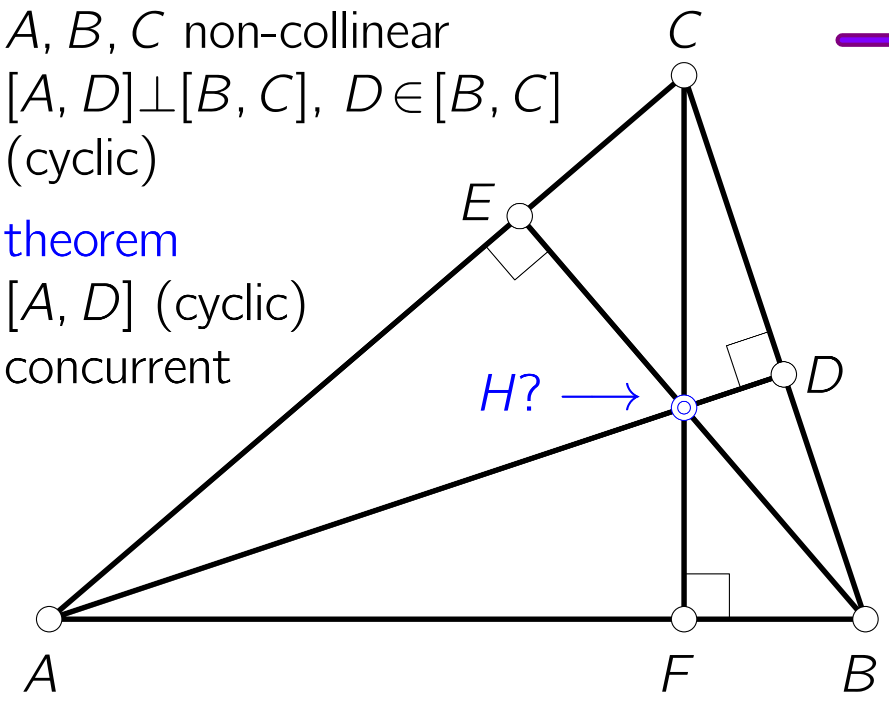
$[A, D] \perp [B, C]$, $D \in [B, C]$

(cyclic)

theorem

$[A, D]$ (cyclic)

concurrent



deduces random sample premises

A, E, D, B concyclic

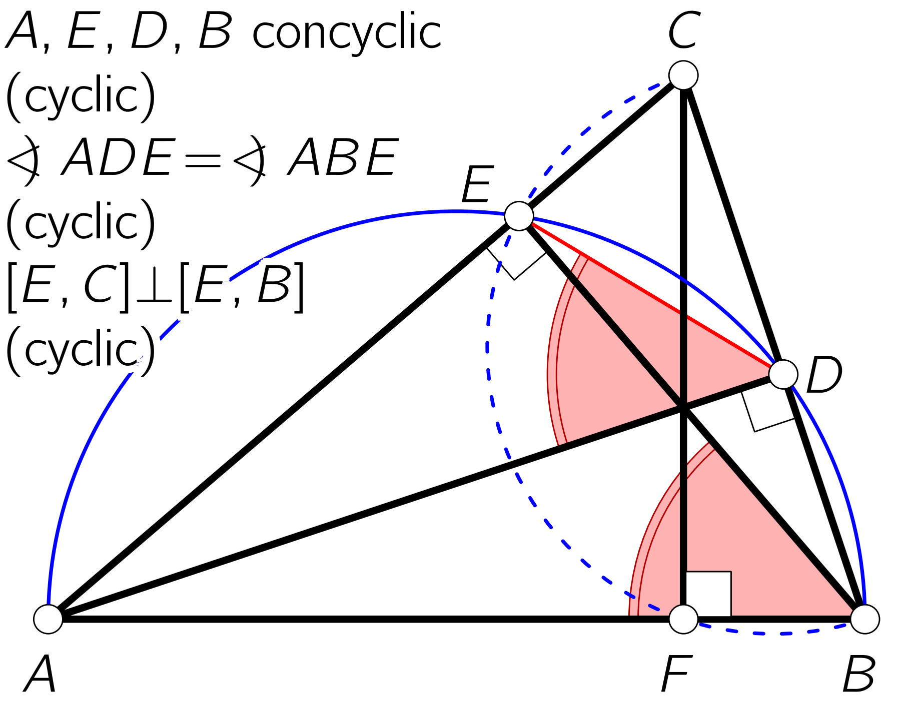
(cyclic)

$\sphericalangle\, ADE = \sphericalangle\, ABE$

(cyclic)

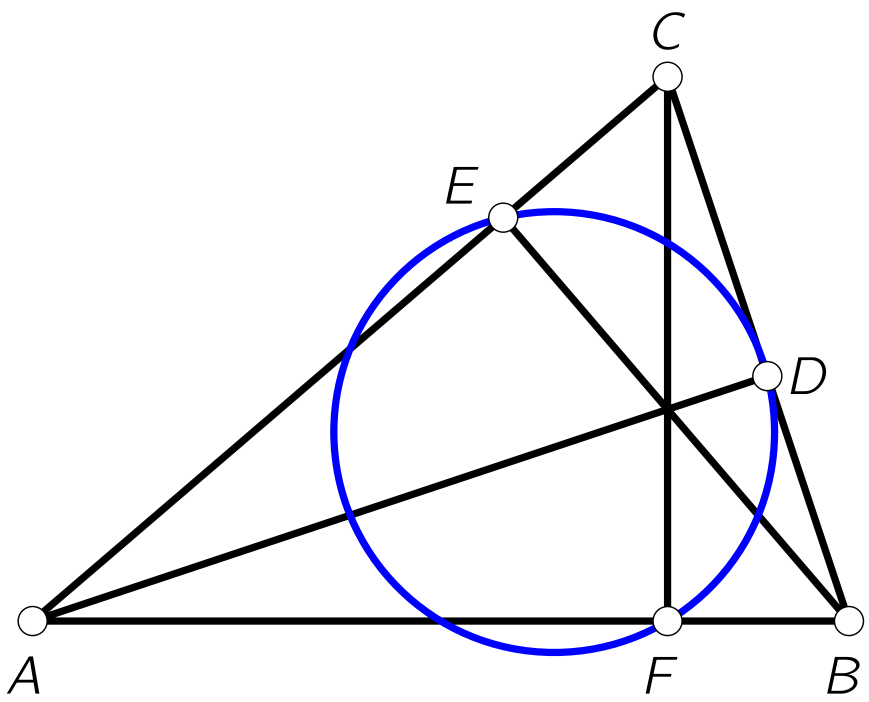
$[E, C] \perp [E, B]$

(cyclic)



The deductive database then consist of a huge variety of premises (facts) hidden in the initial configuration.

AlphaGeometry - algebraic reasoning



Deduction and algebraic reasoning produce (well-)known and new results.

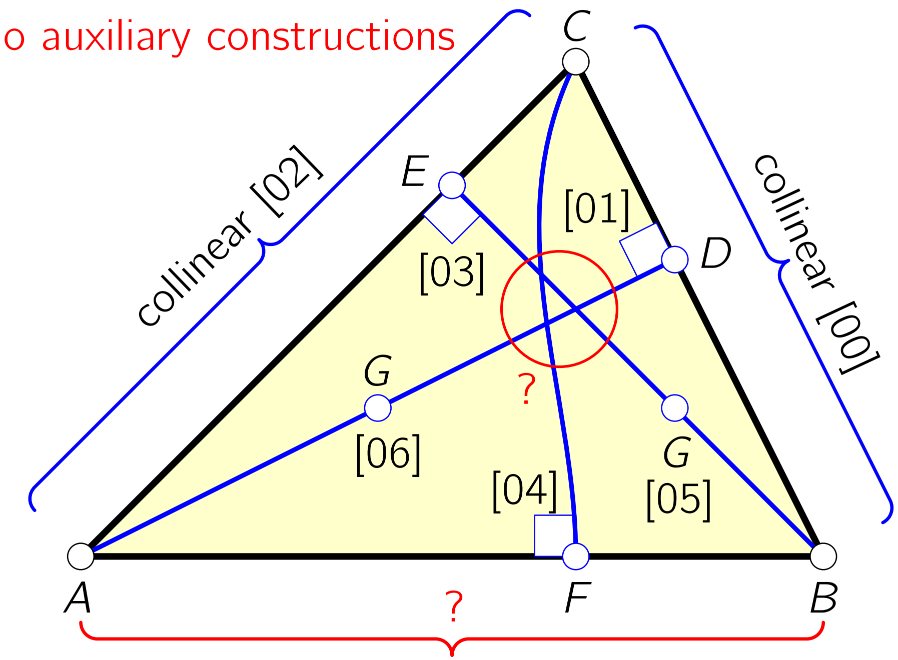
(*e.g.*, Feuerbach's nine point circle including all metric properties and incidences)

It also yields results lacking (cyclic) symmetry, *i.e.*, something humans would not take into account. Results can be of any complexity: number of steps for the proof can be large.

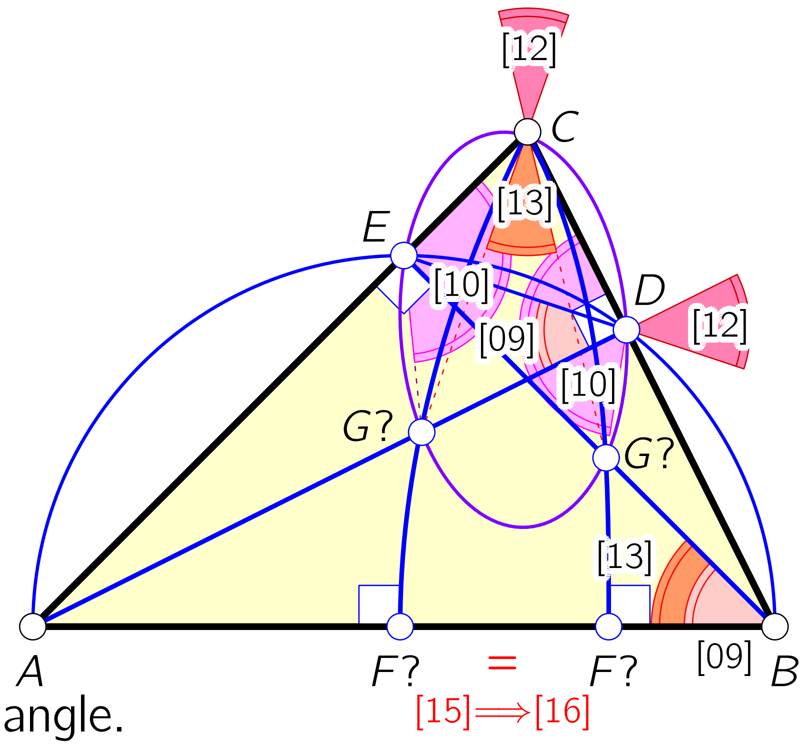
- System is able to introduce auxiliary points and lines (as necessary for the orthocenter proof).
- User is allowed to give hints (rules to use).

Theorem: The three altitudes of a triangle are concurrent.
We forced the machine to prove without auxiliary construction.
Nevertheless, it failed to find one before.

premises from the theorem
no auxiliary constructions



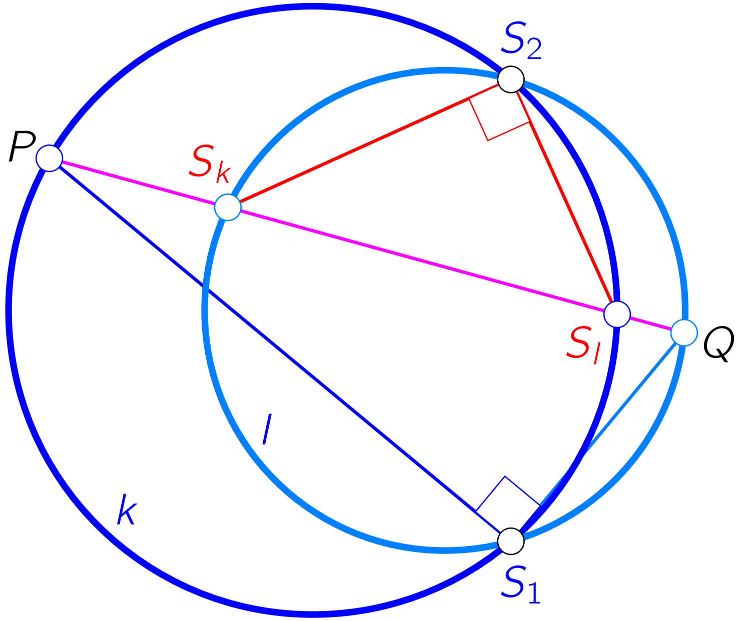
proof: only the important steps



The usual proof uses the anticomplementary triangle.

Let's see how it performs!

variatio delectat: an Ebisui miniature



Assumptions:

given two circles k, l with 2 real intersections $S_1 \neq S_2$

$P \in k$, $P \neq S_1, S_2$, find Q s. t. $[P, S_1] \perp [S_1, Q]$

$S_k := [P, Q] \cap k$, $S_l := [P, Q] \cap l$

Theorem:

$[S_2, S_k] \perp [S_1, S_l]$

translation into machine language

points							
M_k	S_1	S_2	M_l	P	Q	S_k	S_l
a	b	c	d	e	f	g	h

machine readable language	meaning
a b = segment a b;	line segment $M_k S_1$
c = on_circle a b;	S_2 chosen on circle $k(M_k, S_1)$
d = on_bline b c;	M_l chosen on the bisector of $S_1 S_2$
e = on_circle a b;	choose (assume) $P \in k$
f = on_tline f b e b, on_circle d b;	$[Q, S_1] \perp [P, S_1]$ and $Q \in l$
g = on_line e f, on_circle ab;	$S_k = [P, Q] \cap k$
h = on_line e f, on_circle d b	$S_l = [P, Q] \cap l$
? perp g c h c;	the question

machine is working - reading the premises

producing premises read from the theorem

machine	my symbols	number
AC = AB	$\overline{M_k S_2} = \overline{M_k S_1}$	[00]
DB = DC	$\overline{M_l S_1} = \overline{M_l S_2}$	[01]
AE = AB	$\overline{M_k P} = \overline{M_k S_1}$	[02]
BF ⊥ BE	$[S_1, Q] \perp [S_1, P]$	[03]
DF = DB	$\overline{M_l Q} = \overline{M_l S_1}$	[04]
AG = AB	$\overline{M_k S_k} = \overline{M_k S_1}$	[05]
F, G, E are collinear	Q, S_k, P collinear	[06]
DH = DB	$\overline{M_l S_l} = \overline{M_l S_1}$	[07]
H, F, E are collinear	S_l, Q, P collinear	[08]

No auxiliary constructions needed.

machine is working - deducing (step by step), part 1

1.	AC=AB [00] $\overline{M_kS_2}=\overline{M_kS_1}$	&	AG=AB [05] $\overline{M_kS_k}=\overline{M_kS_1}$	&	AE=AB [02] $\overline{M_kP}=\overline{M_kS_1}$	$\Longrightarrow$ B, G, C, E concyclic [09] $\Longrightarrow$ S_1, S_k, S_2, P concyclic
2.	B, G, C, E concyclic [09] S_1, S_k, S_2, P concyclic					$\Longrightarrow$ $\measuredangle$ BCG=$\measuredangle$ BEG [10] $\measuredangle\ S_1S_2S_k = \measuredangle\ S_1PS_k$
3.	AC=AB [00] $\overline{M_kS_2}=\overline{M_kS_1}$	&	DB=DC [01] $\overline{M_lS_1} = \overline{M_lS_2}$			$\Longrightarrow$ BC⊥AD [11] $[S_1, S_2]\perp[M_k, M_l]$
4.	BF⊥BE [03] $[S_1, Q]\perp[S_1, P]$	&	BC⊥AD [11] $[S_1, S_2]\perp[M_k, M_l]$			$\Longrightarrow$ $\measuredangle$ AD,BC=$\measuredangle$ EBF [12] $\measuredangle\ [M_k, M_l], [S_1, S_2] = \measuredangle\ PS_1Q$
5.	DF=DB [04] $\overline{M_lQ} = \overline{M_lS_1}$	&	DH=DB [07] $\overline{M_lS_l} = \overline{M_lS_1}$	&	DB=DC [01] $\overline{M_lS_1} = \overline{M_lS_2}$	$\Longrightarrow$ B, F, C, H concyclic [13] S_1, Q, S_2, S_l concyclic
6.	B, F, C, H concyclic [13] S_1, Q, S_2, S_l concyclic					$\Longrightarrow$ $\measuredangle$ BFH=$\measuredangle$ BCH [14] $\measuredangle\ S_1QS_l = \measuredangle\ S_1S_2S_l$

machine is working - deducing (step by step), part 2

7. $\measuredangle$ AB,BC=$\measuredangle$ EBF $\quad$ $\measuredangle$ BFH=$\measuredangle$ AD,HC $\Longrightarrow$ $\measuredangle$ AD,HC=$\measuredangle$ BE,HF

[12] & [14] [15]

$\measuredangle[M_k, M_l],[S_1,S_2]=\measuredangle PS_1Q,\ \measuredangle S_1QS_l=\measuredangle[M_k,M_l],[S_l,S_2] \Longrightarrow \measuredangle[M_k,M_l],[S_l,S_2]=\measuredangle[S_1,P],[S_l,Q]$

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8. $\measuredangle$ BCG=$\measuredangle$ BEG $\quad$ F, G, E collinear $\quad$ $\measuredangle$ AD,HC=$\measuredangle$ BE,HF $\quad$ H, F, E collinear $\Longrightarrow$ $\measuredangle$ AD,HC=$\measuredangle$ BCG

[10] & [06] & [15] & [08] [16]

$\measuredangle S_1S_2S_k=\measuredangle S_1PS_k \quad Q, S_k, P$ collinear $\quad \measuredangle[M_k,M_l],[S_l,S_2]=\measuredangle[S_1,P],[S_l,P] \quad S_l, Q, P$ collinear

$\Longrightarrow \measuredangle[M_k,M_l],[S_l,S_2]=\measuredangle S_1S_2S_k$

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9. $\measuredangle$AB,HC=$\measuredangle$BCD $\quad$ BC⊥AD $\Longrightarrow$ **CG⊥CH**

[16] & [11] [17]

$\measuredangle[M_k,M_l],[S_l,S_2]=\measuredangle S_1S_2S_k \quad [S_1,S_2]\perp[M_k,M_l] \quad \Longrightarrow [S_2,S_k]\perp[S_2,S_l]$

AlphaGeometry - aftermath

translating the theorem $\longrightarrow$ partly automatic
reading hidden premises & building the database $\longrightarrow$ automatic
finding formal correct proof $\longrightarrow$ sometimes automatic
reading hints from LLM $\longrightarrow$ sometimes automatic
finding new (hidden) results $\longrightarrow$ automatic

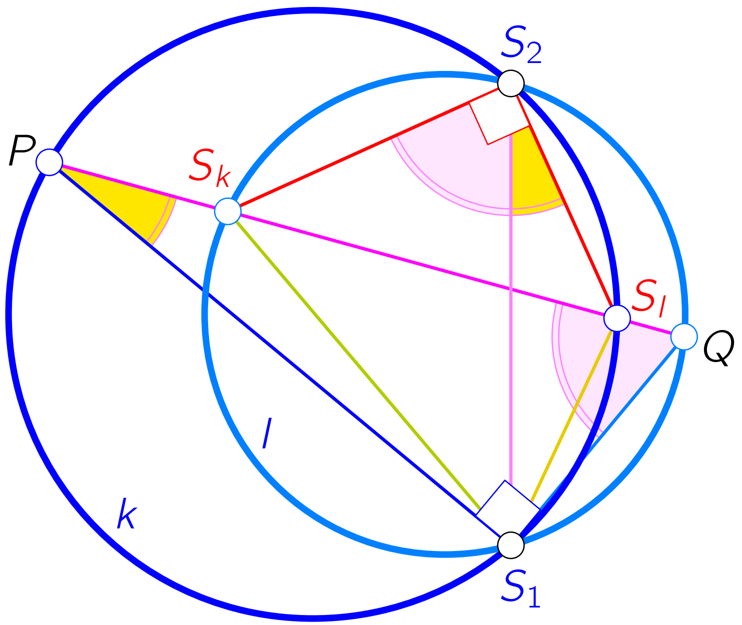
sometimes lacking natural symmetries, eventually "step consuming",
no. of steps not constant in each run

writing readable formulation $\longrightarrow$ partly automatic

limited to Euclidean results, auxiliary constructions are only found sometimes
Projective geometry rules are not incorporated, user could define own rules.

Is the method applicable to other domains of mathematics or reasoning?
Symbolic engines rely on domain-specific rules.

Statistics not accessible to ordinary users.



Proof:

2 × theorem of the angle of circumference:

$\sphericalangle\ S_1PQ = \sphericalangle\ S_1PS_k = \sphericalangle\ S_1S_2S_k$

$\sphericalangle\ S_1QP = \frac{\pi}{2} - \sphericalangle\ S_1PQ = \sphericalangle\ S_1QS_l = \sphericalangle\ S_1S_2S_l$

$\Longrightarrow \sphericalangle\ S_1PQ + \sphericalangle\ S_1QP = \frac{\pi}{2}$ □

What remains?

automated: many steps of proofs are already automated, *i.e.*,
depending on the experience, the arguments themselves, coordinatization,
parametrization, calculation, algebraic formulation, reasoning,
sometimes auxiliary constructions, finding analogies, . . .

not automated: formulation, proper language, translation (in both directions),
proper approach (which kind of geometry?),
the Ansatz, sometimes auxiliary constructions, finding analogies,
finding superordinate standpoints and concepts, . . .

In the end: We still need to feed the machine.

Thank You For Your Attention!

some references

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