19th Scientific-Professional Colloquium on Geometry and Graphics

September 4–8, 2016, Starigrad-Paklenica, HR

On algebraic minimal surfaces

Boris Odehnal

University of Applied Arts Vienna, Austria
di: ˈʌngewʌndtə

Department of Geometry
Abteilung für Geometrie

Contents of this talk

- overview on classical results
- techniques: Björling, Weierstraß, ...
- Enneper, Bour, Henneberg, Richmond
- Möbius type minimal surface, minimal double surfaces
- low degree examples
- open problems

Classical results

[1,4,6,7,8,9,20,22,25,27,28,29,31]

not algebraic: Helicoid, Catenoid, Scherk, ... though related to some algebraic surfaces

Geiser's surface $\mathcal{G}$ and Lie's surface $\mathcal{L}$: lowest possible degree, **but not real!**

$$\mathcal{L}:\ 2(x-iy)^3-6i(x-iy)z-3(x+iy)=0, \quad \mathcal{G}:\ (x-iy)^4+3(x^2+y^2+z^2)=0$$

their duals

$$\mathcal{L}^\star: 27w_0(w_2+iw_1)^2+9i(w_1^2+w_2^2)w_3-4iw_3^3=0, \quad \mathcal{G}^\star: 9w_0^2(w_1-iw_2)^4-(w_1^2+w_2^2+w_3^2)^3=0$$

Their classes and degrees do not follow the rules for real minimal surfaces:

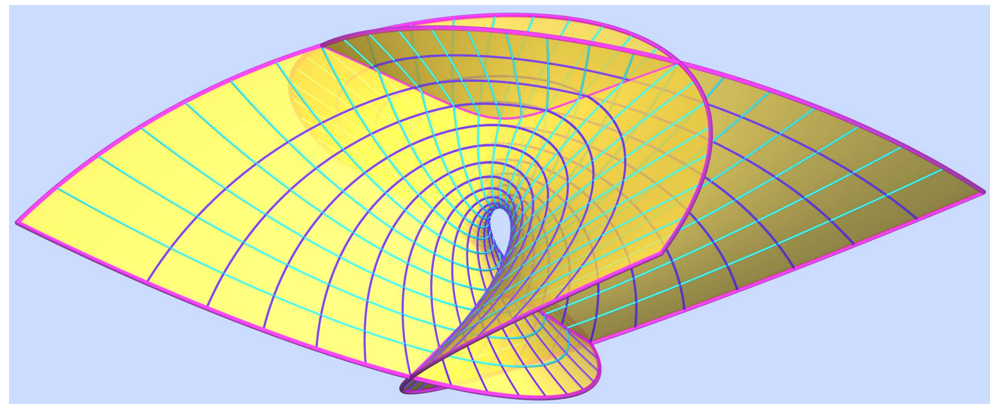
$$\deg \mathcal{L} = \operatorname{cl} \mathcal{L} = \deg \mathcal{L}^\star = 3 \quad \deg \mathcal{G} = 4,\ \operatorname{cl} \mathcal{G} = \deg \mathcal{G}^\star = 6.$$

for real minimal surfaces [22]:

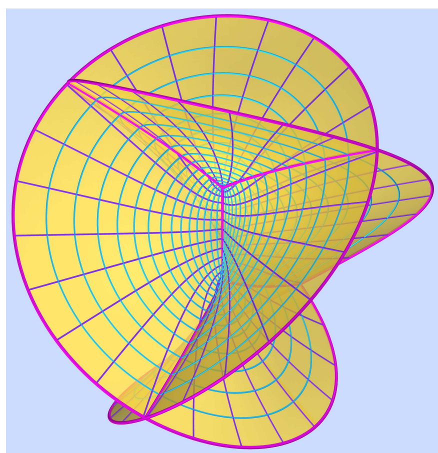
Theorem. The sum of the degree and class of a **real** minimal surface is at least 15.

Classical results

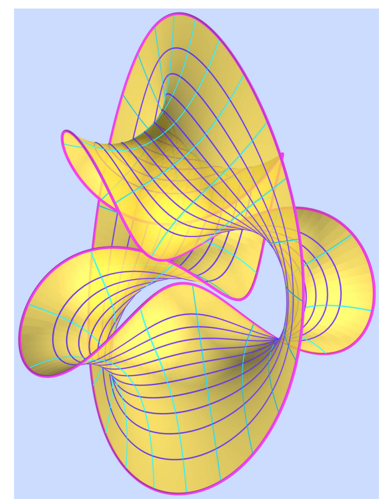
Enneper's surface is minimal in two ways: deg = 9, cl = 6; 6+9=15; and minimal in the differential geometric sense [5,22]



Henneberg's surface is of minimal class (among the real minimal surfaces): cl = 5, deg = 15 [10,11,22]



Richmond's surface: up to equiform transformations the only real minimal surface of deg = 12 = cl [28]



Parametrization techniques - formulae by Weierstraß

[2,16,20,25,32]

$A, B: \ D \subset \mathbb{C} \to \mathbb{C}$ meromorphic functions, $w = u + \mathrm{i}v$ ... complex parameter in D

$$f(u,v) = \mathfrak{Re} \int \begin{pmatrix} A(1-B^2) \\ \mathrm{i}A(1+B^2) \\ 2AB \end{pmatrix} \mathrm{d}w \quad \text{or} \quad f(u,v) = \mathfrak{Re} \int \begin{pmatrix} G^2 - H^2 \\ \mathrm{i}(G^2+H^2) \\ 2GH \end{pmatrix} \mathrm{d}w$$

with $A = G^2$ and $B = HG^{-1}$ ($G \not\equiv 0$) yields parametrizations of minimal surfaces.

Merely inserting algebraic functions does not necessarily result in algebraic minimal surfaces.

Bour's surfaces: $A(w) = cw^{n-2}$, $c \in \mathbb{C} \setminus \{0\}$, $n \in \mathbb{Z} \setminus \{0\}$ and $B(w) = w$

Enneper's surfaces: $A(w) = 1$, $B(w) = w^n$

Richmond's surfaces: $A(w) = \frac{1}{w^2}$, $B(w) = w^{2n}$

Parametrization techniques - formulae by Weierstraß

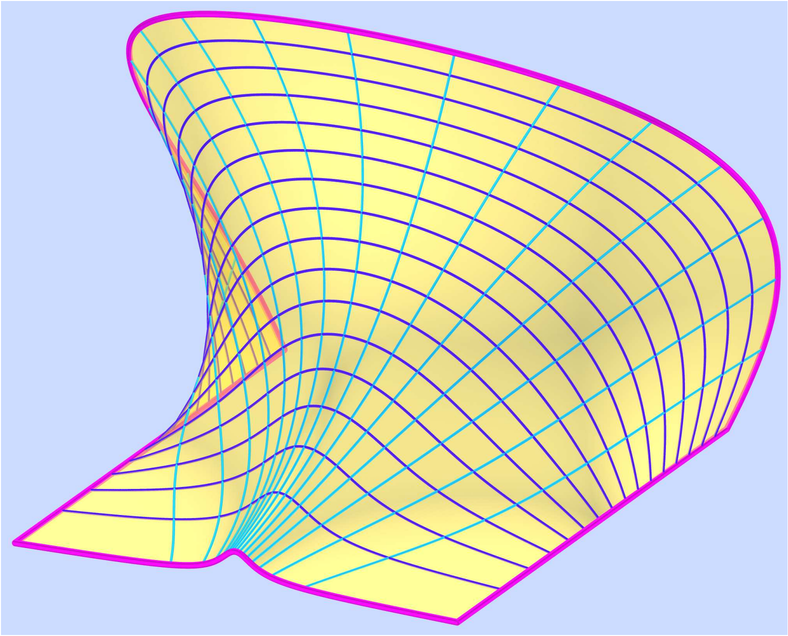
$\mathcal{B}(1,0) = \mathcal{E}(-1)$ ... catenoid, $\mathcal{B}(\mathrm{i},0)$ ... helicoid

$\mathcal{B}(1,2)$, $\mathcal{B}(\mathrm{i},2)$ ... Enneper's surface $\mathcal{E}(1)$ up to equiform transformations

$\mathcal{E}(-2) = \mathcal{R}(1)$

$\mathcal{E}(0)$, $\mathcal{R}(0)$ ... flat

$\mathcal{B}(1,1)$, $\mathcal{B}(\mathrm{i},1)$ ... not algebraic, but worth an inspection



Parametrization techniques - formulae by Weierstraß

[20,22,32]

integral free representation of minimal surfaces

$$f(u,v)=\Re\mathfrak{e}\begin{pmatrix}(1-w^2)A''+2\ wA'-2\ A\\ \mathrm{i}(1+w^2)A''-2\mathrm{i}wA'+2\mathrm{i}A\\ 2wA''-2A'\end{pmatrix}\quad(\star)$$

$A(w): D\subset\mathbb{C}\to\mathbb{C}$ ... meromorphic function

with derivatives $A'=\frac{\mathrm{d}A}{\mathrm{d}w}$, $A''=\frac{\mathrm{d}^2A}{\mathrm{d}w^2}$, $A'''=\frac{\mathrm{d}^3A}{\mathrm{d}w^3}$

$$\mathbf{i}=\begin{pmatrix}1-w^2\\ \mathrm{i}(1+w^2)\\ 2w\end{pmatrix}$$ ... isotropic vector in $\mathbb{R}^3$, *i.e.*, $\langle\mathbf{i},\mathbf{i}\rangle=0$

define $\mathbf{j}=A''\mathbf{i}-A'\mathbf{i}'+A\mathbf{i}''$; elementary to verify: $\langle\mathbf{j}',\mathbf{j}'\rangle=0$, and thus, $\mathbf{j}'$ is isotropic

Therefore, $\Re\mathfrak{e}\mathbf{j}=f$ from $(\star)$ is a real parametrization of a real minimal surface.

Theorem. Each algebraic function $A: D\subset\mathbb{C}\to C$ with $A'''\not\equiv 0$ in D yields an algebraic minimal surface parametrized by $(\star)$.

Parametrization techniques - Björling's Formula

$\gamma: \ I \subset \mathbb{R} \to \mathbb{R}^3$ ...spine curve, $\nu: \ I \to \mathrm{S}^1$ ...unit normal vector field along γ

Both are considered to have a complex continuation.

$\varphi(t) = \gamma - \mathrm{i} \int_{t_0}^{t} \nu(\tau) \times \mathrm{d}\gamma(\tau)$ ...isotropic curve, *i.e.*, a curve of constant slope $\pm\mathrm{i}$

Complex continuation: Let $t = u + \mathrm{i}v$ and then

$$f(u, v) = \mathfrak{Re}(\varphi)$$

is a real parametrization of the minimal surface on the scroll (γ, ν).

$$f^{\perp}(u, v) := \mathfrak{Im}(\varphi)$$

is the adjoint minimal surface to f.

Associate family f_τ of minimal surfaces

$$f_\tau = \mathfrak{Re}(\mathrm{e}^{\mathrm{i}\tau}\varphi(t)), \qquad f = f_0,\ f^{\perp} = f_{\frac{\pi}{2}}$$

Theorem. The associate minimal surfaces to an algebraic minimal surface are algebraic.

Parametrization techniques - Björling's Formula

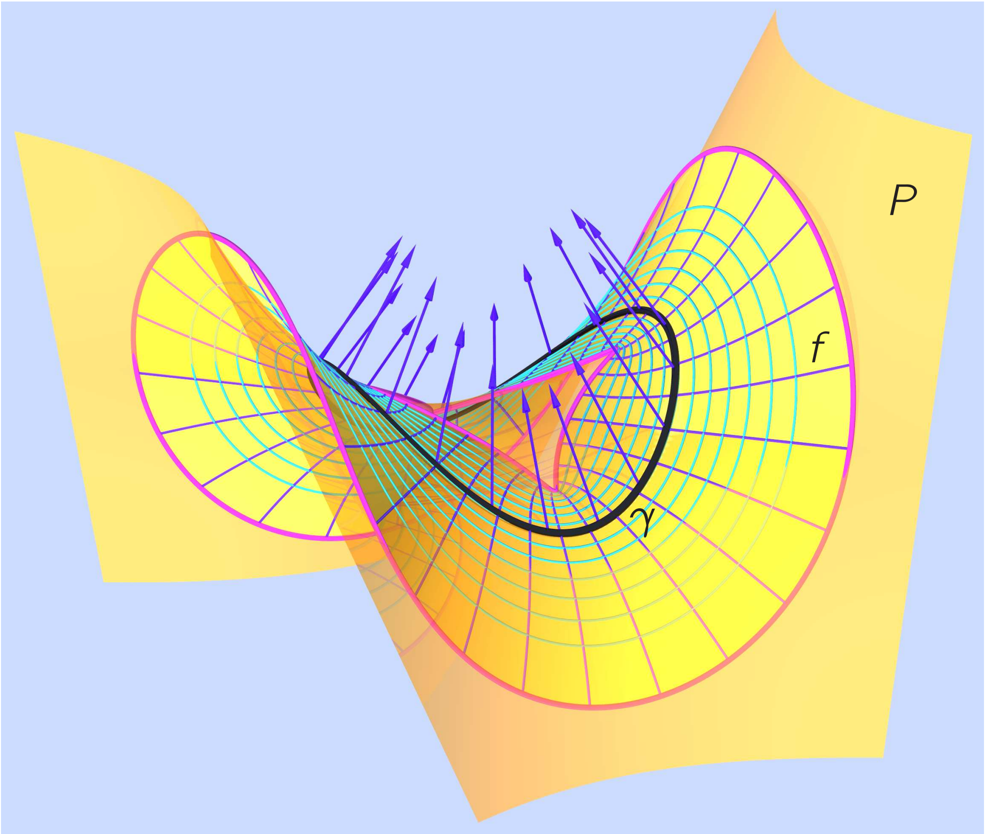
Theorem. Let γ be the evolute of an algebraic curve contained in an algebraic cylinder Z. The minimal surface tangent to Z along γ is algebraic. [12,20,22]

This theorem gives only a few exmaples, but:

Replace Z by an orthogonal hyperbolic paraboloid $P:(1-b^2)xy=2bz$, $b\neq\pm1$ and $\gamma=P\cap Z$ with $Z:x^2+y^2=1$.

Such curves γ are curves of constant Gaussian curvature on P.

Theorem. The minimal surfaces f that touch an orthogonal hyperbolic paraboloid P along the curves of constant Gaussian curvature are rational (and thus algebraic) minimal surfaces of degree 30 with the parametrization . . .



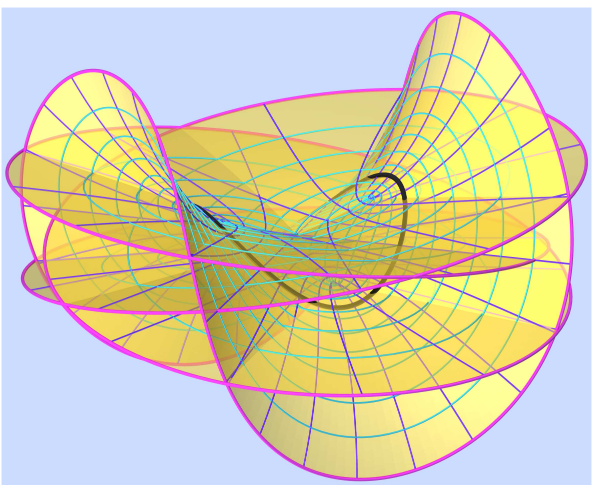
Parametrization techniques - Björling's Formula

$$\ldots f(u,v) = \frac{1}{12b\beta_1}\begin{pmatrix} \beta_2^2 \mathrm{c}_{3u}\mathrm{S}_{3v} + 3\mathrm{c}_u(\beta_3\mathrm{S}_v + 4b\beta_1\mathrm{C}_v) \\ -\beta_2^2 \mathrm{s}_{3u}\mathrm{S}_{3v} + 3\mathrm{s}_u(\beta_3\mathrm{S}_v + 4b\beta_1\mathrm{C}_v) \\ 3\beta_2\mathrm{s}_{2u}(\beta_1\mathrm{C}_{2v} + 2b\mathrm{S}_{2v}) \end{pmatrix} \text{ with}$$

$\beta_1 := 1 + b^2$, $\beta_2 := 1 - b^2$, $\beta_3 := b^4 + 6b^2 + 1$.

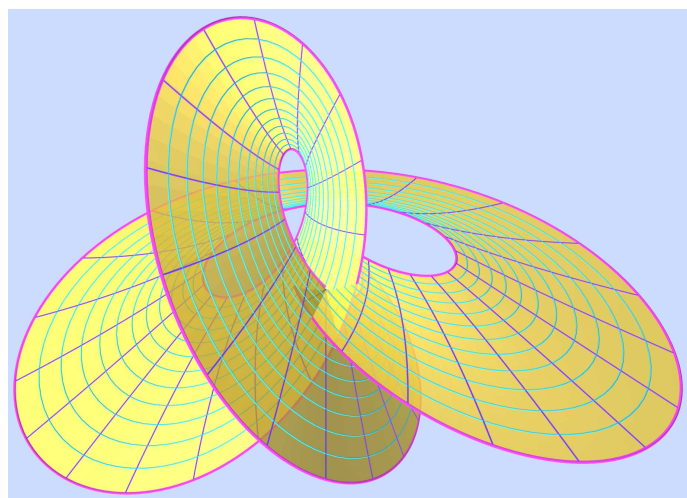
For a curve to a prescribed Gaussian curvature $K < K_{\max} = -\frac{\beta_2^2}{16b^4}$,

the radius r of Z fullfils $$r^2 = \frac{1}{\beta_2\sqrt{-K}} - \frac{4b^2}{\beta_2^2}.$$



$\longleftarrow f$

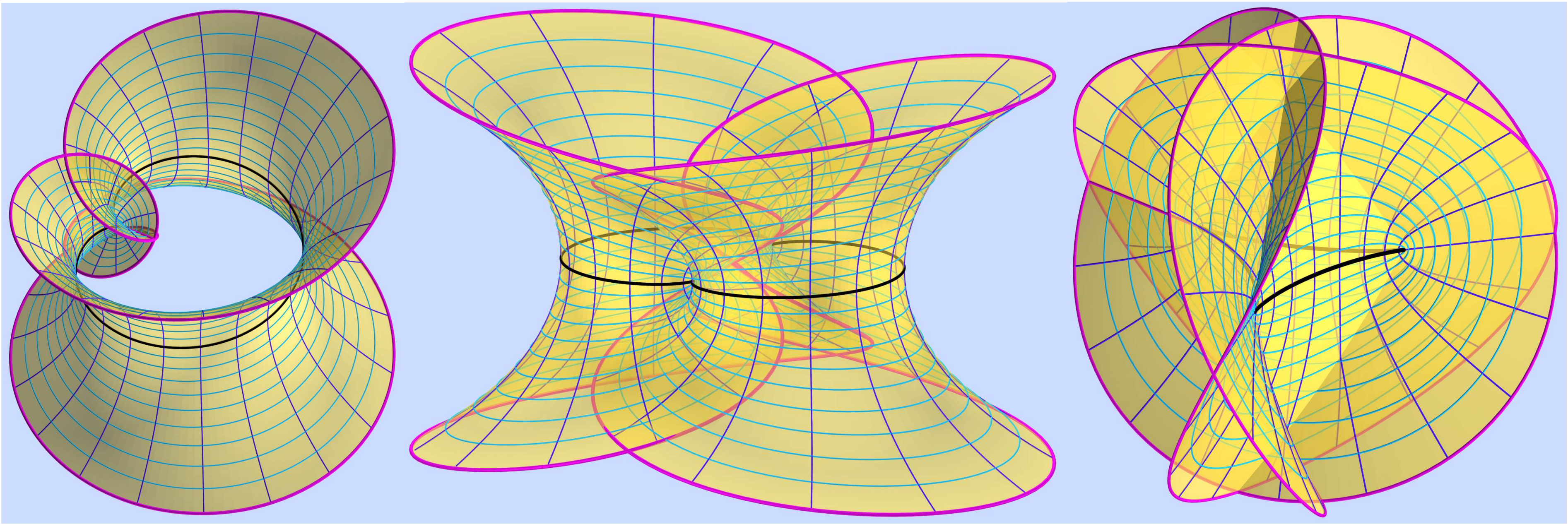
$f^{\perp} \longrightarrow$



Parametrization techniques - Björling's Formula

Theorem. Let γ, ν_γ be a planar algebraic curve and its unit normal vector field. The minimal surface S on the scroll (γ, ν_γ) is algebraic, γ is a planar geodesic on the minimal surface, and the plane of γ is a plane of symmetry of S. [20]

This leads to cycloidal minimal surfaces (apparently new):



Not to be mixed up with Catalan's surface which is not algebraic!

Parametrization techniques - Björling's Formula - cycloidal minimal surfaces

Theorem. Let $r, R \in \mathbb{R} \setminus \{0\}$ be real constants with $R + 2r \neq 0$ and $R + r \neq 0$.
The minimal surfaces on the scroll (ζ, ν) with $\zeta \subset \pi_3$ and $\nu \in \mathrm{S}^1$

$$\zeta(t) = \begin{pmatrix} (R+r)\mathrm{c}_t + r\mathrm{c}_{\frac{(R+r)t}{r}} \\ (R+r)\mathrm{s}_t + r\mathrm{s}_{\frac{(R+r)t}{r}} \\ 0 \end{pmatrix}, \quad \nu(t) = \frac{1}{2\mathrm{c}_{\frac{Rt}{2r}}} \begin{pmatrix} -\mathrm{c}_t - \mathrm{c}_{\frac{(R+r)t}{r}} \\ -\mathrm{s}_t - \mathrm{s}_{\frac{(R+r)t}{r}} \\ 0 \end{pmatrix}$$

can be parametrized by

$$f(u,v) = \begin{pmatrix} (R+r)\mathrm{c}_u \mathrm{C}_v + r\mathrm{c}_{\frac{(R+r)u}{r}} \mathrm{C}_{\frac{(R+r)v}{r}} \\ (R+r)\mathrm{s}_u \mathrm{C}_v + r\mathrm{s}_{\frac{(R+r)u}{r}} \mathrm{C}_{\frac{(R+r)v}{r}} \\ -\frac{4r(R+r)}{R}\mathrm{c}_{\frac{Ru}{2r}} \mathrm{S}_{\frac{Rv}{2r}} \end{pmatrix}.$$

These minimal surfaces are algebraic, rational, and closed if, and only if, $R, r \in \mathbb{Q} \setminus \{0\}$.
In any case, the cycloid $\zeta \subset \pi_3$ is a geodesic on the minimal surface.
The surfaces with $R, r \in \mathbb{Q} \setminus \{0\}$ contain at least one straight line.

Cycloidal minimal surfaces & curves of constant slope

[3,5,13,15,26,36]

The cycloid ζ is of constant slope ($\sigma = 0$):

Theorem.

Moving trough the associate family bends the curve ζ smootly into curves of constant slope on the quadrics

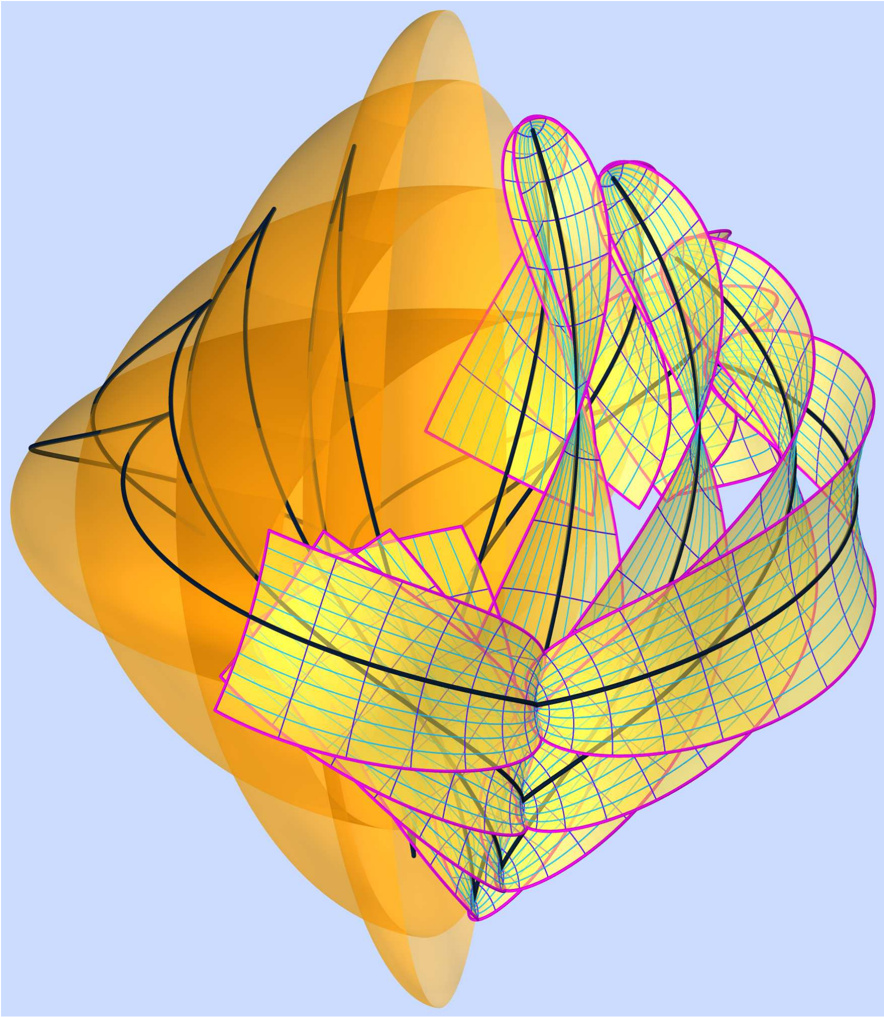
$$Q: x^2+y^2+\frac{R^2 \mathrm{ctg}^2 \tau}{4r(r+R)} z^2 = (2r+R)^2 \cos^2 \tau.$$

The slope angle σ is independent of R and r and is related to τ by

$$\cos\sigma = -\sin\tau \Longleftrightarrow \sigma = \tau + \frac{\pi}{2} \mod 2\pi.$$

Theorem.

The u-lines on the cycloidal minimal surfaces are generalized oscillation curves. [26]

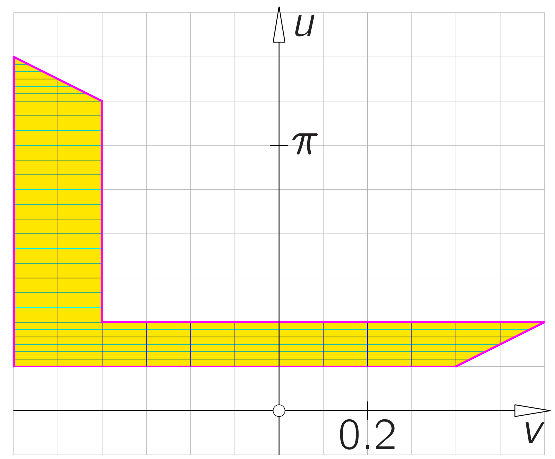


Minimal Möbius strips

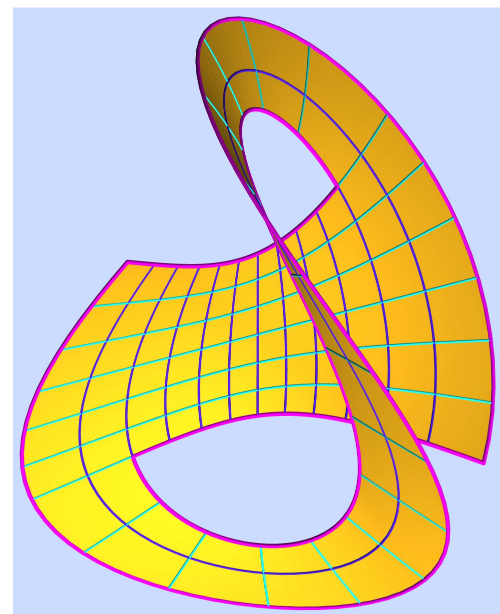
Henneberg's surface is not orientable!

minimal double surface

(cf. [10,11,12])

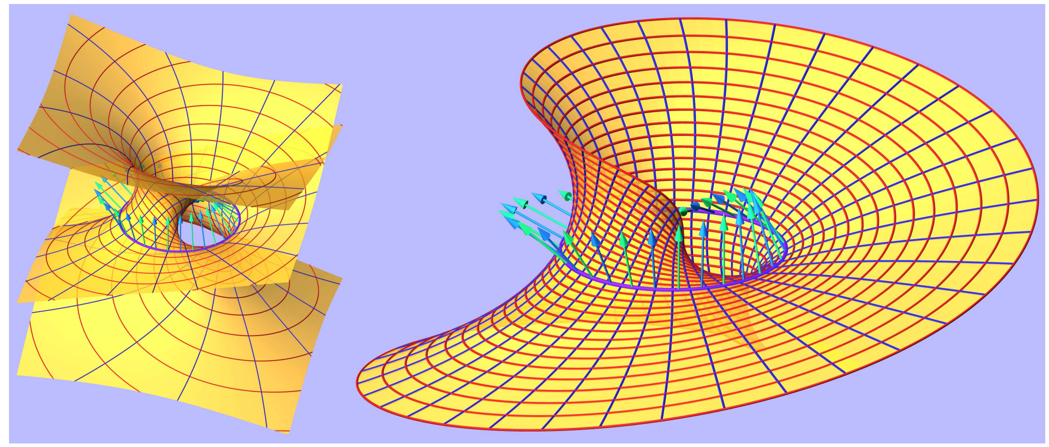


Mapping an L-shaped domain in the parameter plane to the surface yields a Möbius strip on Henneberg's surface. (cf. [22])

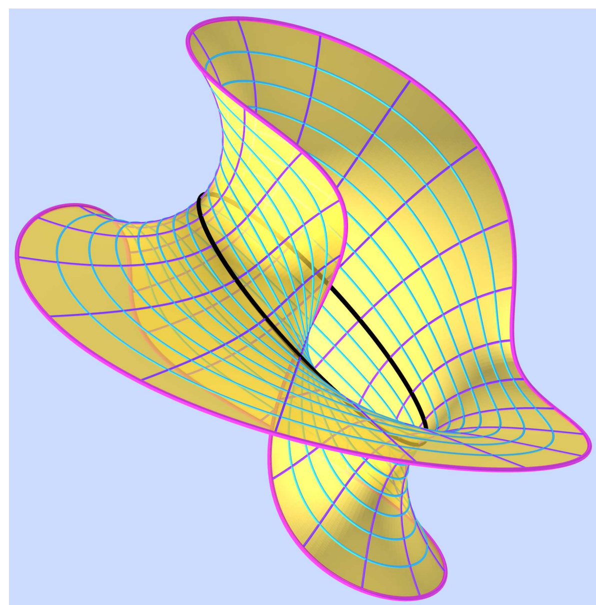


Minimal Möbius strips - via Björling

$\gamma(t) = (\cos t, \sin t, 0)$,
$\nu(t) = (\cos t \cos \frac{t}{2} \sin t \cos \frac{t}{2}, \sin \frac{t}{2})$
yields a rational minimal surface
of degree 11, class 22, and genus 1.
details: see [23]
tangent to Krames's surface cf. [18]
rotoidal cf. [13,23]



$\gamma(t) = (\cos t, \sin t, \sin t)$,
$\nu(t) = (\cos t \cos \frac{t}{2} \sin t \cos \frac{t}{2}, \sin \frac{t}{2})$
yields a rational minimal surface
of degree 35, class $\approx$70, and genus 1.



Some remarks

- equation ($\star$) with $A = \sqrt{w} \implies f$ is algebraic, degree ?, implicit equation ??
- relations to isotropic congruences of lines: central envelope is minimal, rational, ... [14,17,24,27,30]
- degrees grow rapidly:
  $\deg \mathcal{B}(c, n) = (n+1)^2$, $\deg \mathcal{E}(n) = (2n+1)^2$, $\deg \mathcal{R}(n) = 2n(n+1)$
- Parmetrization formulae involving integrals should be handled with care.
  Antiderivatives of algebraic functions need not be algebraic.
- Only Björling's formula with geometric ideas leads to low degree examples.
  Proper curves may be found in [19,21,33–36].

Thank You For Your Attention!

Related work

[1] G. Andrews: *The primitive double minimal surface of the seventh class and its conjugate.* Michigan Historical Reprint Series, Michigan, 2005.

[2] W. Blaschke: *Vorlesungen über Differential Geometrie und geometrische Grundlagen von Einsteins Relativitätstheorie.* Springer, Berlin, 1921.

[3] W. Blaschke: *Bemerkungen über allgemeine Schraublinien.* Mh. Math. Phys. **19** (1908), 188–204.

[4] E. Bour: *Théorie de la déformationi des surfaces.* J. Êcole Imp. Polytech. **22** (1862), cahier 39, 1–148.

[5] A. Enneper: *Über Loxodromen.* Ber. Sächs. Ges. Wiss. Leipzig, 1902.

[6] A.S. Gale: *On the rank, order and class of algebraic minimum curves.* Yale University, Yale, 1902.

[7] C.F. Geiser: *Zur Erzeugung von Minimalflächen durch Scharen von Kurven vorgegebener Art.* Sb. Königl. Preuss. Akad. Wiss. Berlin, phys.-math. Classe, 1904, 677–686.

[8] C.F. Geiser: *Notiz über die algebraischen Minimalflächen.* Math. Ann. **3** (1871), 530–534.

[9] L. Henneberg: *Determination of the lowest genus of the algebraic minimum surfaces.* Brioschi Ann. (2) **IX** (1878), 54–57.

[10] L. Henneberg: Über diejenige Minimalfläche, welche die Neilsche Parabel zur ebenen geodätischen Linie hat. Wolf Z. XXI. 17-21 (1876).

[11] L. Henneberg: *Über solche Minimalflächen, welche eine vorgeschriebene ebene Curve zur geodätischen Linie haben.* PhD thesis, Zrich, 1876.

[12] L. Henneberg: *Über die Evoluten der ebenen algebraischen Curven.* Wolf Z, XXI. 22-23 (1876).

[13] F. Hohenberg: *Über die Zusammensetzung zweier gleichförmiger Schraubungen.* Mh. Math. **54** (1950), 221–234.

[14] J. Hoschek: *Liniengeometrie.* Bibliographisches Institut, Zürich, 1971.

[15] M. Huth: *Kurven konstanter Steigung auf Flächen.* Jber. Realschule zu Stollberg, 1983, 22 p.

[16] H. Karcher: *Construction of minimal surfaces.* In: *Surveys in Geometry*, Univ. of Tokyo, 1989, Lecture Notes No. 12, SFB 256, Bonn, 1989.

[17] R. Koch: *Über die Mittelfläche einer isotropen Geradenkongruenz.* J. Geom. **23** (1984), 152–169.

[18] J. Krames: *Die Regelfläche dritter Ordnung, deren Striktionslinie eine Ellipse ist.* Sb. Akad. Wiss. Wien **127** (1918), 1–22.

Related work

[19] J.D. Lawrence: *A Catalog of Special Plane Curves.* Dover Publications, New York, 1972.

[20] S. Lie: *Gesammelte Abhandlungen.* Friedrich Engel, Poul Heegaard, eds., B.G. Teubner, Leipzig, 1922–1960.

[21] G. Loria: *Spezielle algebraische und transcendente ebene Kurven.* B.G. Teubner, Leipzig, 1911.

[22] J.C.C. Nitsche: *Vorlesungen über Minimalflächen.* Springer-Verlag, 1975.

[23] B. Odehnal: *A rational minimal Möbius strip.* Proc. 17^{th} ICGG, August 4–8, 2016, Beijing, P.R. China, article no. 070.

[24] B. Odehnal: *On rational isotropic congruences of lines.* J. Geom. **81** (2005), 126–138.

[25] R. Osserman: *A Survey of Minimal Surfaces.* Dover Publications, New York, 2nd edition, 1986.

[26] H. Pottmann: *Zur Geometrie höherer Planetenumschwungbewegungen.* Mh. Math. **97** (1984), 141–156.

[27] A. Ribaucour: *Étude des Élassoïdes ou Surfaces A Courbure Moyenne Nulle.* Brux., Mém. cour. in **XLV**/4, 1882.

[28] H.W. Richmond: *On the simplest algebraic minimal curves, and the derived real minimal surfaces.* Trans. Cambridge Phil. Soc. **19** (1904), 69–82.

[29] H.A. Schwarz: *Gesammelte mathematische Abhandlungen.* Springer, Berlin, 1890.

[30] N.K. Stephanidis: *Minimalfllächen und Strahlensysteme.* Arch. Math. **41** (1983), 544–554.

[31] E. Study: *Über einige imaginäre Minimalflächen.* Ber. königl. sächs. Ges. Wiss., math.-phys. Kl. **LXIII** (1911), 14–26.

[32] K. Weierstraß: *Mathematische Werke.* Bd. 3, Berlin, 1903.

[33] H. Wieleitner: *Spezielle Ebene Kurven.* G.J. Göschen'sche Verlagshandlung, Leipzig, 1908.

[34] E. Wölffing: *Über Pseudotrochoiden.* Zeitschr. f. Math. u. Phys. **44** (1899), 139–166.

[35] W. Wunderlich: *Ebene Kinematik.* Bibliograph. Inst. Mannheim, 1970.

[36] W. Wunderlich: *Höhere Radlinien.* Österr. Ingen. Archiv **1** (1947), 277–296.