Universal Porisms and Yff Conics

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overview

motivation	a question, Cayley's criterion
Yff conics	in- and circumscribed, independent of triangle sides
rational parametrization	unversality: arbitrary fields
examples	various characteristics, different behaviour
more universal porisms	orbit of tangent triangle, exponential pencil
some Euclidean remarks	orbits of centers, . . .

Cayley's criterion

$\mathcal{M} : \mathbf{x}^\top \mathbf{M}\mathbf{x} = 0 \ldots$ circumconic, $\mathcal{N} : \mathbf{x}^\top \mathbf{N}\mathbf{x} = 0 \ldots$ inconic

power series: $S(t) = \sqrt{\det(t \cdot \mathbf{M} + \mathbf{N})} = a_0 + a_1 t + a_2 t^2 + \ldots$

Cayley's criterion: $\mathcal{M}$ and $\mathcal{N}$ allow for interscribed n-gon families if

$$a_2 = 0, n = 3; \quad \det \begin{pmatrix} a_2 & a_3 \\ a_3 & a_4 \end{pmatrix} = 0, n = 5; \ldots$$

$$a_3 = 0, n = 4; \quad \det \begin{pmatrix} a_3 & a_4 \\ a_4 & a_5 \end{pmatrix} = 0, n = 6; \ldots$$

J. Chipalkatti: *On the Poncelet triangle condition over finite fields.* Finite Fields Appl. **45** (2017), 59–72.

uses Cayley's criterion to discuss the propability of finding Poncelet porisms (of triangles) in finite planes, if there are some

This does neither answer the question if there are some nor what they look like.

pencil of Yff conics – real and complex case

$\mathcal{M}: xy+yz+zx=0 \ldots$ circumconic

$\mathcal{N}: x^2+y^2+z^2-2(xy+yz+zx)=0 \ldots$ inconic

in $\mathbb{P}^2(\mathbb{R})$: pencil of the 3rd kind

$X_1 := 1:1:1 \ldots$ common pole,

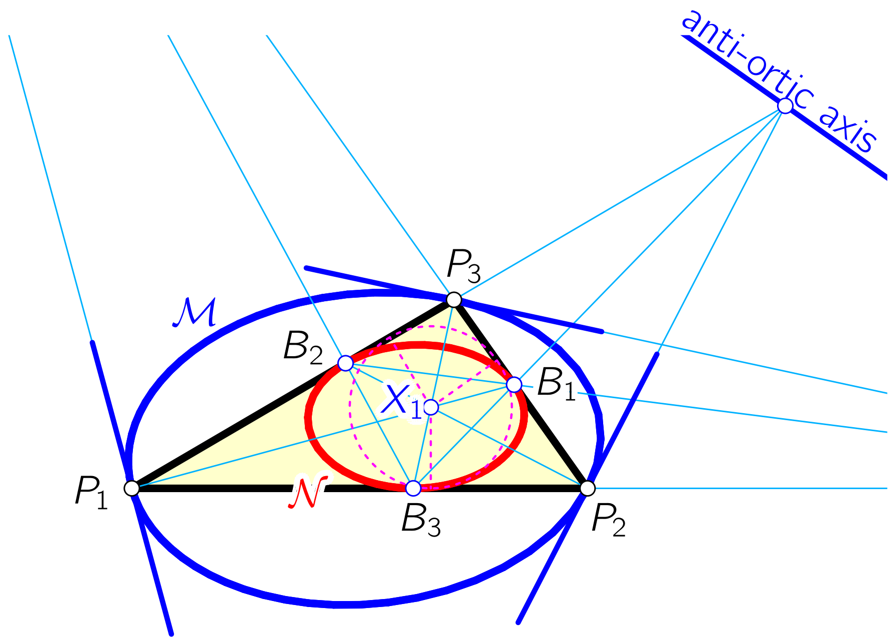
$\mathcal{L}_1 : x+y+z=0 \ldots$ common polar

polynomial parametrization of $\mathcal{M}$

$P_1 = -uv : u(u+v) : v(u+v)$ yields

$P_2 = v(u+v) : -uv : u(u+v)$,

$P_3 = u(u+v) : u(u+v) : -uv$.



also for the contact triangle $B_1B_2B_3$: $B_1 = (u+v)^2 : v^2 : u^2, \ldots$

$u : v \neq 0 : 0$, $u, v \in \mathbb{F}$ the choice of $\mathbb{F}$ matters!

Yff porism

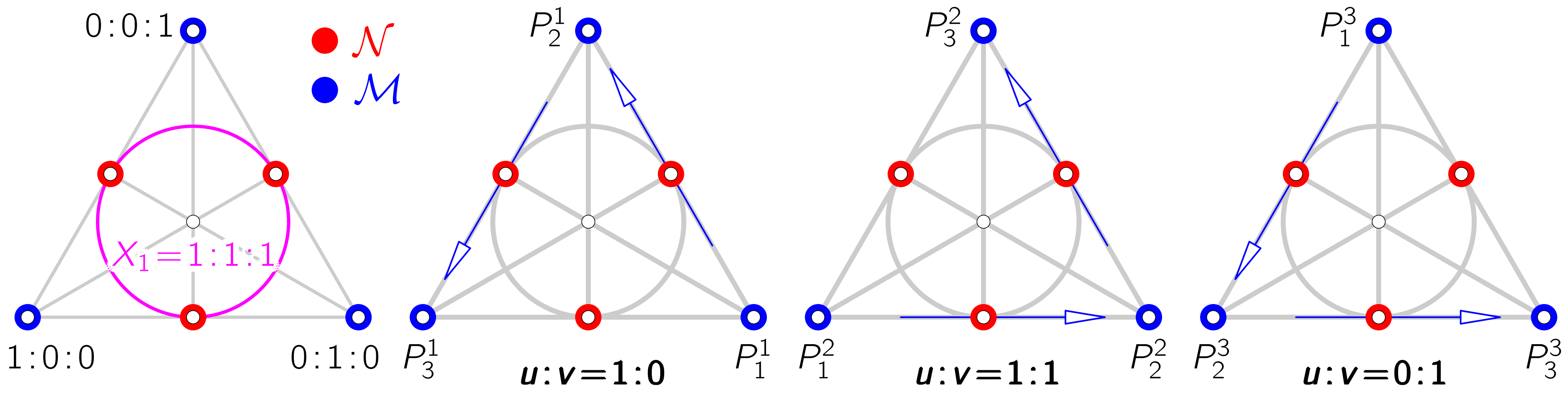
Yff porism (YP) = family of triangles interscribed between $\mathcal{M}$ and $\mathcal{N}$
We have seen: YP contains triangles whose vertices have rational coordinates.

Coordinate functions of P_i and B_i make sense even in the case of finite fields.

Finite fields:
Triangles $P_1P_2P_3$ become degenerate if $\delta^3 := (u^2 + uv + v^2)^3 = 0$.
The same holds true also for the contact triangle $B_1B_2B_3$.

$\mathrm{char}\,\mathbb{F} = 2, 5, 11, 17, 23, 29, 41, 47, 53, 59, 71, 83, 89,$ ...... no degenerate triangle
$\mathrm{char}\,\mathbb{F} = 3$ .... exactly one degenerate triangle
$\mathrm{char}\,\mathbb{F} = 7, 13, 19, 31, 37, 43, 61, 67, 73, 79, 97,$ .... two degenerate triangles

examples – projective minimal plane, Fano plane



in $\mathbb{P}^2(\mathrm{GF}(2))$:

$\mathcal{M}$ singular, consists of three non-collinear points with nucleus X_1

$\mathcal{N}$ regular, although it consists of three collinear points

n.b. $\mathcal{N} = \mathcal{L}_1$ as point sets!

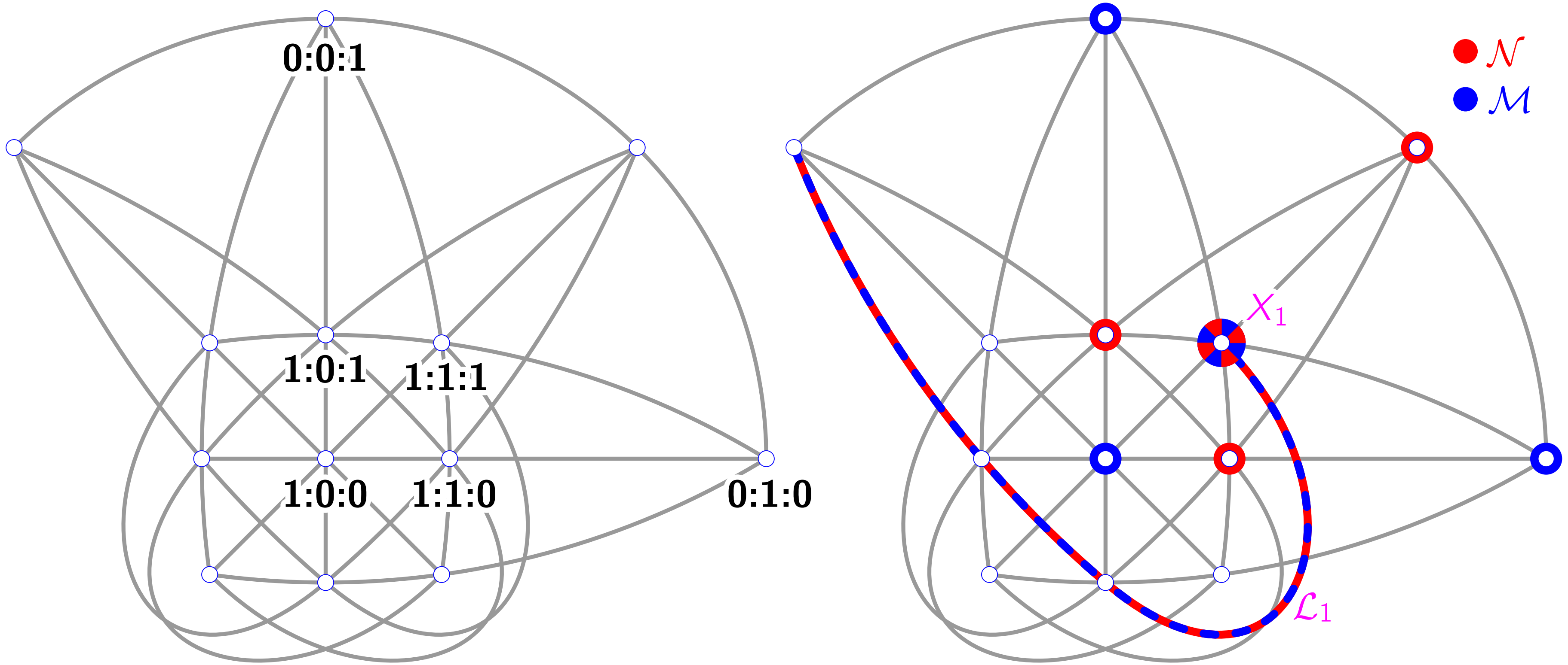
$\mathcal{M} \cap \mathcal{N} = \emptyset$

only regular triangles, triangle sides not tangent to $\mathcal{N}$, here: **polarity = null-polarity!**

Superscripts inidicate the pose.

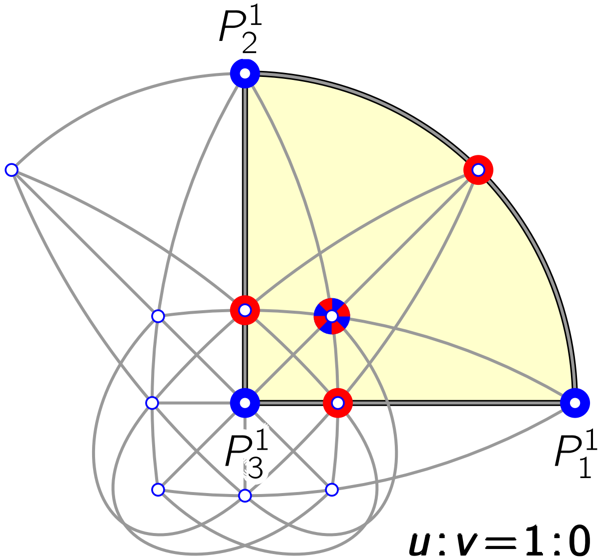
examples – 13 point plane

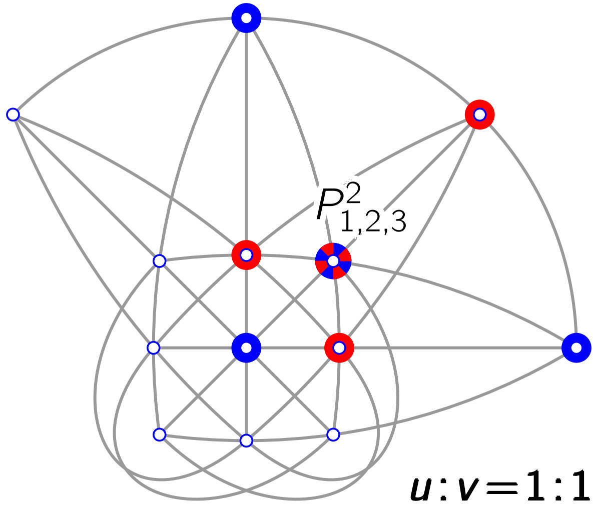
The 13 point plane is isomorphic to $\mathbb{P}^2(\mathbf{GF}(3))$.

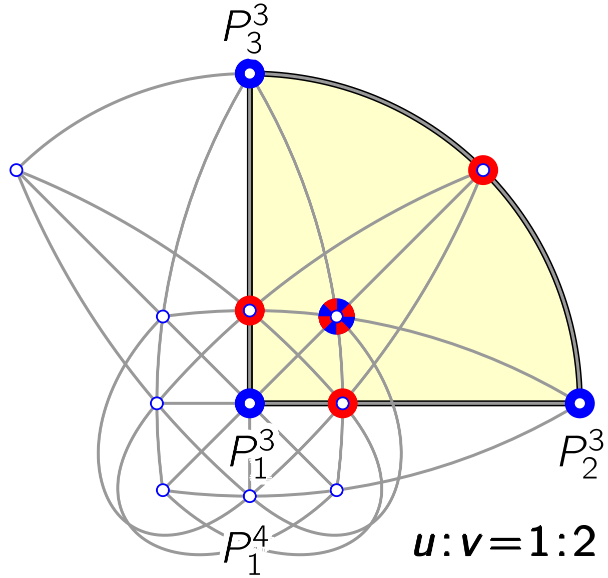


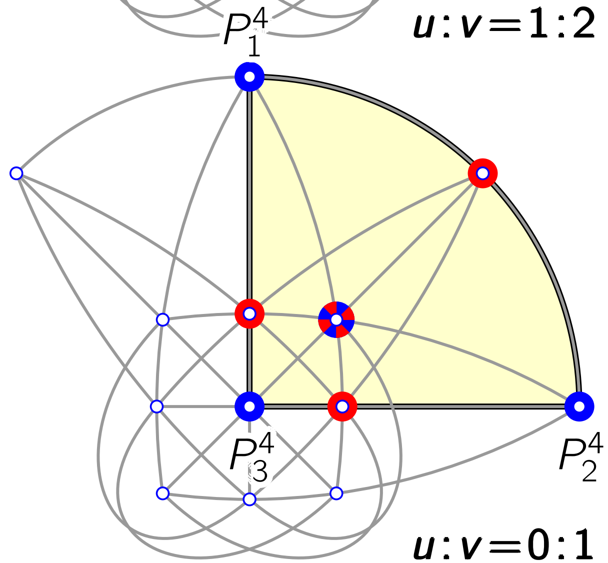
$\mathcal{M}$ and $\mathcal{N}$ are both regular and share the line element $(X_1, \mathcal{L}_1)$.
$\Longrightarrow$ pencil of the 2nd kind?

examples – 13 point plane









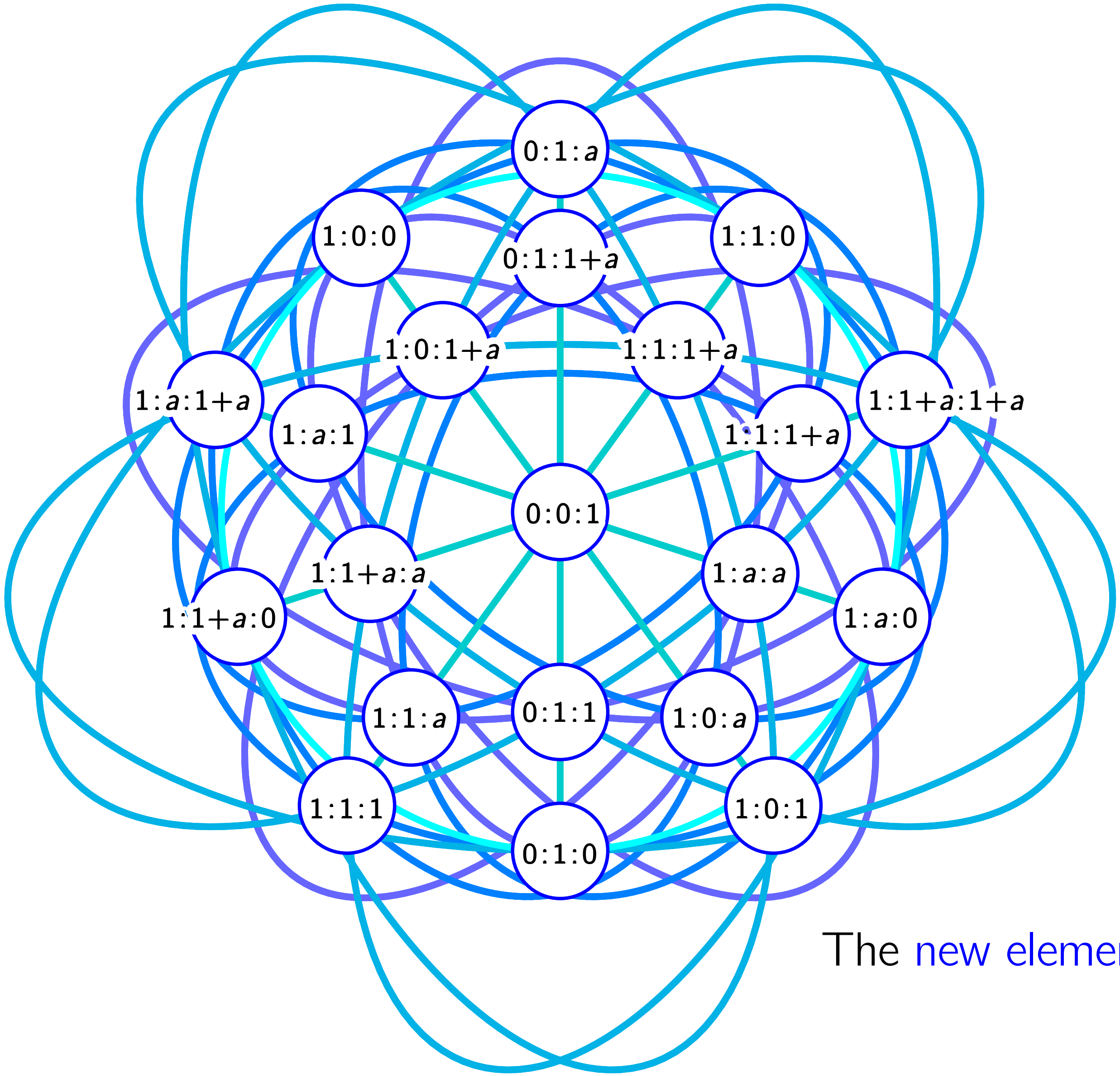
The YP in the 13 point plane contains

one regular triangle - labelled in three different ways and

one degenerate triangle - the point X_1, since

$u^2 + uv + v^2 = u^2 - 2uv + v^2 = (u - v)^2$ in GF(3)

examples – plane of order 4



Mind your field extensions!

21 point plane = $\mathbb{P}^2(\mathrm{GF}(4))$

$\mathrm{GF}(4) = \mathrm{GF}(2)[x]/(x^2+x+1)$

usual quadratic field extension

Note: $\operatorname{char} \mathrm{GF}(4) = 2$, *i.e.*,

order $\neq$ characteristic

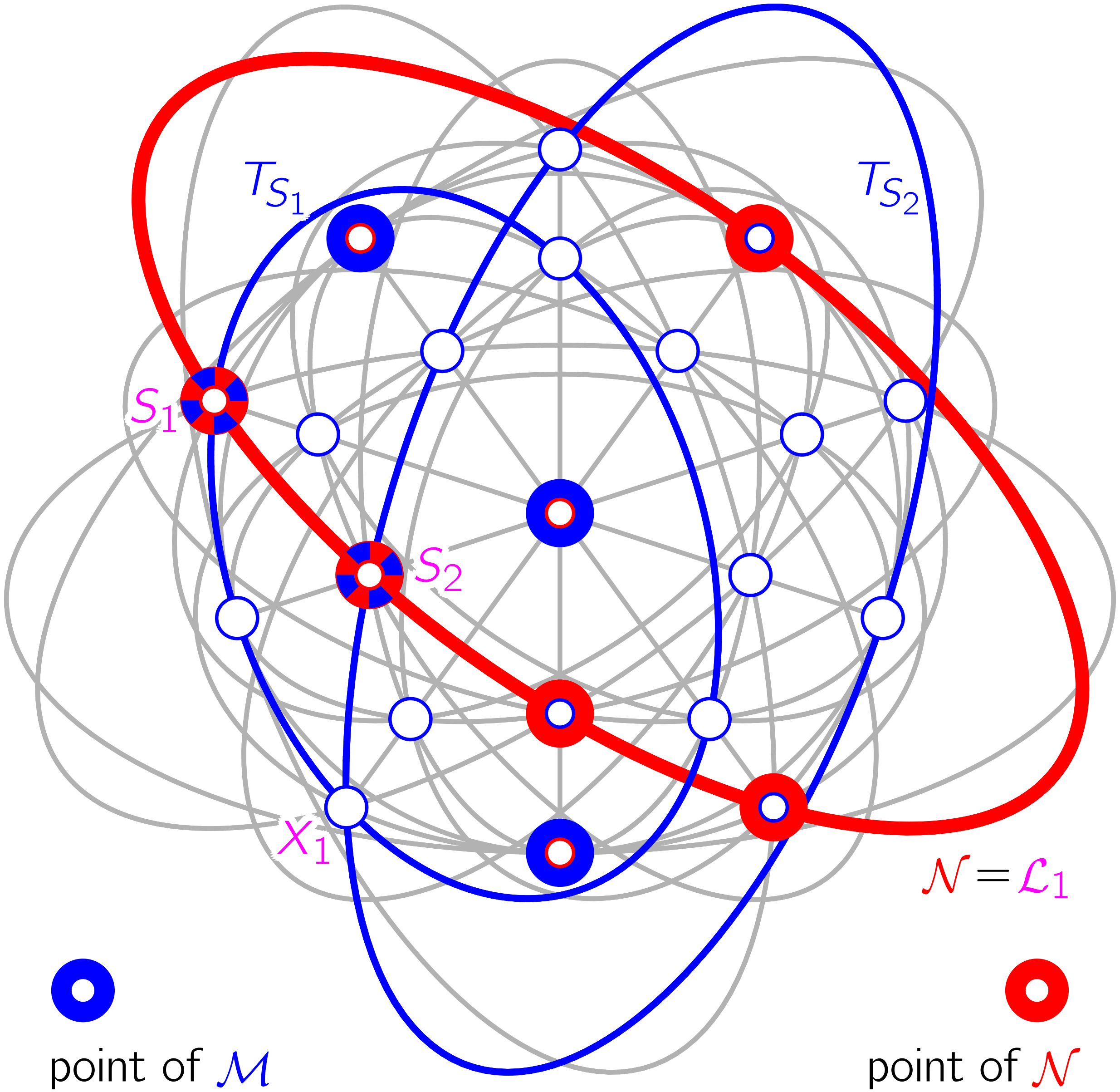
Unfortunately:

(1) $x^2 + x + 1 = \frac{v^2}{u^2} + \frac{v}{u} + 1$

(2) unique field extension $\Longrightarrow$

The new elements cause degenerate triangles!

examples – plane of order 4



$\mathcal{M}$ singular, consists of
5 points – no 3 collinear
with nucleus X_1

$\mathcal{N}$ regular, consists of
five collinear points

n.b. $\mathcal{N} = \mathcal{L}_1$ as point sets!

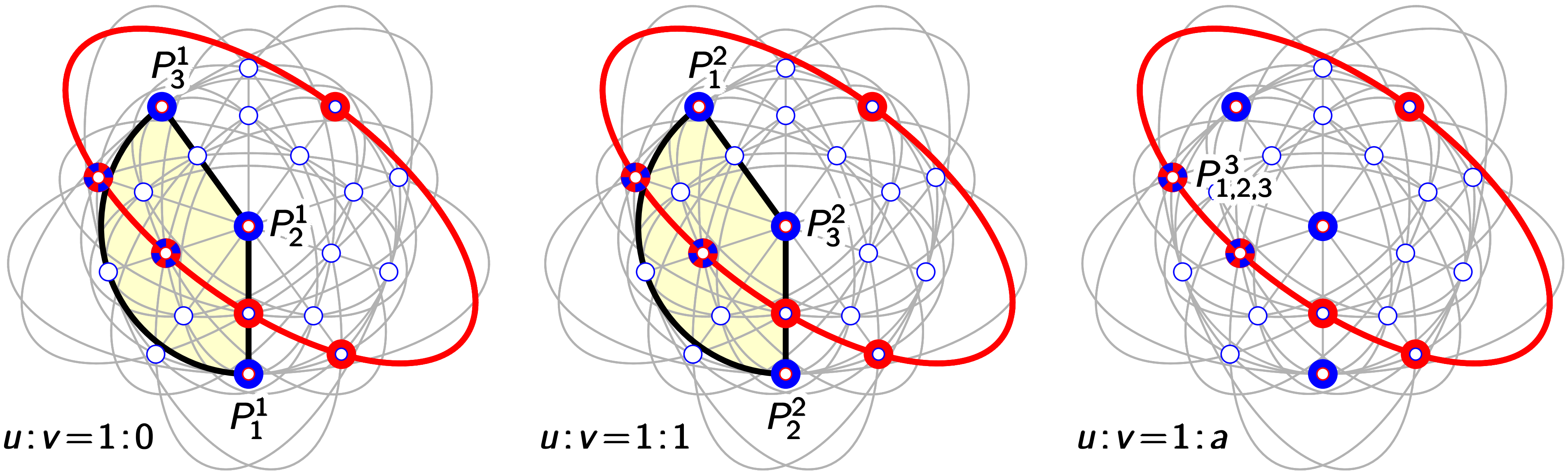
$\mathcal{M} \cap \mathcal{N} = \{S_1, S_2\}$ with

$S_1 = 1 : a : 1+a$,

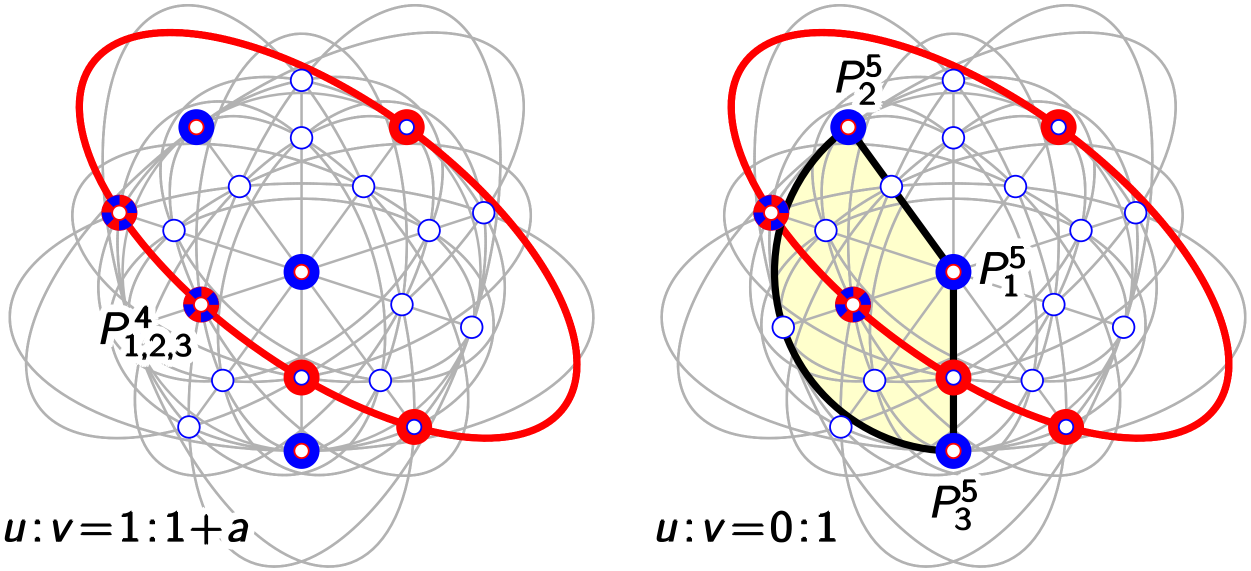
$S_2 = 1 : 1+a : a$

and $T_{S_1} \cap T_{S_1} = X_1$
(nucleus of $\mathcal{M}$)

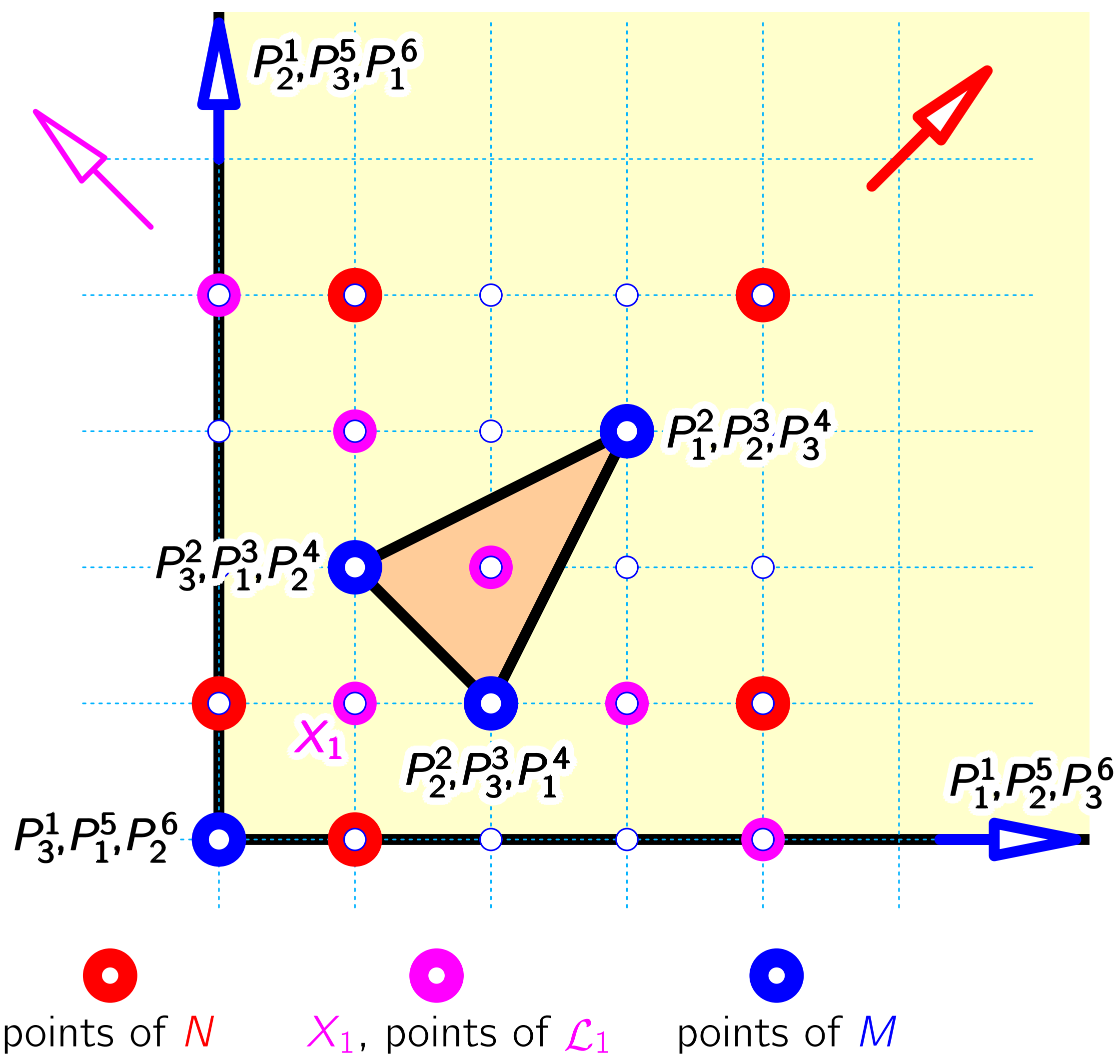
examples – plane of order 4



The YP in $\mathbb{P}^2(\mathrm{GF}(4))$:
two degenerate triangles and
one regular triangle –
labelled in 3 different ways
triangle sides not tangent to
$\mathcal{N}$, polarity = null-polarity



examples – plane of order 5



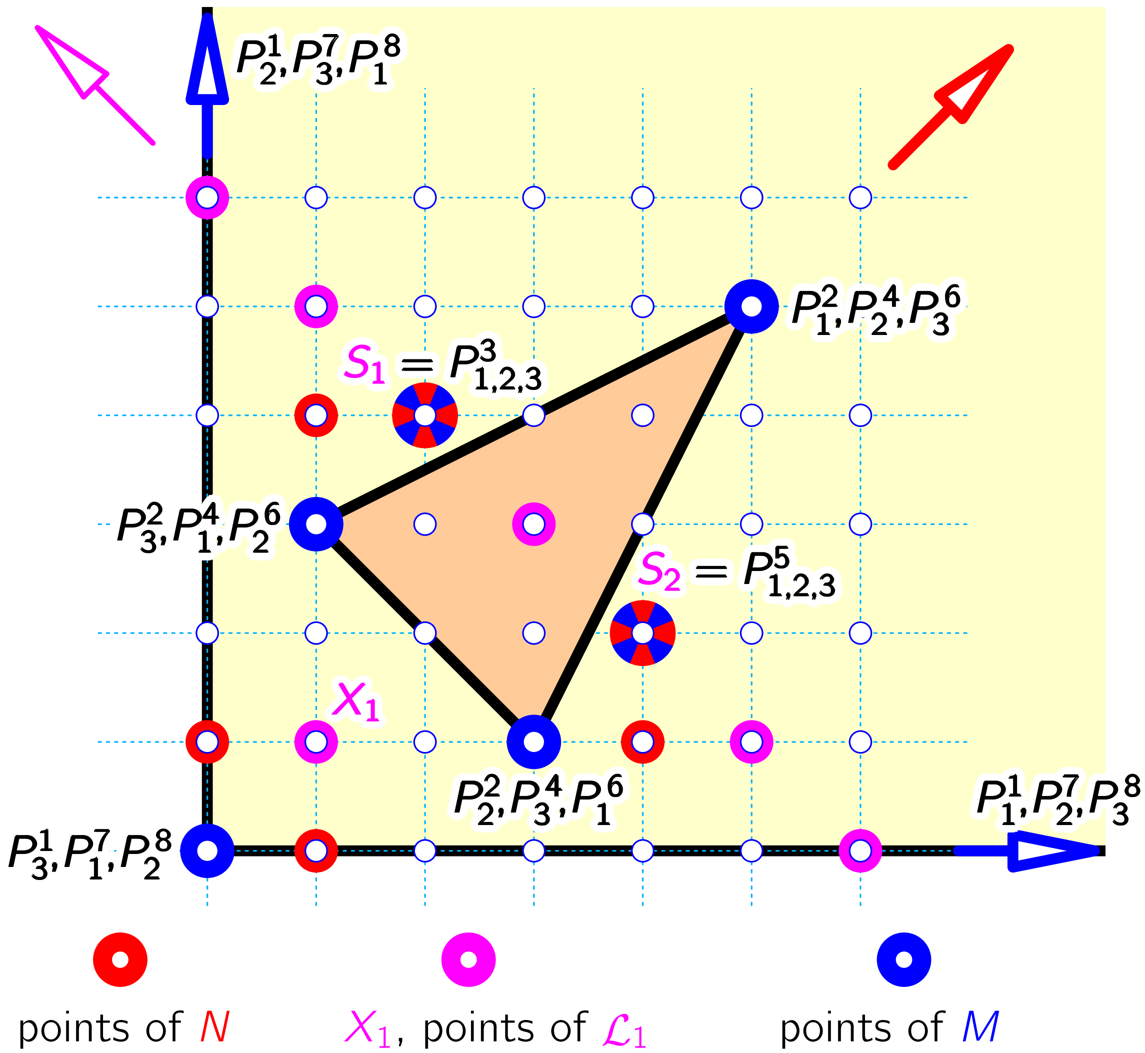
$\mathcal{M}$, $\mathcal{N}$ regular

$\mathcal{M} \cap \mathcal{N} = \emptyset$

no degenerate triangle

The number of (regular) triangles in the poristic family equals the number points on a line:
2 different triangles each plays a threefold role.

examples – plane of order 7



$\mathcal{M}$, $\mathcal{N}$ regular, $\mathcal{M} \cap \mathcal{N} = \{S_1, S_2\}$ with

$$S_1 = 1 : 2 : 4, \; S_2 = 1 : 4 : 2$$

with common tangents there $\Longrightarrow$ resembles a pencil of the 3rd kind

S_1, S_2 = two degenerate triangles

The number of (regular + singular) triangles in the poristic family equals the number points on a line: 2 different triangles each plays a threefold role.

examples – plane of order 8

$\mathbb{P}^2(\mathsf{GF}(8))$: $\mathsf{GF}(2^3) \cong \mathsf{GF}(2)[x]/(x^3+x+1)$ (cubic field extension)

Any other cubic field extension yields isomorphic copies of $\mathsf{GF}(8)$ and $\mathbb{P}^2(\mathsf{GF}(8))$.

order = 8, characteristic = 2

$u:v$	$1:0$	$1:1$	$1:a$	$1:1+a$	$1:a^2$	$1:1+a^2$	$1:a+a^2$	$1:1+a+a^2$	$0:1$
triangle	BCA	ABC	$R_1S_1T_1$	$R_2S_2T_2$	$S_1T_1R_1$	$T_2R_2S_2$	$T_1R_1S_1$	$S_2T_2R_2$	CAB

$$R_1 = 1 : a^2 : 1+a, \quad S_1 = 1 : a : 1+a+a^2, \quad T_1 = 1 : a+a^2 : 1+a^2,$$
$$R_2 = 1 : 1+a+a^2 : a, \quad S_2 = 1 : 1+a : a^2, \quad T_2 = 1 : 1+a^2 : a+a^2.$$

$\mathcal{M}$, $\mathcal{N}$ singular, …see $\mathbb{P}^2(\mathsf{GF}(2))$, 3 non-degenerate triangles (each threefold)

examples – plane of order 9

$\exists$ 4 non-isomorphic copies of projective planes of order 9,
but only one that is Desarguesian and Pappian.

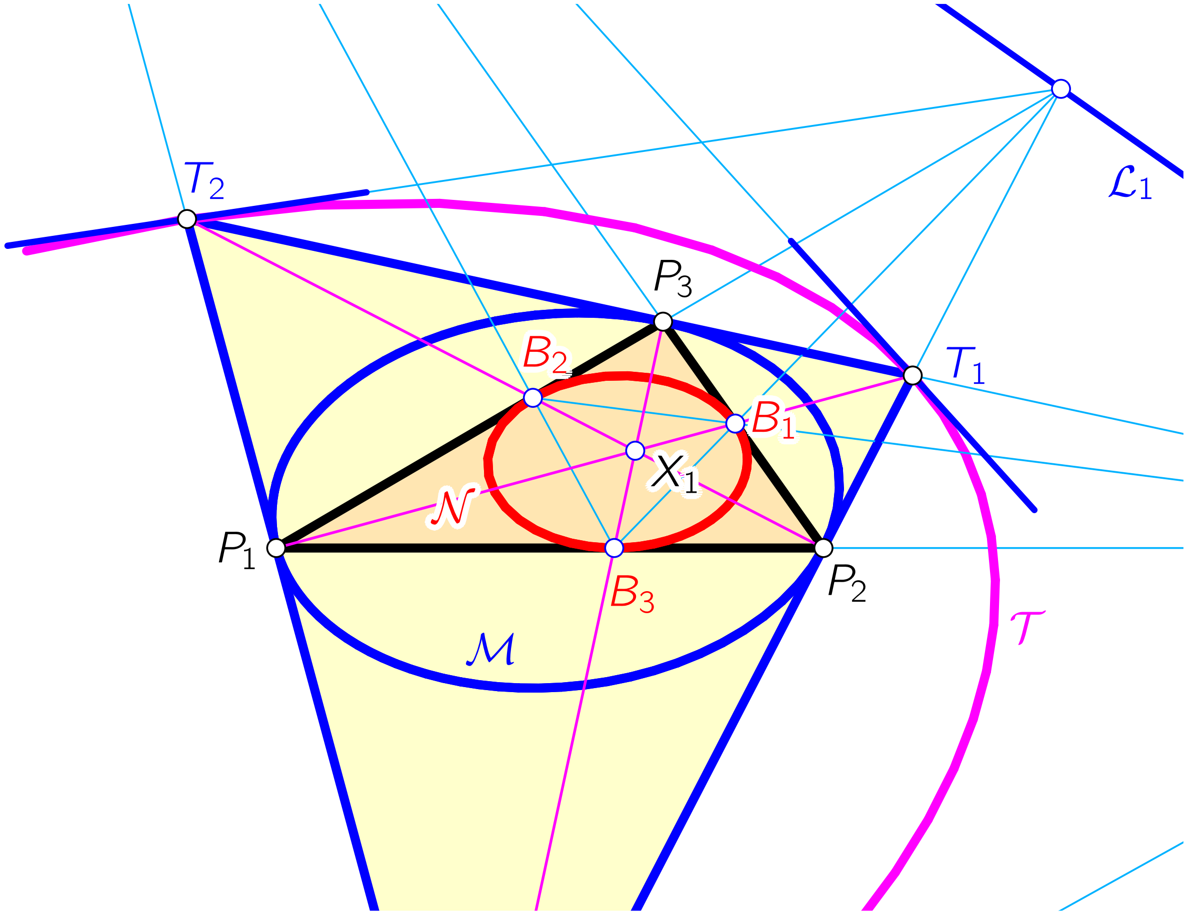
$\mathbb{P}^2(\mathsf{GF}(9))$: $\mathsf{GF}(9) \cong \mathsf{GF}(3)[x]/(x^2+1)$ (quadratic field extension)
order = 9, characteristic = 3

$u:v$	$1:0$	$1:1$	$1:2$	$1:a$	$1:1+a$	$2+a$	$2a$	$1+2a$	$2+2a$	$0:1$
triangle	BAC	$1:1:1$	ACB	$R_1S_1T_1$	$R_2S_2T_2$	$T_1R_1S_1$	$T_2R_2S_2$	$S_1T_1R_1$	$S_2T_2R_2$	CAB

$$R_1 = 1:2+a:2+2a, \quad S_1 = 1:a:1+a, \quad T_1 = 1:1+2a:2a,$$
$$R_2 = 1:2a:1+2a, \quad S_2 = 1:1+a:a, \quad T_2 = 1:2+2a:2+a,$$

$\mathcal{M}$, $\mathcal{N}$ regular, . . . see $\mathbb{P}^2(\mathsf{GF}(3))$, 3 non-degenerate triangles (each threefold)
+ a single degenerate triangle

more YPs



The orbit of the tangent triangle $T_1T_2T_3$ is a conic $\mathcal{T}$ from the Yff pencil.

$\mathcal{T}$ lies in the linear / exponential pencil spanned by $\mathcal{M}$, $\mathcal{N}$.

Coefficient matrices satisfy:

linear $5 \cdot \mathbf{M} + 2 \cdot \mathbf{N} = \mathbf{T}$ or

exponential $\mathbf{M}(\mathbf{N}^{-1}\mathbf{M})^{t-1} = \sigma \cdot \mathbf{T}$,

and since $\mathbf{M} \cdot \mathbf{N} = -2 \cdot \mathbf{I}_3 \Longrightarrow$

$\mathbf{T} = \mathbf{M}^{2t-1}$ with $t = 2$ ($\mathrm{char}\, \mathbb{F} \neq 2$)

Thm.:

$\mathcal{T}$ is the image of $\mathcal{N}$ under a harmonic homology with center X_1 and axis $\mathcal{L}_1$.

more YPs

harmonic homology $\Longrightarrow$

Thm. ($\mathbb{F} \cong \mathbb{C}$):

There exist infinitely many pairs of consecutive conics in the Yff pencil that allow for porisms with regular $3n$ gons.

There are no harmonic homologies in $\mathsf{GF}(2)$.

Thm. ($\mathbb{F} \cong \mathsf{GF}(p)$, $p \geq 3$ prime):

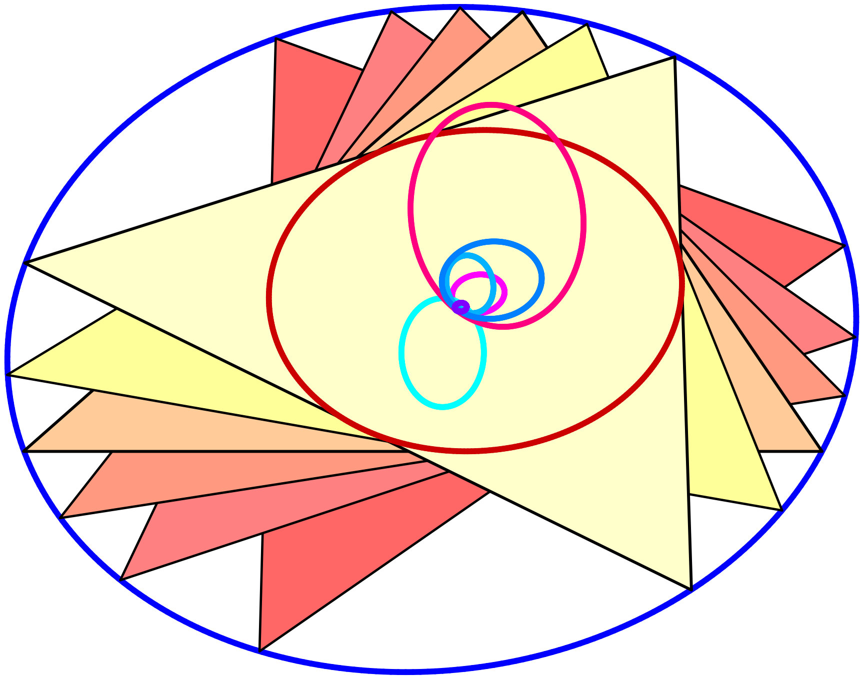
There exist $p+1$ pairs of consecutive conics in the Yff pencil that allow for porisms with regular $3n$ gons.

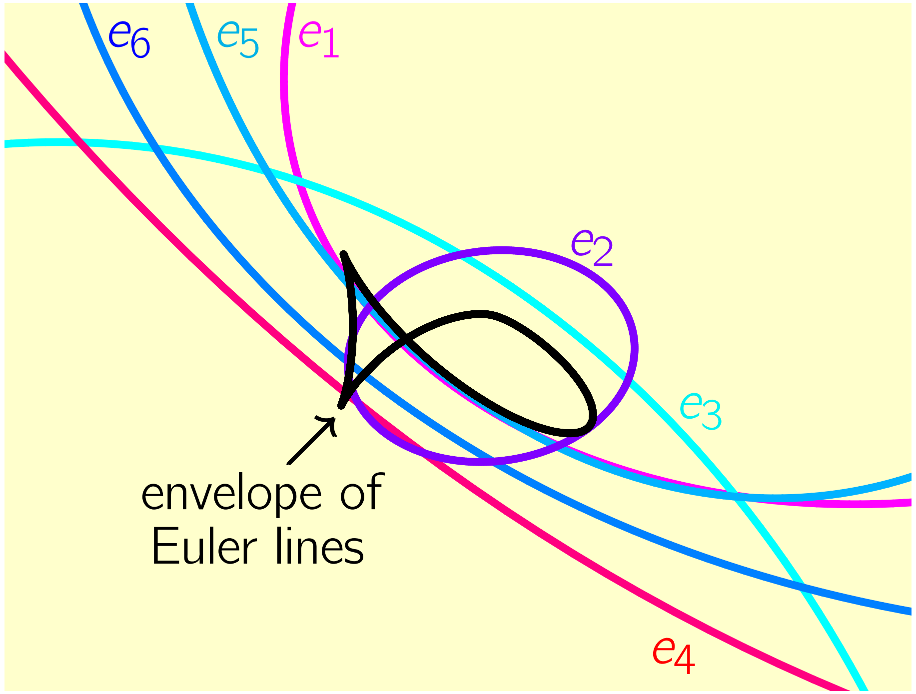
In $\mathsf{GF}(p^k)$ (with p prime and $k > 1$), the number of consecutive pairs of conics allowing for poristic families of $3n$ gons does not change.

Field extensions do not lead to new members in the family.

Euclidean facts

Thm.:
The orbits of the triangle centers X_i with Kimberling indices $i = \{1, 2, 3, 4, 5, 6\}$ are ellipses e_i.





Thm.:
Vertices of tangent triangles move on conics, and also for all iterated tangent triangles.

Euclidean facts

in the following i = center number in Kimberling's encyclopedia

124 triangle centers trace $\mathcal{M}$ thrice:

$i \in \{88, 100, 162, 190, 651, 653, 655, 658, 660, 662, 673, 771, 799, 823, 897, 1156, 1492, 1821, 2349, 2580, 2581, 3257, 4598, 4599, 4604, 4606, 4607, 8052, 20332, 23707, 24625, 27834, 29059, 32680, 34085, 34234, 36083 - 36102, 37128 - 37143, 37202 - 37223, 38340, 40110, 43069, 43192, 43757 - 43764, 45875, 46116 - 46122, 55321, 55325, 55328, 55331, 60055 - 60057, 61240, 62535\}$.

24 triangle centers trace $\mathcal{N}$ thrice:

$i \in \{244, 678, 2310, 2632, 2638, 2643, 3248, 4094, 4117, 10501, 24012, 41211, 42074, 42075, 42076, 42077, 42078, 42079, 42080, 42081, 42082, 42083, 42084, 52302\}$.

Euclidean facts

204 centers do not escape from $\mathcal{L}_1$:

$i \in \{44, 649, 650, 652, 654, 656, 657, 659, 661, 672, 770, 798, 822, 851, 896, 899, 910, 1155, 1491, 1575, 1635, 1755, 2173, 2182, 2183, 2225, 2227-2240, 2243-2247, 2252-2254, 2265, 2272, 2290, 2312-2315, 2348, 2483, 2484, 2503, 2509, 2511, 2515, 2516, 2522, 2526, 2578, 2579, 2590, 2591, 2600, 2610, 2624, 2630, 2631, 2635, 2637, 2641, 2642, 3000, 3013, 3287, 3330, 3768, 4394, 4724, 4782, 4784, 4790, 4813, 4893, 4979, 7655, 7659, 8043, 8061, 9356, 9360, 9393, 9404, 9508, 9511, 10495, 13401, 14298, 14299, 14300, 15586, 17410, 17418, 17420, 18116, 20331, 20979, 21127, 21894, 22108, 22443, 23503, 24533, 24750, 25143, 29357, 29361, 30600, 38472, 39690, 40109, 40137, 40338, 44151, 44319, 45877, 45881-45886, 46380-46393, 47777, 47810, 47811, 47826-47828, 47842, 48019-48033, 48160, 48162, 48193, 48194, 48213, 48226, 48244, 48544, 48572, 50328, 50335, 50336, 50349, 50350, 50358, 50359, 50454, 50455, 50505, 50525, 53300, 54258, 54277, 54278, 55216, 57164, 58288, 58374, 58773, 58842\}.$

Thank You For Your Attention!

some references

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