10^{th} SIAM Conference on Geometric Design and Computing

November 4–8, 2007, San Antonio, TX, USA

On two-parameter families of spheres with rational envelopes

Boris Odehnal

(joint work with Martin Peternell)

Vienna University of Technology

aims & results

- circumvent constructions using square roots
- find rational parameterizations of envelopes
- understand the geometry behind
- give examples
- generalize the construction

related work

[1] J. Kosinka, B. Jüttler: MOS surfaces: Medial surface transforms with rational domain boundaries. The Mathematics of Surfaces XII, Springer, 2007, 245–262.

[2] W. Lü: Rationality of offsets to algebraic curves and surfaces. Applied Mathematics 9 (Ser. B), 1994, 265–278.

[3] H. Pottmann: Rational curves and surfaces with rational offsets. CAGD 12 (1995), 175–192.

[4] M. Peternell, H. Pottmann: A Laguerre geometric approach to rational offsets. CAGD 15 (1998), 223–249.

[5] M. Peternell, B. Odehnal, L. Sampoli: Quadratic two-parameter families of spheres with rational envelopes. CAGD, to appear.

[6] M. Peternell, B. Odehnal: Two-parameter families of spheres with rational envelopes. (in preparation)

what are we doing?

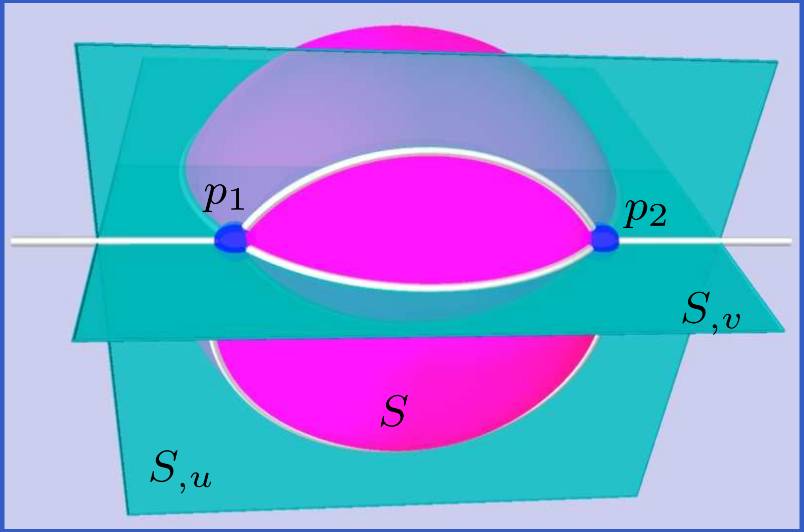
- study the geometry behind
- using the cyclographic model and some line geometry
- compute two-dim. surfaces S in $\mathbb{R}^{3,1}$ with prescribed normal planes
- apply cyclographic mapping
- compute the envelope in a constructive way using the cyclographic mapping

envelope construction: direct approach

2-parameter family of spheres

$$\langle x - s(u,v), x - s(u,v)\rangle = 0 \qquad (1.0)$$
$$\langle x - s, s_{,u}\rangle = 0, \quad \langle x - s, s_{,v}\rangle = 0 \quad (1.1, 2)$$

contact points of
sphere and envelope
=
intersection of line
and sphere

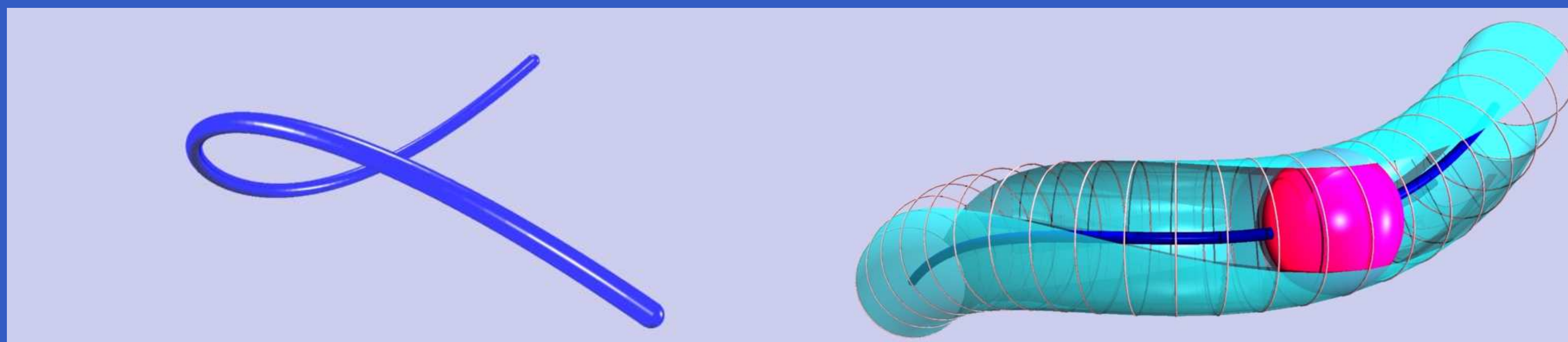


cyclographic mapping

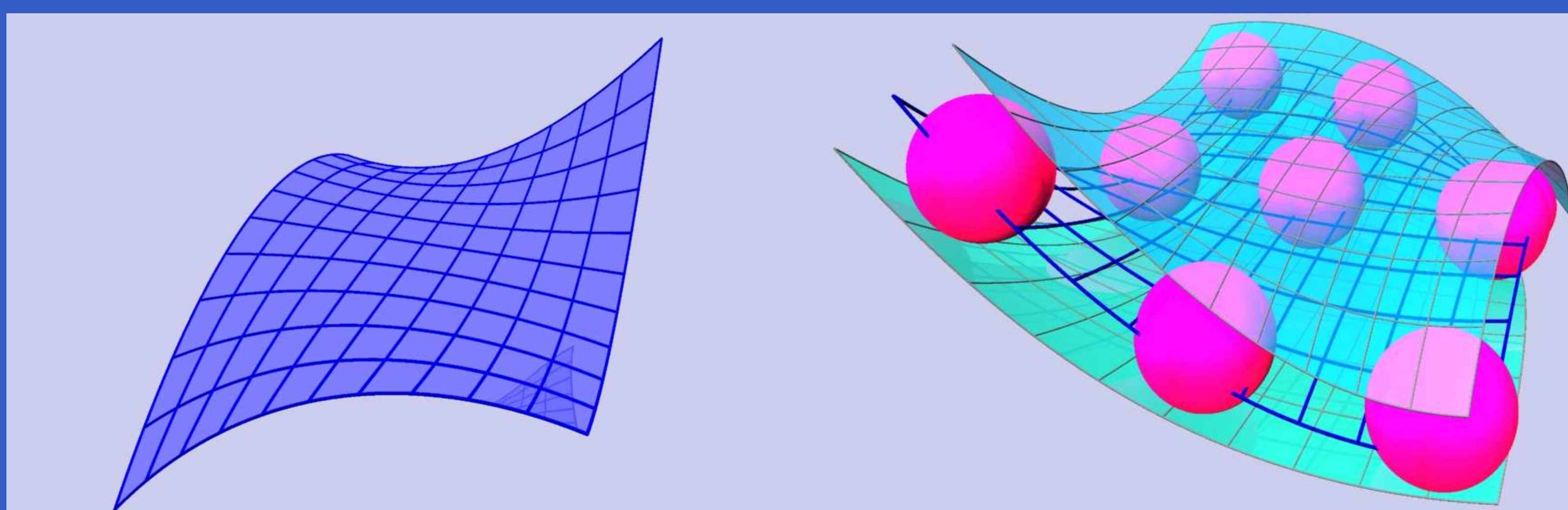
- sphere $S = (c, r)$ in Euclidean 3-space $\mathbb{R}^3$,
  $c \in \mathbb{R}^3$ . . . center, $r \in \mathbb{R}$ . . . radius
  S likewise oriented according to $\operatorname{sgn} r$
- cyclographic mapping
  $z^{-1} : \; S \mapsto s := (c, r) \in \mathbb{R}^4$
- metric in $\mathbb{R}^4$
  $\langle x, y \rangle := x^T D y$, $D :=$ diag $(1, 1, 1, -1)$
  $\Longrightarrow$ $\mathbb{R}^4$ becomes a pseudo-Euclidean, or Minkowski space $\mathbb{R}^{3,1}$
- S, T touching spheres $\Longleftrightarrow \langle s, t \rangle = 0$

cyclographic mapping

curve $c \subset \mathbb{R}^{3,1} \xrightarrow{z}$ canal surface in $\mathbb{R}^3$



2-surface $S \subset \mathbb{R}^{3,1} \xrightarrow{z}$ 2-param. family of spheres



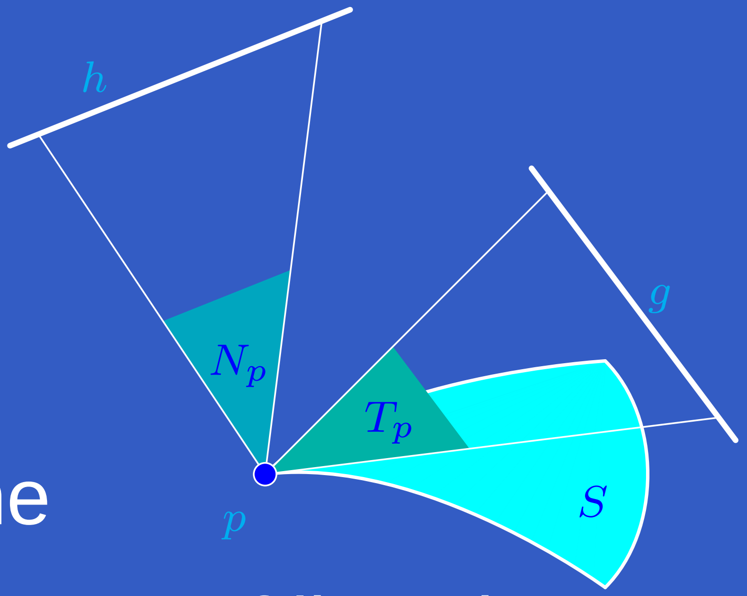
envelope construction: the other way

- $\mathbb{P}^4$ . . . projective closure of $\mathbb{R}^{3,1}$
- $\omega: \; y_0 = 0$ . . . ideal space,
  $y_4 = 0$ . . . base space $\mathbb{R}^3$
- metric defines a quadric in ω

$$\Omega: \; \langle y, y \rangle = y_1^2 + y_2^2 + y_3^2 - y_4^2 = 0$$

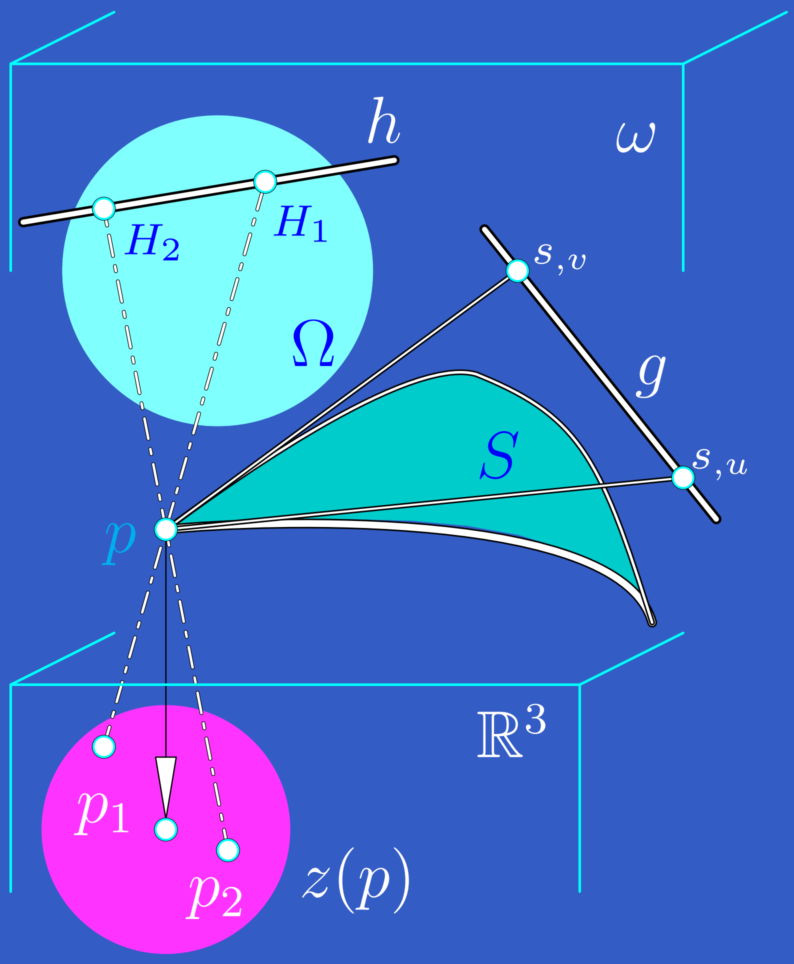
- polar system of Ω defines orthogonality in $\mathbb{R}^{3,1}$
  Ω-conjugate points are ideal points of
  pe-orthogonal directions

envelope construction: the other way



- $S \subset \mathbb{R}^{3,1}$ . . . 2-dim. surface
- T_p . . . tangent plane at p
- $g := T_p \cap \omega$ . . . T_p's ideal line
- $G := \{g | \forall p \in S\}$ . . . congruence of lines in ω
- h . . . Ω-polar image of $g \iff h$ is ideal line of normal plane N_p at p (H = congruence of . . .)

envelope construction: contact points



- $h \cap \Omega = \{H_1, H_2\}$
- Isotropic lines $H_i p$ of $\mathbb{R}^{3,1}$ intersect the base space at the contact points p_i of the sphere $z(p)$ with the envelope.
- $z(p)$'s center is obtained by the canonical projection to $\mathbb{R}^3$.

main theorem: a generalization of the LN-property

Theorem
If the ideal lines of S's normal planes form a fibration of ω then the cyclographic image of S can be parameterized rationally.

Remark
H is a fibration of ω. $\Longleftrightarrow$ H sends a unique line through each point of ω.

computing rational parameterizations = proof of the theorem

- start with the rational parameterization of Ω
  $w = (2x, 2y, 1 - x^2 - y^2, 1 + x^2 + y^2)$
- solve the system of equations (1.1,2)

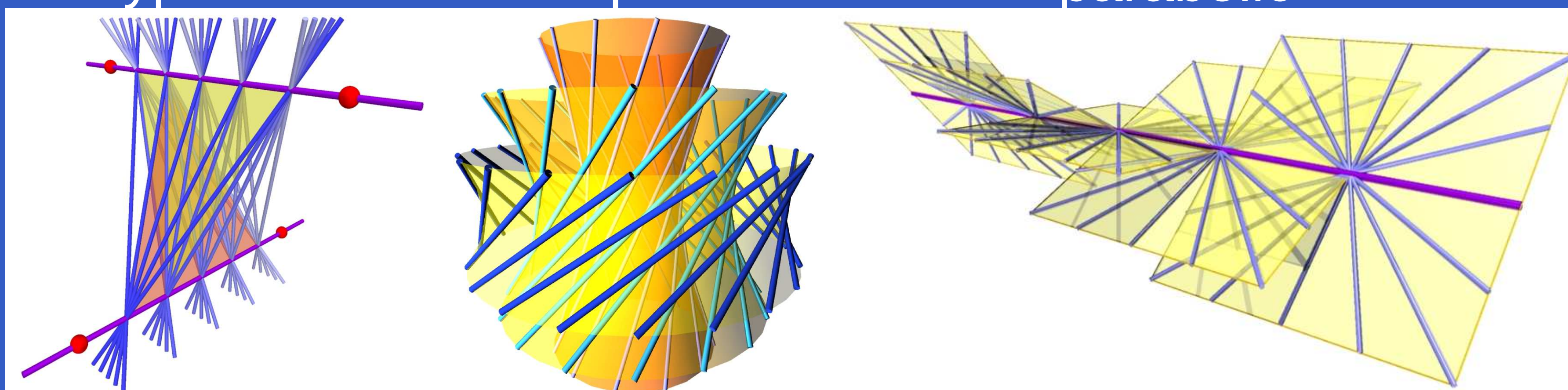
  $$\langle w, s_{,u} \rangle = 0 \quad \text{and} \quad \langle w, s_{,v} \rangle = 0$$

  in order to reparameterize S such that the cyclographic image becomes rationally parameterized
- H is a fibration of ω. $\iff$ There exists a unique solution $\{u(x, y), v(x, y)\}$.

quadratic triangular Bézier surfaces I

G, H are linear line congruences in ω

hyperbolic elliptic parabolic



$S = (u^2, v^2, u, v)$ $S = (u^2 - v^2, uv, u, v)$ $S = (u^2, uv, u, v)$

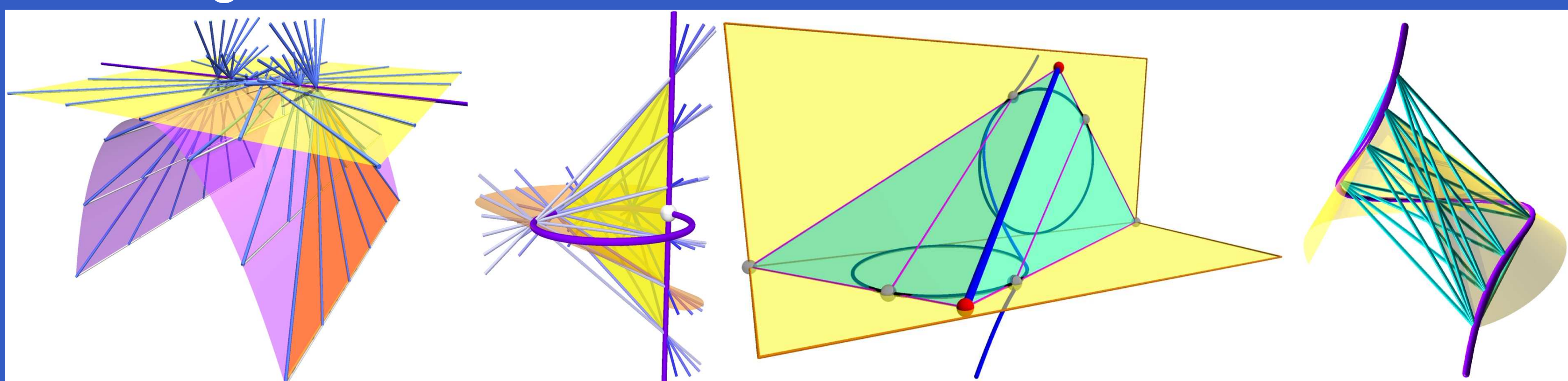
5 types in $\mathbb{R}^4$ acccording to:

[7] J. Peters, U. Reif: 42 equivalence classes of quadratic triangular Bézier surfaces. CAGD 15 (1998), 459–473.

quadratic triangular Bézier surfaces II

G, H are chords of spatial cubic curves in ω

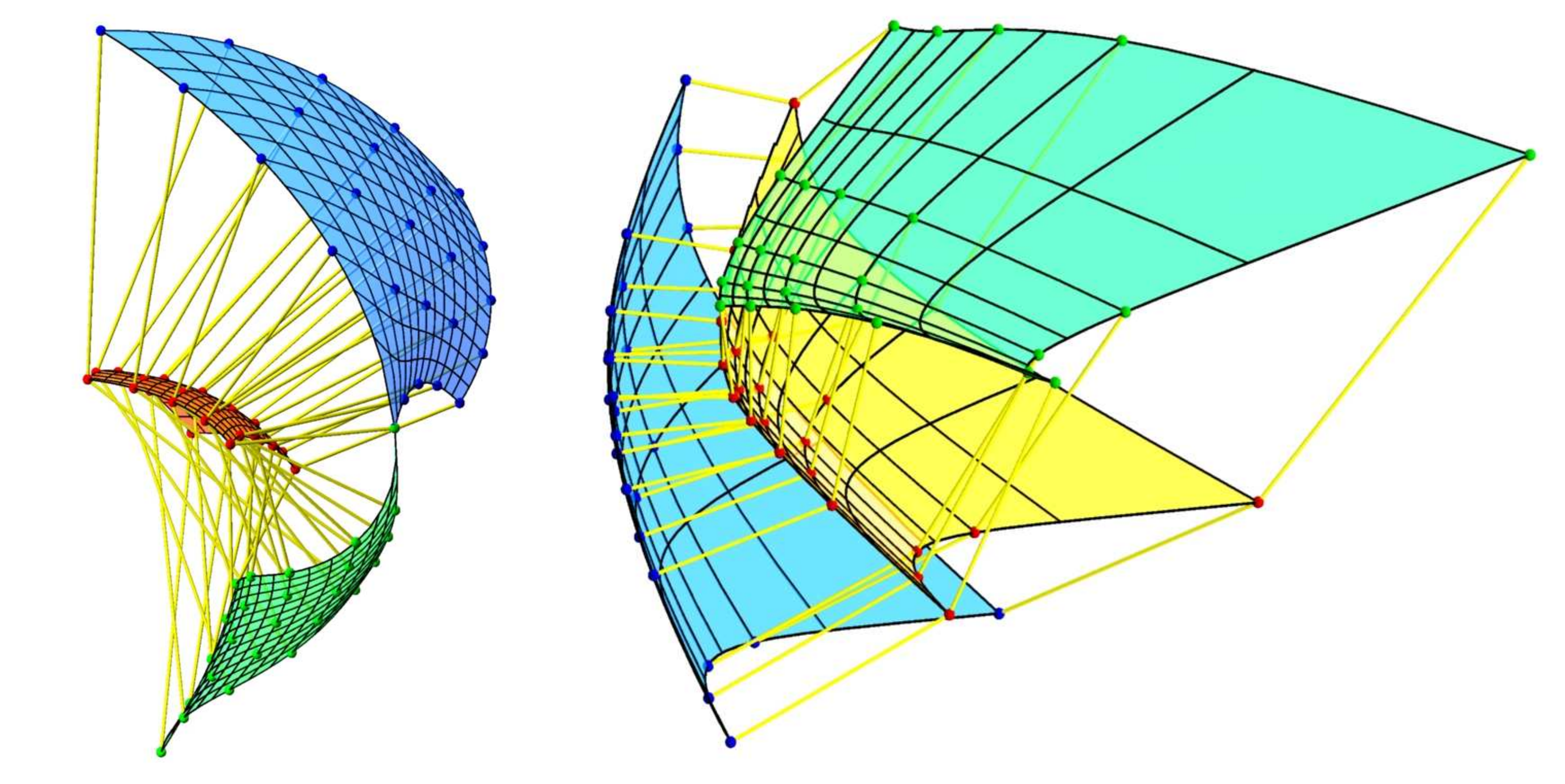
degenerate cubic | twisted cubic



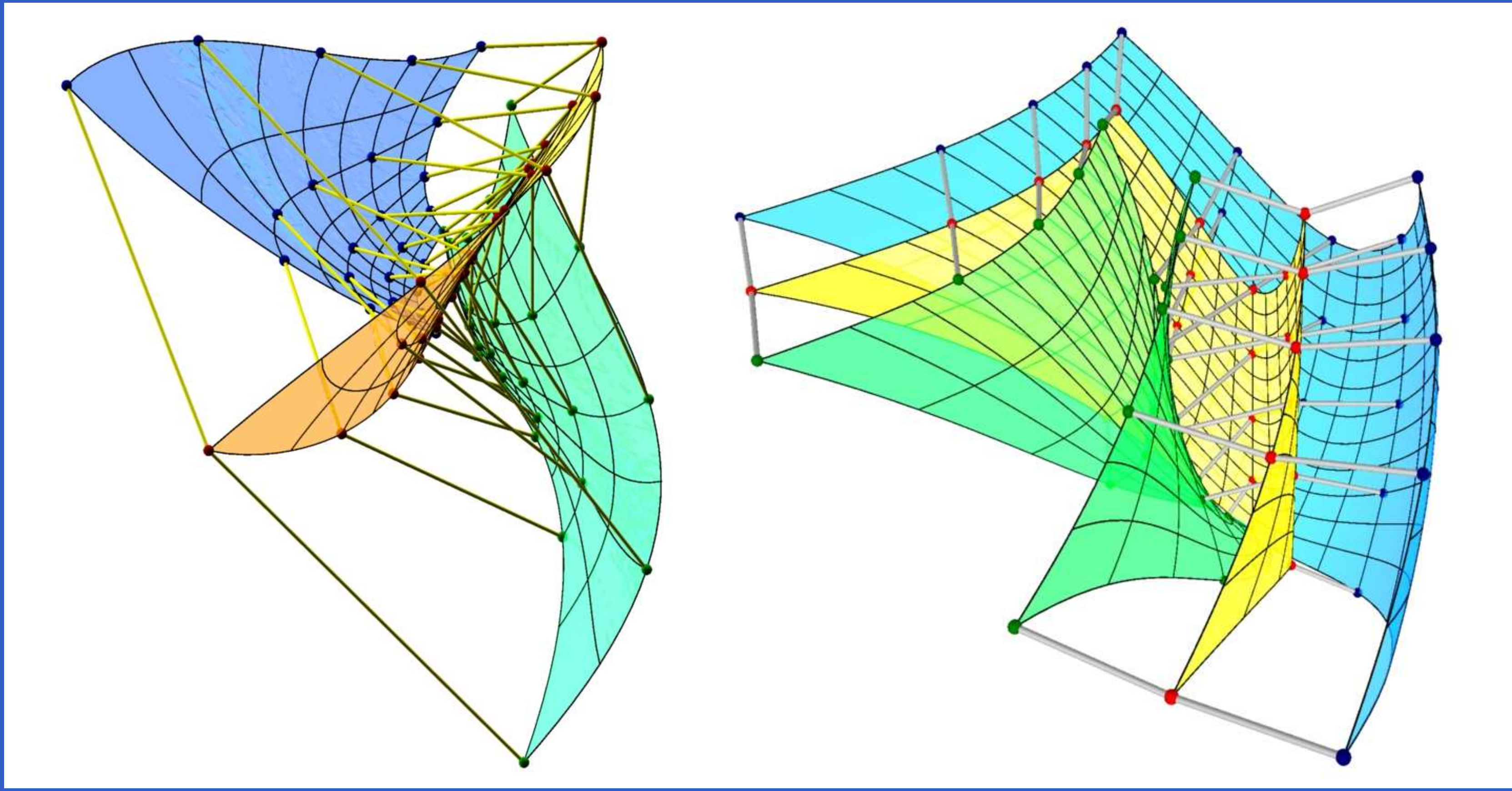
$S = (u^2 + v, v^2, uv, u)$ | $S = (u^2, v^2, uv, u)$

These five types are examples of low degree rational fibrations of ω.

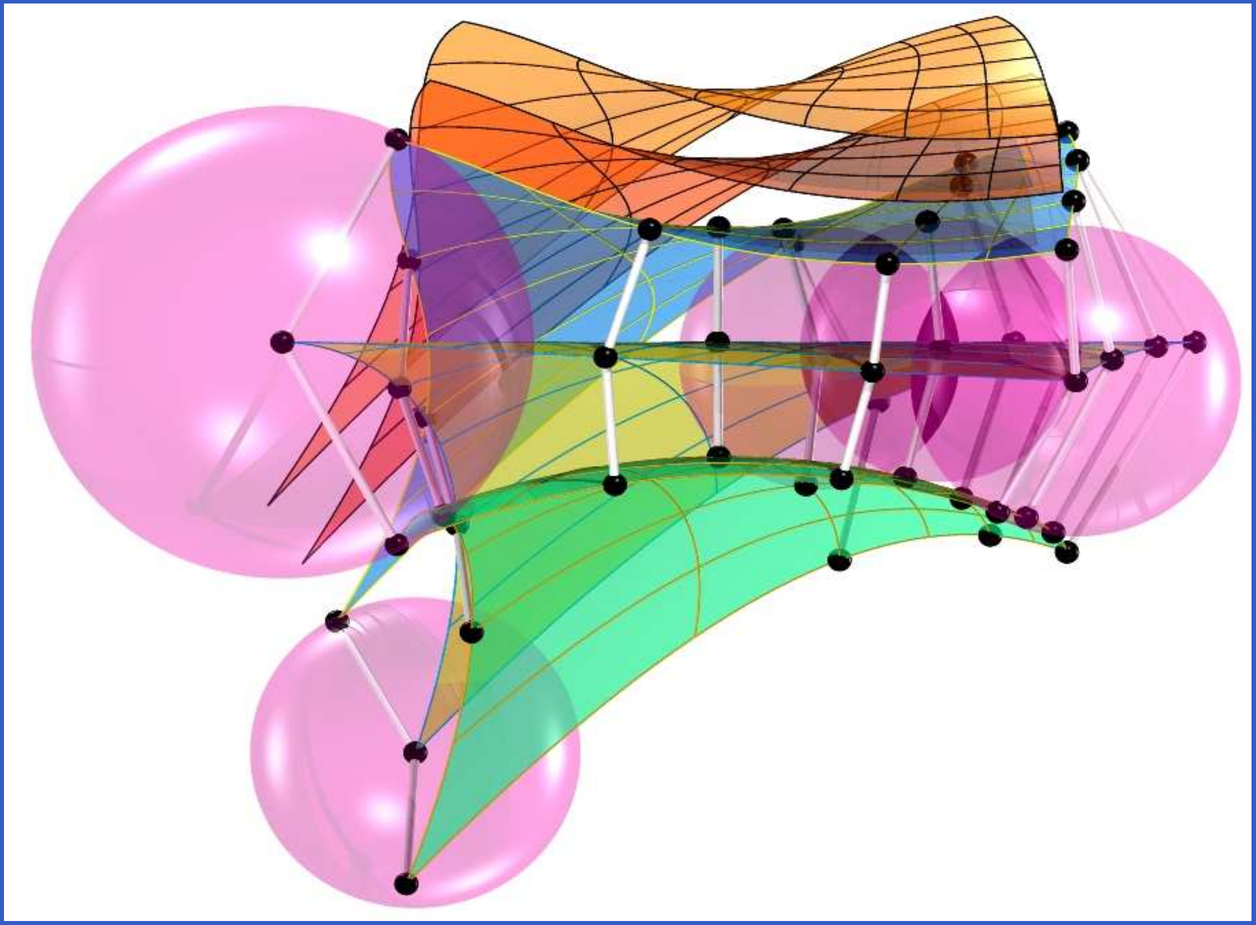
examples I



examples II



examples III



Offsets of envelopes also admit rational parameterizations.

generalizations = future work

- start with a fibration of the ideal space

  consider each line as intersection of planes from certain families

- prescribe two support functions

  for hyperplanes in $\mathbb{R}^{3,1}$ through the given ideal lines/planes

- support functions

  cannot be chosen independently

- compute S as envelope of its tangent planes

  by intesecting two hyperplanes

- determine the cyclographic image of S and compute the reparameterization

Thank You For Your Attention!