# Remarks on Algebraic Geodesics on Quadrics

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## comming soon

| | |
|---|---|
| geodesics | definitions, equations |
| general results | for geodesics on quadrics, tangents, umbilics, symmetry, asymptotic behavior |
| algebraic geodesics | how to find, different approaches |
| some results | on various quadric types |
| examples | algebraic, rational |
| open problems | questions, results to be verified / falsified |

## Geodesics

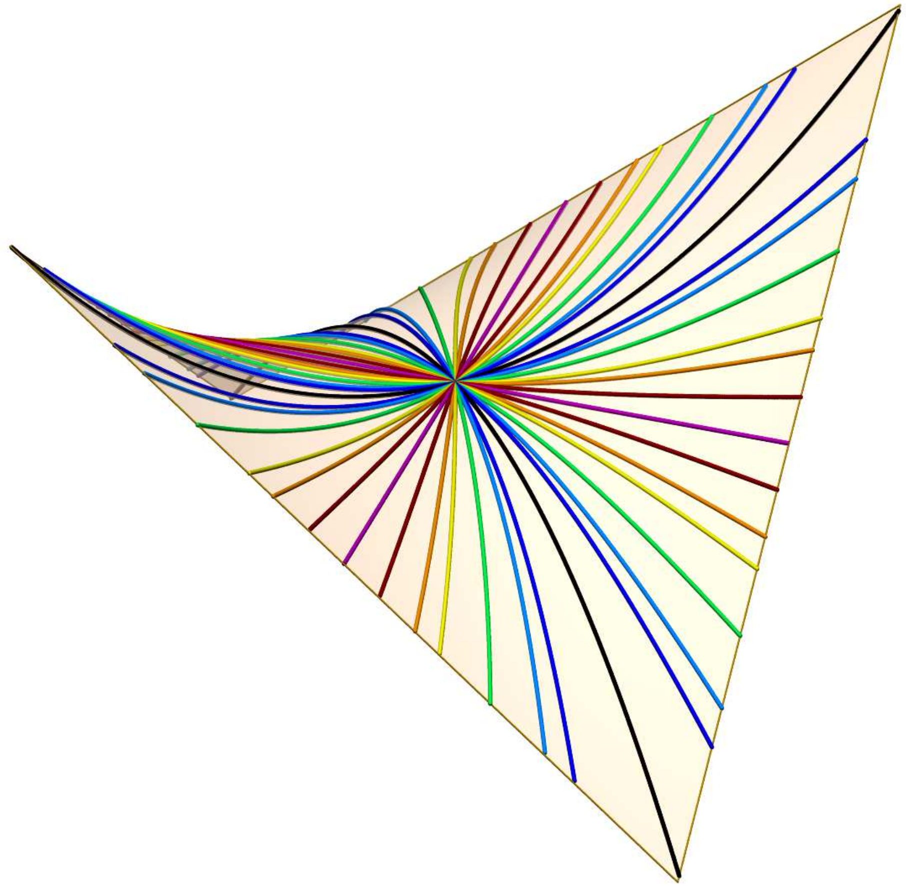
A geodesic (curve) $g$ on a surface $\mathcal{S}$ is a curve whose principal normals coincide with the surface normals of $\mathcal{S}$ at each point $P \in g \subset \mathcal{S}$.

$$\Longrightarrow \det(\mathbf{n_g}, \dot{\mathbf{g}}, \ddot{\mathbf{g}}) = 0$$

Locally, $g$ is the shortest curve between to different points (on $\mathcal{S}$), provided sufficient proximity of endpoints.

In terms of local coordinates, the geodesics on $\mathcal{S}$ with metric $\mathrm{d}s^2 = g_{ij}\mathrm{d}u^i\mathrm{d}u^j$ are solutions of the second order differential equations

$$\ddot{c}^k + \Gamma^k_{ij}\dot{c}^i\dot{c}^j = 0, \quad i, j, k \in \{1, 2\}.$$



These equations turn out to be helpful only in connection with elliptic coordinates!

## Geodesics on quadrics - general results

The principal sections are geodesics. [1,2,5]

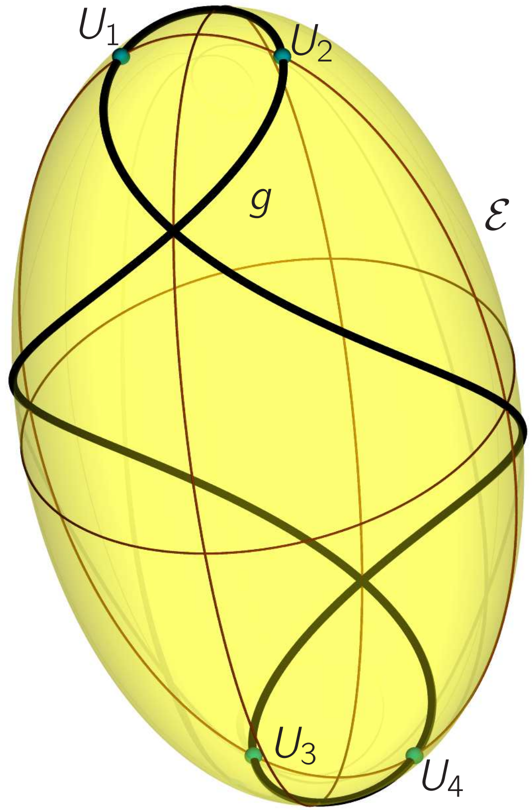
Sphere: All great circles are geodesic, and *vice versa*.

Cylinders, cones, ruled quadrics: rulings are geodesic.

Asymptotic behavior: Geodesics converge to rulings or principal sections. [2,5]

If an umbilic of a quadric lies on a geodesic, then so does the opposite umbilic. [8]

In general: Parametrizations of geodesics involve elliptic integrals of the third kind. [1,2,3,6,11]

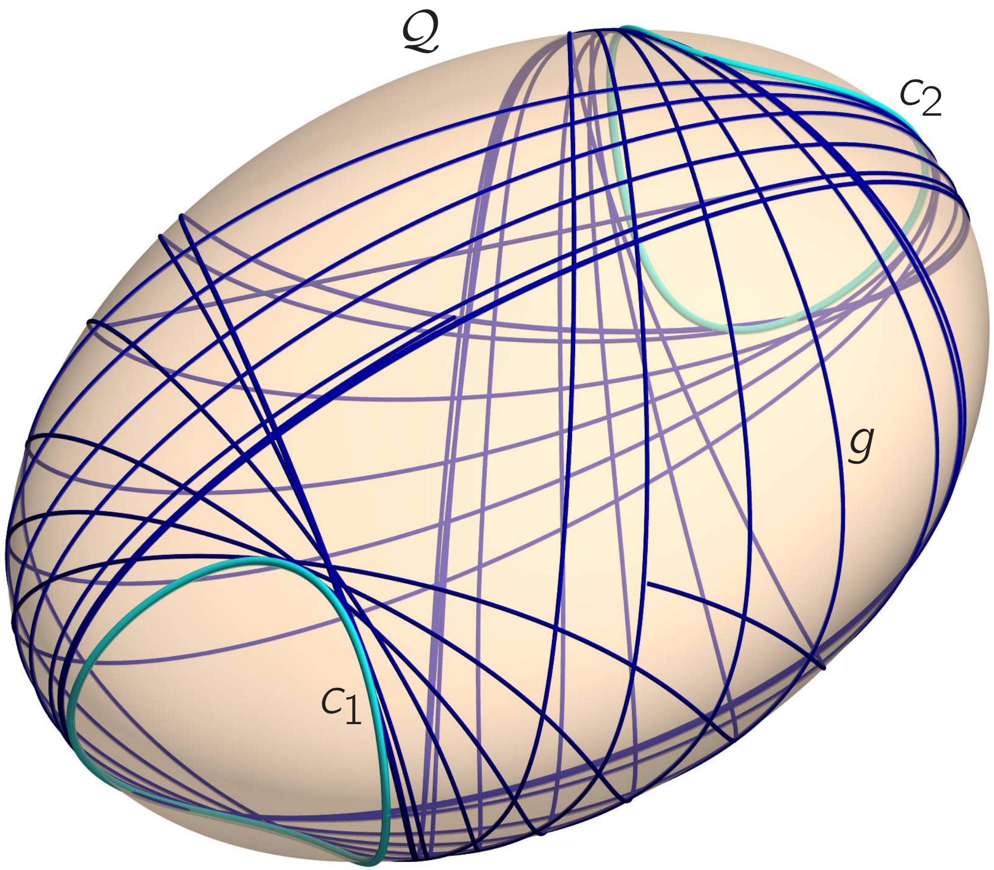


## Geodesics on quadrics - general results

A generic (non-algebraic) geodesic $g$ on a quadric $\mathcal{Q}$ oscillates between a pair $(c_1, c_2)$ of symmetric lines of curvature.

The geodesic $g$ makes infinitely many rounds, and enevlopes the lines $c_1$ and $c_2$ of curvature. The curve $g$ fills the strip between $c_1$ and $c_2$. [5,7]

Geodesics on an ellipsoid (of revolution) can be transpolar, circumpolar, . . . . . . and look completely different on prolate and oblate ellipsoids.



## Elliptic coordinates

Elliptic coordinates allow a simplification of the differential equations of geodesics.
*e.g.* metric of the ellipsoid:

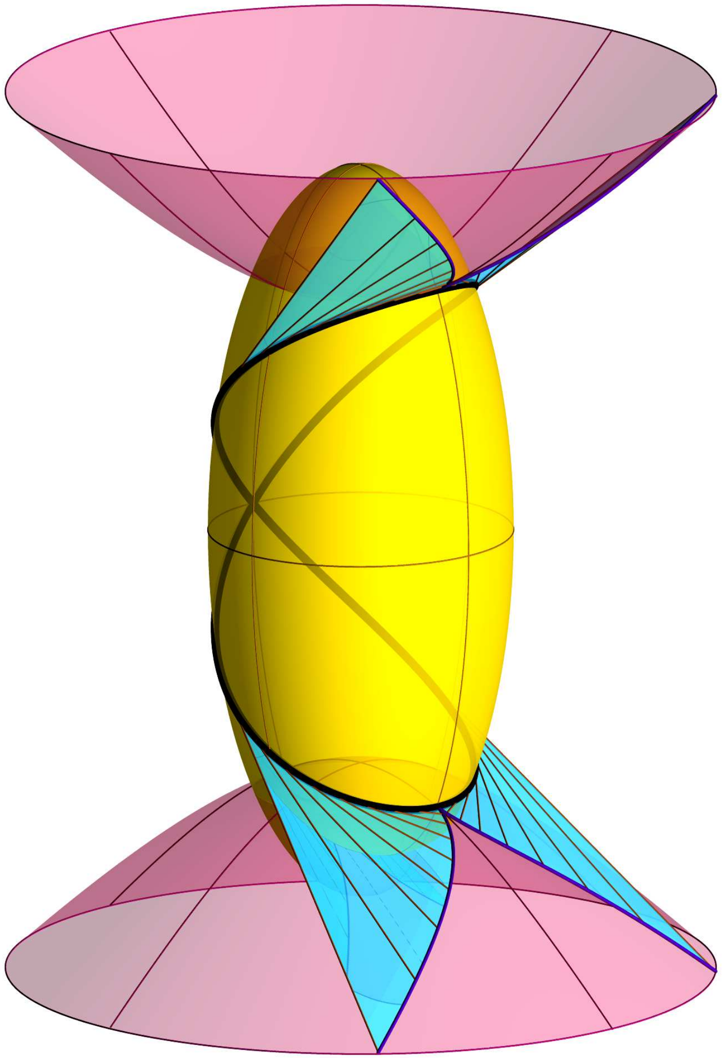
$$ds^2 = \frac{u-v}{4}\left(\frac{v\,du^2}{(a^2-u)(b^2-u)(c^2-u)} - \frac{u\,dv^2}{(a^2-v)(b^2-v)(c^2-v)}\right)$$

has a diagonal coefficient matrix.

coordinate lines = lines of curvature

$\Longrightarrow$ Family of confocal quadrics enters the scene.

Tangents of geodesics (on a quadric) touch one of the confocal quadrics. [4,6]



## How to find algebraic geodesics?

following Perelomov [9,10]

hyperbolic paraboloid $\mathcal{P}: \frac{x^2}{a} - \frac{y^2}{b} = 2z$

in terms of elliptic coordinates

$$x^2 = -\tfrac{a}{a+b}(u+a)(v+a), \quad y^2 = -\tfrac{b}{a+b}(u-b)(v-b),$$
$$z = -\tfrac{1}{2}(u+v+a-b)$$

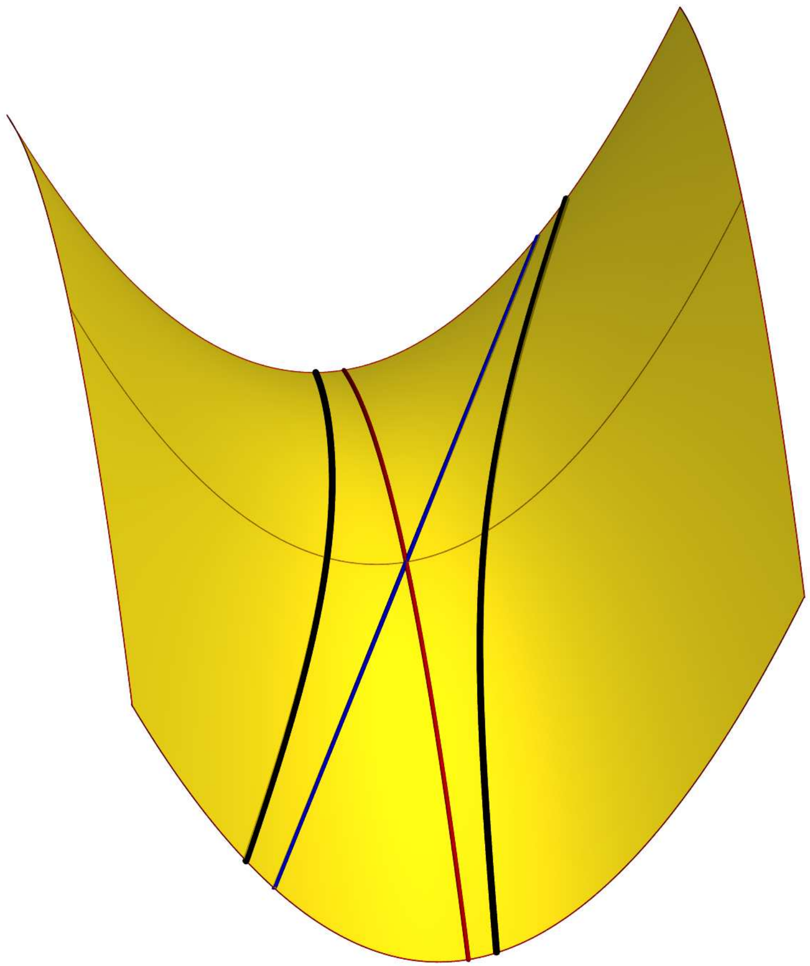
Euler-Lagrange equations of $\int \mathrm{d}s \to \min$ simplify to

$$\int \sqrt{\frac{u}{(u+a)(u-b)(u+c)}}\mathrm{d}u = \int \sqrt{\frac{v}{(v+a)(v-b)(v+c)}}\mathrm{d}v$$

with an integration constant $c \in \mathbb{R}$

unfortunately: left- and right-hand side are

elliptic integrals of the third kind



## How to find algebraic geodesics?

following Y.N. Fedorov [3], A.M. Perelomov [9,10]

The geodesics obtained in this way are algebraic only if $r = \frac{p}{q} = \sqrt{1 + \frac{b}{a}}$ is rational.

That is a condition on the quadric!

With $r \in \mathbb{Q}$, the integrals simplify to

$$\int \frac{1}{a+u}\sqrt{\frac{u}{u-b}}\mathrm{d}u = \int \frac{1}{v+a}\sqrt{\frac{v}{v-b}}\mathrm{d}v$$

and (with a further constant of integration $d \in \mathbb{R}$) evaluates to

$$\frac{2ic}{\sqrt{b(a-c)}}\left(\mathrm{E}\left(\frac{\sqrt{(a-c)u}}{\sqrt{a(u+c)}}, \frac{\sqrt{a(b+c)}}{\sqrt{b(a-c)}}\right) - \Pi\left(\frac{\sqrt{(a-c)u}}{\sqrt{a(u+c)}}, \frac{a}{a-c}, \frac{\sqrt{a(b+c)}}{\sqrt{b(a-c)}}\right)\right) = \dots$$ in general not algebraic!

$\sqrt{1 + \frac{b}{a}} \in \mathbb{Q} \implies$ Both sides can be expressed in terms of logarithms only.

$\implies$ yields an implicit algebraic equation in $u$ and $v$.

## How to find algebraic geodesics?

Subsequently changing from elliptic coordinates $(u, v)$ to Cartesian coordinates $(x, y)$ yields

$$\mathcal{Z}: \ (1-x)^p(1-y)^p(r+x)^q(r+y)^q - d(1+x)^p(1+y)^p(r-x)^q(r-y)^q = 0$$

$\mathcal{Z}$ is a cylinder erected over a planar curve in the $[x, y]$-plane.

At the same time: $\mathcal{Z}$ is an algebraic curve in the parameter plane.

With $p, q \in \mathbb{Z}$ (and hence $r = \frac{p}{q} \in \mathbb{Q}$), $\mathcal{Z}$ is algebraic.

$\Longrightarrow$ **The geodesics $\mathcal{Z} \cap \mathcal{P}$ are algebraic.**

This also works for an elliptic paraboloid.

## The only cubic geodesic on a hyperbolic paraboloid?

With $a = 1$, $b = 3$, and $d = 1$ we have $r = p = 2$ and $q = 1$.

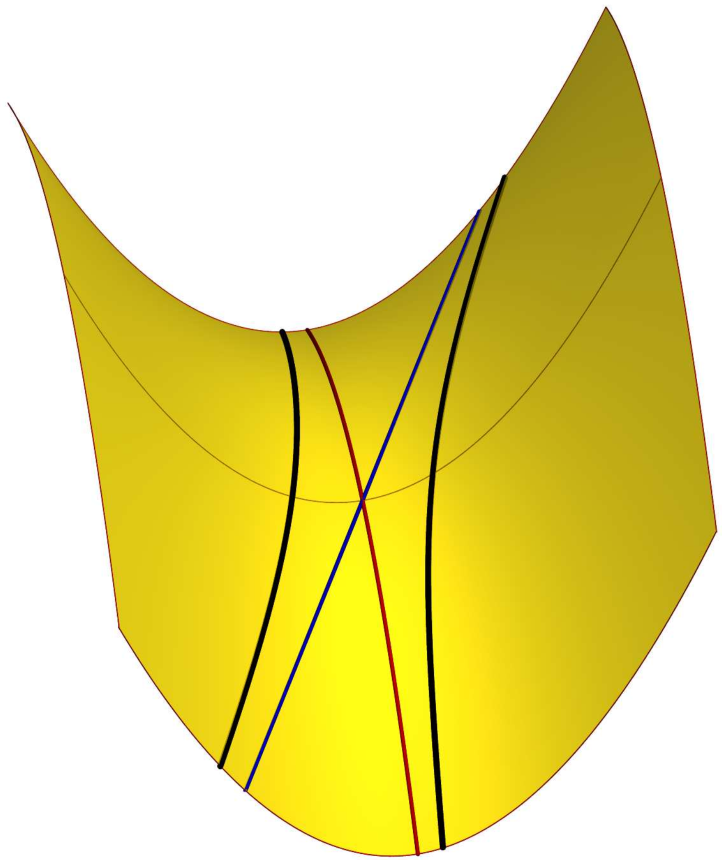
$\Longrightarrow \mathcal{P}: \; x^2 - \frac{y^2}{3} = 2z$ and $\mathcal{Z}$ becomes the union of two quadratic cylinders:

$$\mathcal{Z}: \; x\left(x \pm \frac{1}{\sqrt{3}}y\right) = \frac{1}{2}.$$

The geodesic(s) is (are) the cubic hyperbolic parabola(e)

$$\mathbf{g}(t) = \frac{1}{2t^2}\left(t^3, \mp\sqrt{3}t(t^2 - 2), t^2 - 1\right).$$

Appearantly the only cubic geodesics on hyperbolic paraboloids.



## Geodesics - Perelomov's examples 1

ignota nativitas - aenigma sui temporis

ellipsoid $\mathcal{E}$ : $b_1 x^2 + b_2 y^2 + b_3 z^2 = 1$ with geodesic

$$\mathbf{g}(t) = \frac{1}{a - \sin^2 t} \Big( c_x (b_0 - \sin^2 t), c_y \sin t \cos t, c_z \cos t \Big)$$

where $a \in \mathbb{R} \setminus \{-1, 0, \frac{1}{2}, 2\}$ can be chosen freely, but

$$b_0 = \tfrac{a-2}{2a-1}, \quad b_1 = 4(a^2 - a + 1), \quad b_2 = (2a-1)^2, \quad b_3 = (a-2)^2,$$

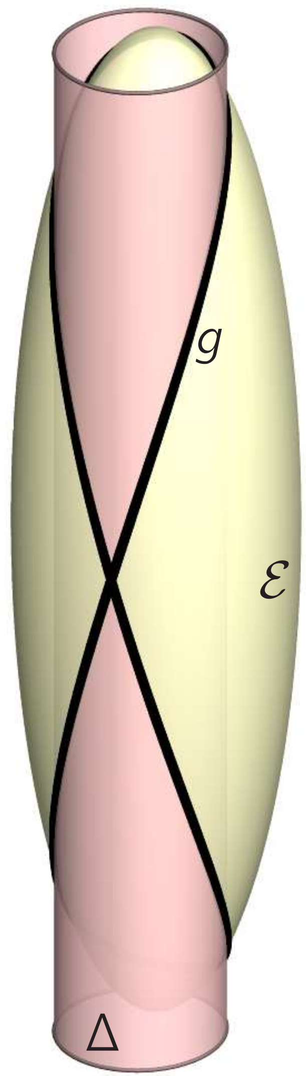
$$c_x = \frac{(a-1)\sqrt{b_2}}{(a+1)\sqrt{b_1}}, \quad c_y = \frac{\sqrt{a b_1 b_3}}{(a+1)\sqrt{b_2}}, \quad c_z = \frac{\sqrt{b_1 b_2}}{(a+1)\sqrt{b_3}}.$$

Definitely not found with the cylinder(s) mentioned before.

Where does it come from?

$\mathbf{g}(t)$ parametrizes a quartic of the first kind, rational, with double point.

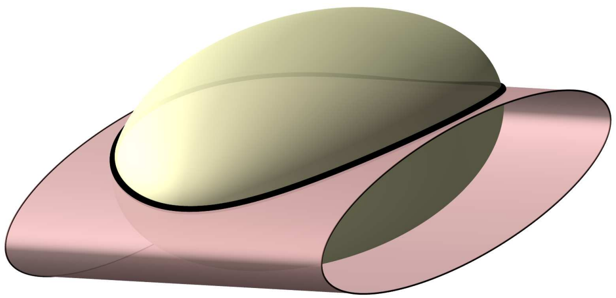
There exists a one-parameter family of algebraic geodesics that determine the surfaces on which they are geodesic!



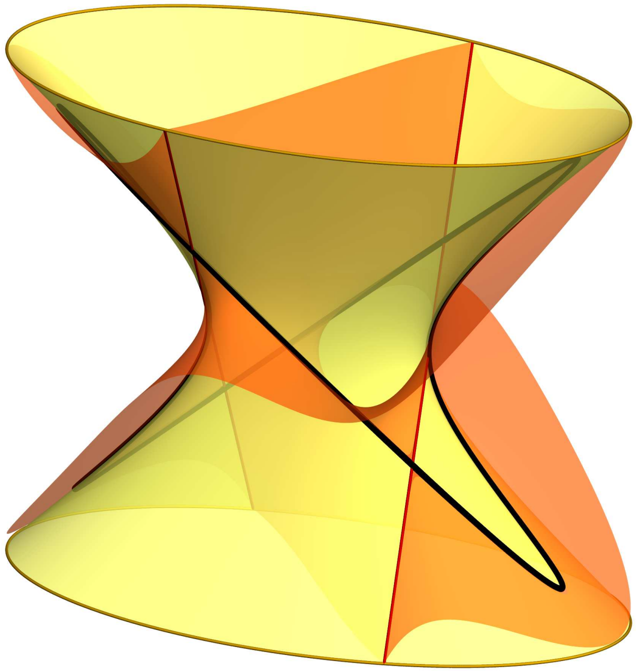
## Geodesics - Perelomov's examples 2

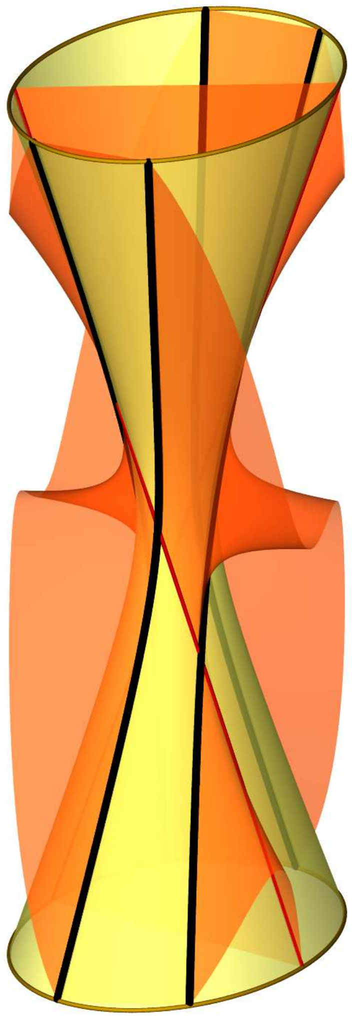
ignota nativitas - aenigma sui temporis

There exist geodesic quartics of the first kind, especially on oval quadrics.



There exist geodesic quartics of the second kind, only on ruled quadrics.





Where do they come from?

## open problems & questions

Fedorov $\longrightarrow$ Perelomov?

Is there a systematic approach to low degee geodesics on quadrics?

correction of the wrong examples in [9, 10]

sparse known results, examples by chance? [3, 7]

## related work


[1] A. von Braunmühl: *Notiz über die geodätischen Linien auf den dreiaxigen Flächen zweiten Grades, die sich durch elliptische Functionen darstellen lassen.* Math. Ann. **26** (1885), 151–153.

[2] M. Chasles: *Les lignes géodésiques et les lignes de courbure des surfaces du second degré.* Journ. de Math. **11** (1846), 5–20.

[3] Y.N. Fedorov: *Algebraic closed geodesics on a triaxial ellipsoid.* Reg. Chaot. Dyn. **10**/4 (2005), 1–23.

[4] G. Glaeser, H. Stachel, B. Odehnal: *The Universe of Quadrics.* Springer Spectrum, Berlin - Heidelberg, 2019, in preparation.

[5] D. Hilbert, S. Cohn-Vossen: *Anschauliche Geometrie.* Springer, Berlin, 1932.

[6] D.G.J. Jacobi: *Note von der geodätischen Linie auf dem Ellipsoid.* Crelle's Journal **19** (1839), 309–313.

[7] H. Knörrer: *Geodesics on the Ellipsoids.* Inventiones math. **59**/2 (1980), 119–143

[8] R. Langenbeck: *Über diejenigen geodätischen Linien auf dem dreiaxigen Ellipsoid,* welche durch einen der Nabelpunkte desselben gehen. Inaug. Diss. Univ. Göttingen, 1877.

[9] A.M. Perelomov: *Some examples of algebraic geodesics on quadrics.* J. Nonlinear Mathem. Physics **16**/1 (2009), 1–5.

[10] A.M. Perelomov: *Some examples of algebraic geodesics on quadrics II.* J. Nonlinear Mathem. Physics **17**/4 (2010), 423–428.

[11] K. Weierstraß: *Über die geodätischen Linien auf dem dreiaxigen Ellipsoid.* Monatsber. Königl. Akad. Wiss., 1861, 257–273 (1861)

Thank You For Your Attention!