Distances and central Projections

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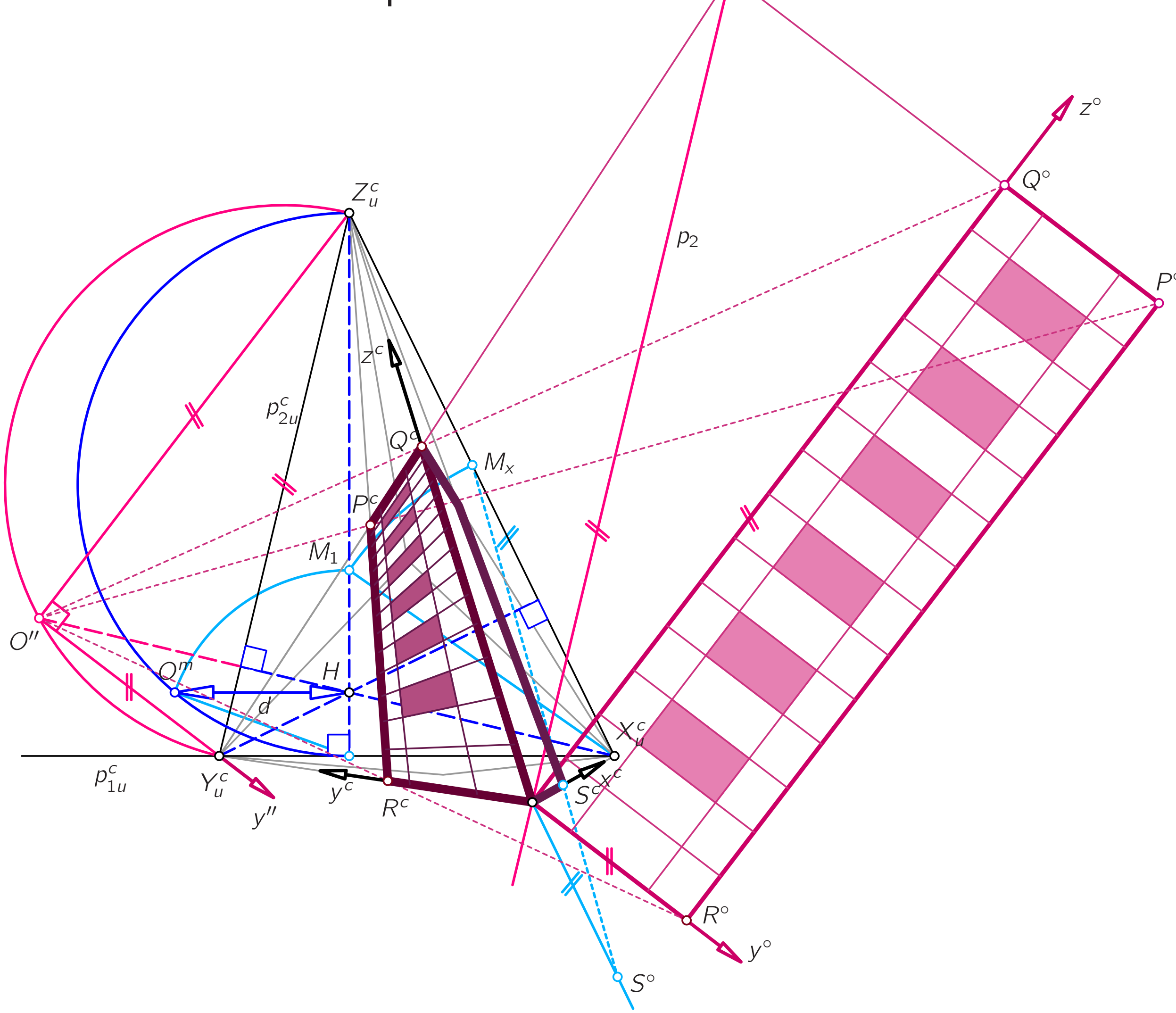
Outline

motivation	—	Why do we measure distances in central projections?
principal lines	—	parallel projection, central projection
constructive approach	—	a quartic surface point by point
algebraic properties	—	singular points and lines
special cases	—	cubic surface, surface of revolution

Motivation

We want to avoid complicated constructions like this:



But we cannot!

And still there is this naive question:

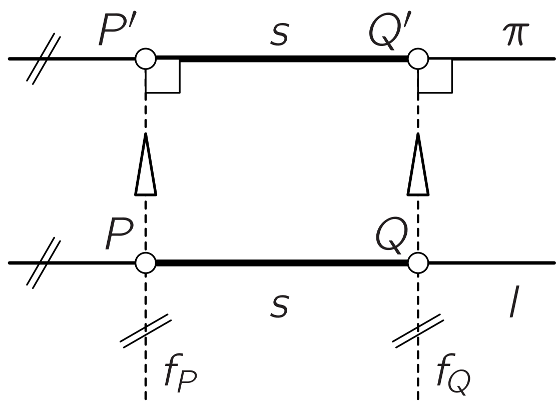
Isn't there a line where we can measure directly on its image and get correct results?

Principal lines: orthogonal and oblique parallel projection

orthogonal projection:

Principal lines are parallel to the image plane.

They are mapped congruent onto their images.

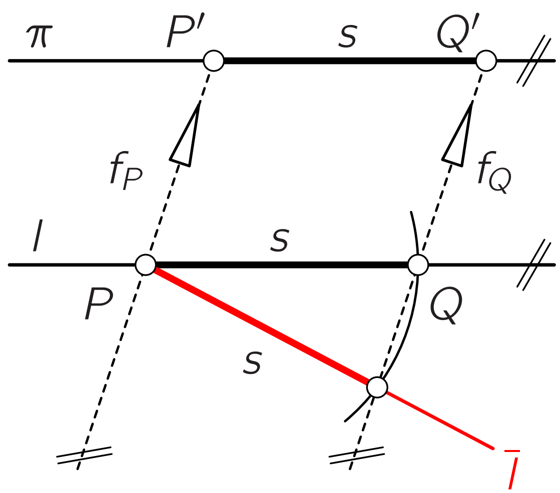


oblique parallel projection:

Principal lines are parallel to the image plane.

They are mapped congruent onto their images.

There is a second kind of principal line $\bar{l}$ which is not parallel to l (also mapped congruent onto its image, not parallel to π)!



Principal lines: central projection

central projection $\kappa : \mathbb{R}^{3\star} \to \pi \cong \mathbb{R}^2$:

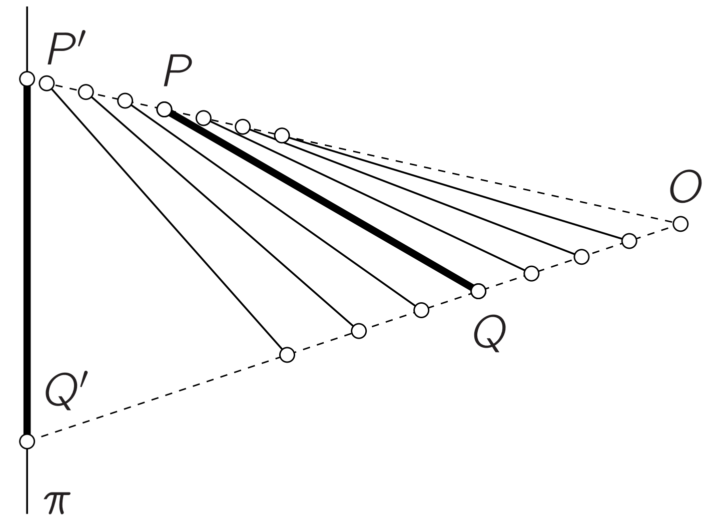
$O \notin \pi$ .... eye point,

π .... image plane,

$\mathbb{R}^{3\star} := \mathbb{R}^3 \setminus \{O\}$, $P' := \kappa(P)$, $Q' := \kappa(Q)$

For any $P \in [O, P']$ we find at least one $Q \in [O, Q']$ such that

$$\overline{P'Q'} = \overline{PQ}.$$



We find two Qs as long as $\overline{P[O,Q]} < \overline{P'Q'}$.

Any such line $[P, Q]$ can be called a principal line, but it is not mapped congruent onto its image!

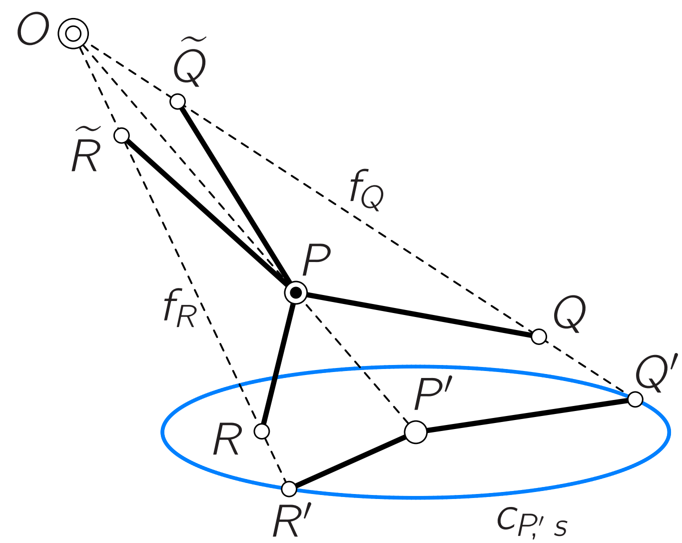
Usualy, principal lines of a central projection are parallel to π and mapped similar onto their images.

Constructive approach

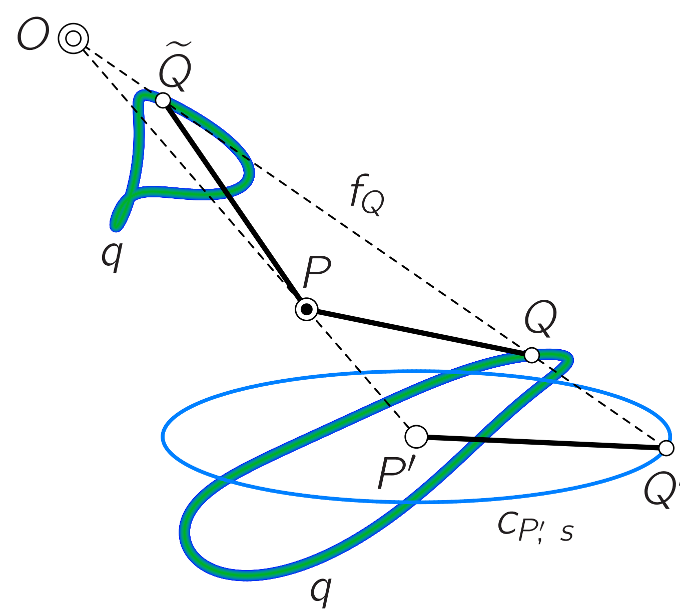
assume $P \in \mathbb{R}^{3\star}$ fixed with image P'

All points $Q' \in \pi$ with $\overline{P'Q'} = s \in \mathbb{R}^+$ (fixed s) gather on a circle $c_{P',s} \subset \pi$.

Endpoints Q of segments PQ with $\overline{PQ} = s$ are located on a Euclidean sphere $\Sigma_{P,s}$ (centered at P, radius s).

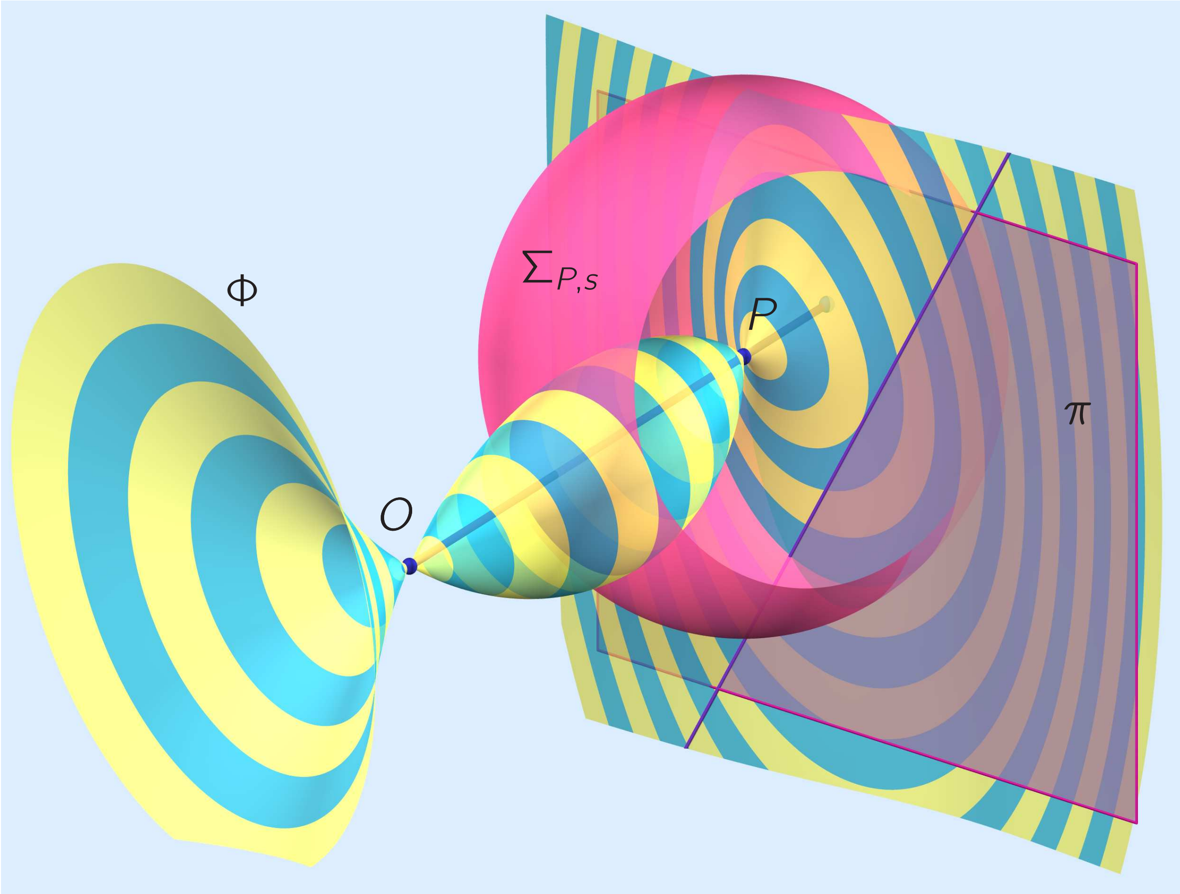


$\Longrightarrow$ The endpoints Q of segments PQ with $\overline{PQ} = s$ such that $\overline{P'Q'} = s$ trace a quartic curve q which is the intersection of the quadratic cone $\Gamma_{P',s} = c_{P',s} \vee O$ and the sphere $\Sigma_{P,s}$.



Constructive approach

s varies (linearly) in $\mathbb{R}^+ \Longrightarrow$
linear family of quartic curves
$q(s) \Longrightarrow$



Theorem:

The set of all endpoints Q of segments PQ that satisfy $\overline{PQ} = \overline{P'Q'}$ form a quartic surface Φ passing through P.

Algebraic equation of Φ

some geometric objects and their equations

$O \ldots \mathbf{o} = (d, 0, 0)^{\mathsf{T}}$ with $d \in \mathbb{R}^{+}$ . . . . eye point

$\pi \ldots x = 0$ . . . . image plane

$\pi_V \ldots x = d$ . . . . vanishing plane

$P \ldots \mathbf{p} = (\xi, \eta, \zeta)^{\mathsf{T}}$ with $\xi \neq d$ . . . . "the point"

$Q \ldots \mathbf{x} = (x, y, z)^{\mathsf{T}}$ with $x \neq d$ . . . . endpoints

$\Lambda_P \ldots (x-\xi)^2+(y-\eta)^2+(z-\zeta)^2=0$ . . . isotropic cone at P

$P' = \kappa(P) = [O, P] \cap \pi$ . . . . κ-image of P

$$\kappa : \ (x, y, z)^{\mathsf{T}} \mapsto \left(\frac{dy}{d-x}, \frac{dz}{d-x} \right)^{\mathsf{T}}$$

Algebraic equation of Φ

looking for points Q with $\overline{PQ} = \overline{P'Q'}$ (for fixed P) yields

$$\begin{aligned}\Phi:\ d^2\left((y(d-\xi)-\eta(d-x))^2+(z(d-\xi)-\zeta(d-x))^2\right) &= \\ = (d-\xi)^2(d-x)^2\left((x-\xi)^2+(y-\eta)^2+(z-\zeta)^2\right)\end{aligned}$$

or, in terns of geometric objects

$$d^2\left(\|\pi_V\mathbf{p}-(d-\xi)\mathbf{x}\|^2-((d-\xi)\pi-\xi\pi_V)^2\right) = (d-\xi)^2\pi_V^2\Lambda_P$$

Theorem (once again):

Φ is a quartic surface passing through O and P.

Algebraic properties of Φ

Theorem:

1. Φ is uni-circular and has π's ideal line p_2 for a double line.
2. planes $x = k$ $(\neq 0, d, \xi, 2d) \cap \Phi$: circles + two-fold ideal line p_2
3. planes $x = 2d$ / $\pi \cap \Phi$: proper lines $m \parallel l$ + three-fold line p_2
4. planes π_v / $x = \xi \cap \Phi$: isotropic lines through O / P + two-fold line p_2.

Proof:

1. homogenize equation of Φ by substituting $x \to X_1X_0^{-1}$, $y \to X_2X_0^{-1}$ $z \to X_3X_0^{-1}$ and insert $X_0 = 0 \Longrightarrow$ ideal curve ϕ: $X_1^2(X_1^2 + X_2^2 + X_3^2) = 0$.
   $\Longrightarrow X_1 = 0$ . . . . π's ideal line (multiplicity 2),
   $X_1^2 + X_2^2 + X_3^2 = 0$ . . . . . . absolute conic of Eucl. geometry (multiplicity 1).
2. $x = k$, $k \in \mathbb{R} \setminus \{0, d, \xi, 2d\}$ are parallel to π:

   substitute $x = k$ into Φ's equation $\Longrightarrow$ equation of circles, centered at

$$\left(k, \frac{\eta(d-k)(d\xi + dk - k\xi)}{(k(d-\xi)(2d-k)}, \frac{\zeta(d-k)(d\xi + dk - k\xi)}{(k(d-\xi)(2d-k)}\right)^{\mathrm{T}}$$

Algebraic properties of Φ

$\Longrightarrow$ Curve of centers is a rational cubic space curve.

3. insert $x = 2d$ / $x = 0$ into Φ's equation:

$$
\begin{aligned}
l: &\quad 2(\xi - d)(\eta y + \zeta z) = (\xi - 2d)\|\mathbf{p}\|^2 + d^2\xi, \\
m: &\quad 2(\xi - d)(\eta y + \zeta z) = \xi\|\mathbf{p}\|^2 - d(2d^2 - 5d\xi + 4\xi^2),
\end{aligned}
$$

   obviously parallel, from Φ's homogeneous equation, X_1^3 splits off.

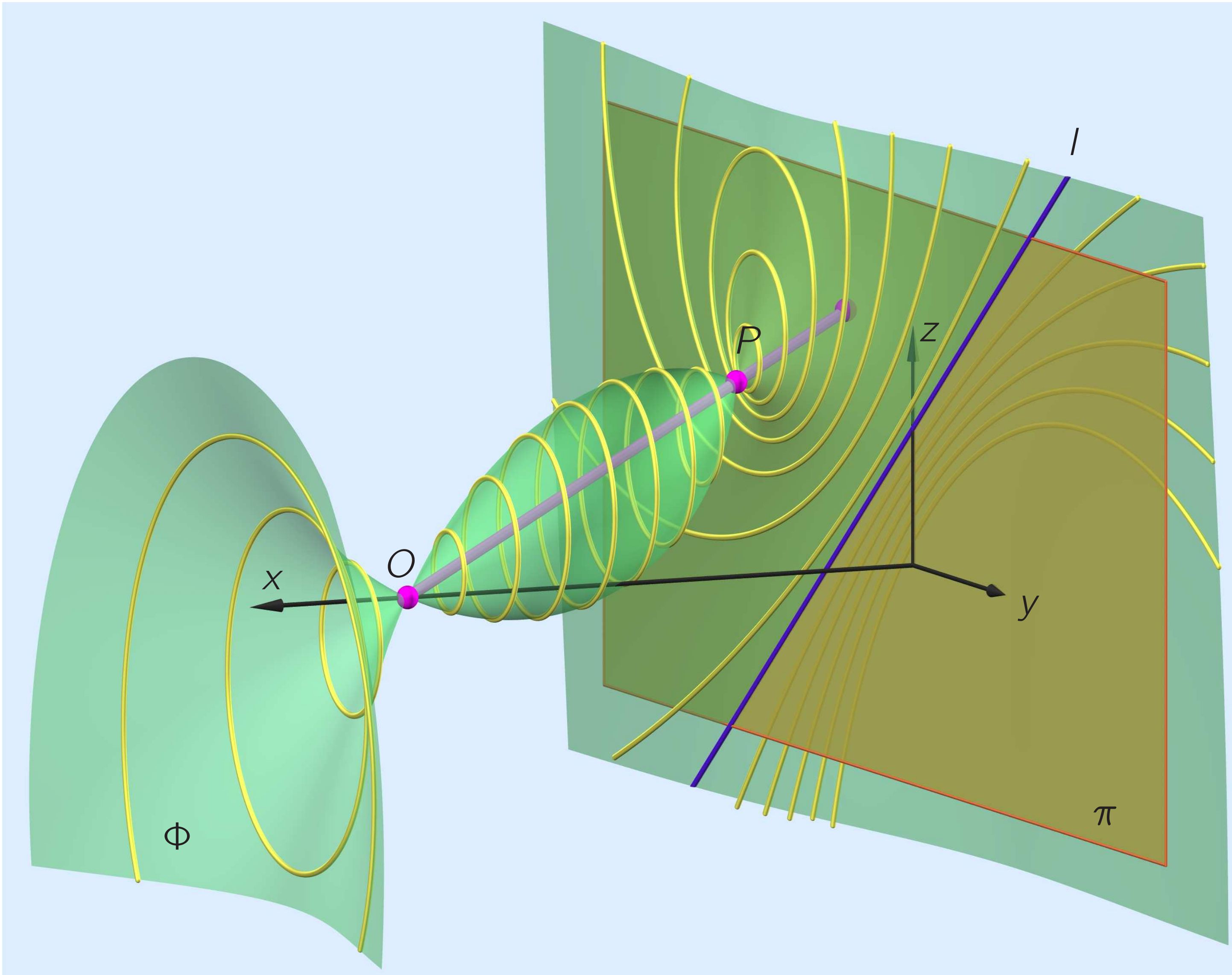
4. insert $x = d$ / $x = \xi$ into Φ's equation:

   besides some non-vanishing factors, we find

   $y^2+z^2=0$ / $(y-\eta)^2+(z-\zeta)^2=0$ . . . . two pairs of isotropic lines

   from Φ's homogeneous equation, X_1^2 splits off.

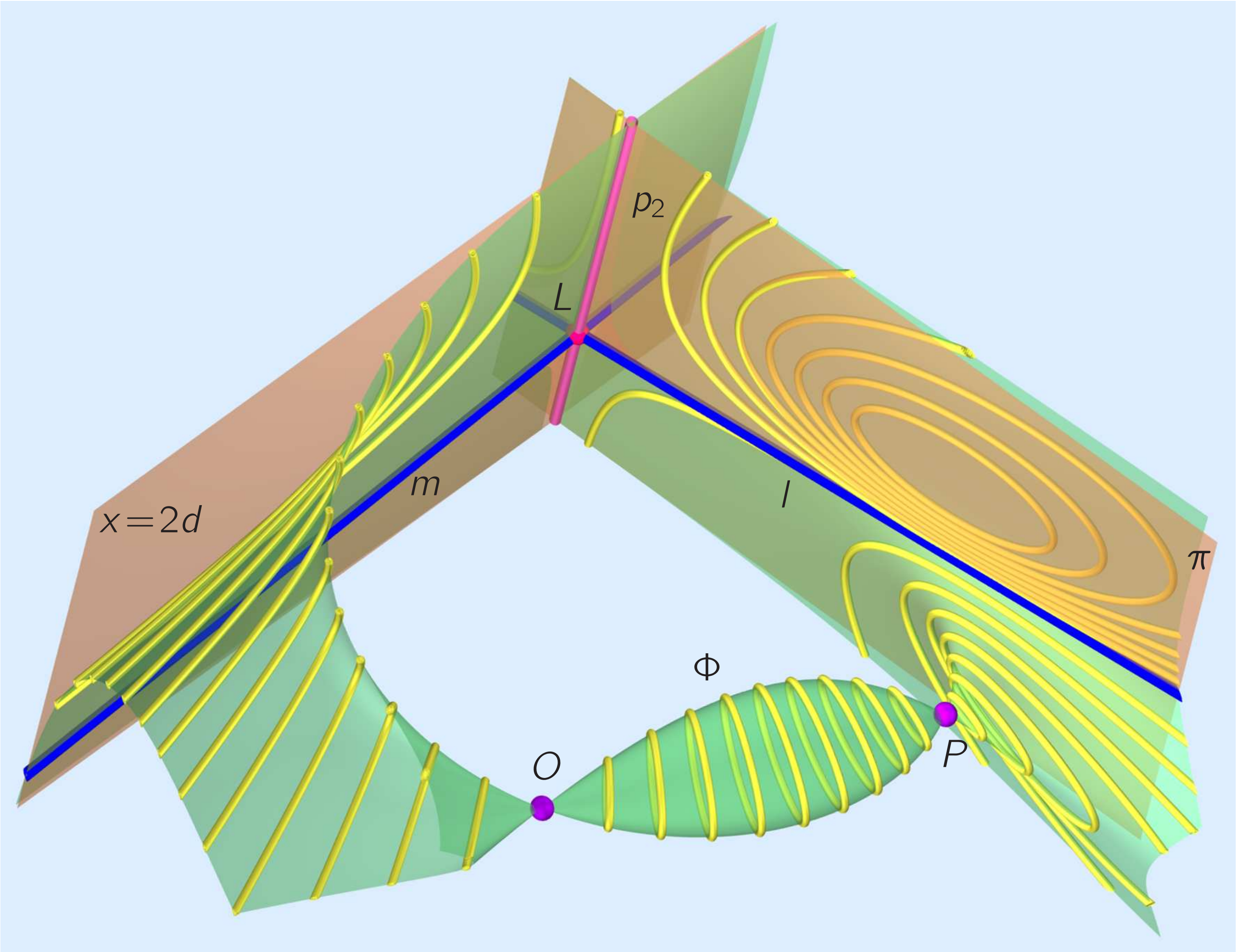
Algebraic properties of Φ



The quartic surface Φ with its circles in planes parallel to π and the line l.

Algebraic properties of Φ

The ideal line p_2 of π is a part of the double curve of Φ. The two planes π and $x = 2d$ serve as tangent planes of Φ along p_2 and meet Φ along p_2 (with multiplicity 3) and l and m appear as the remaining linear part.



Algebraic properties of Φ

Theorem:

O and P are conical singularities on Φ.

Proof:

Singularities at O / P $\Longleftrightarrow$ gradient of Φ vanishes at O / P.

Conical: (for sake of simplicity) translate Φ s.t. $O \mapsto (0,0,0)^{\mathsf{T}}$ / $P \mapsto (0,0,0)^{\mathsf{T}}$

extract coefficient of X_0^2:

$$\Gamma_O : d^2(\xi - d)^2\|\mathbf{x}\|^2 + 2d^2(d-\xi)x(\eta y + \zeta z) = ((d-\xi)^4 + \xi(2d+\xi)\|\mathbf{p}\|^2 + \xi^3(2d-\xi))x^2$$

$$\Gamma_P : \xi(-2d)(\xi - d)^2\|\mathbf{x}\|^2 + 2d^2(\xi - d)x((\eta y + \zeta z) = d^2((d-\xi)^2 - \|\mathbf{p}\|^2 + 2\xi^2)x^2$$

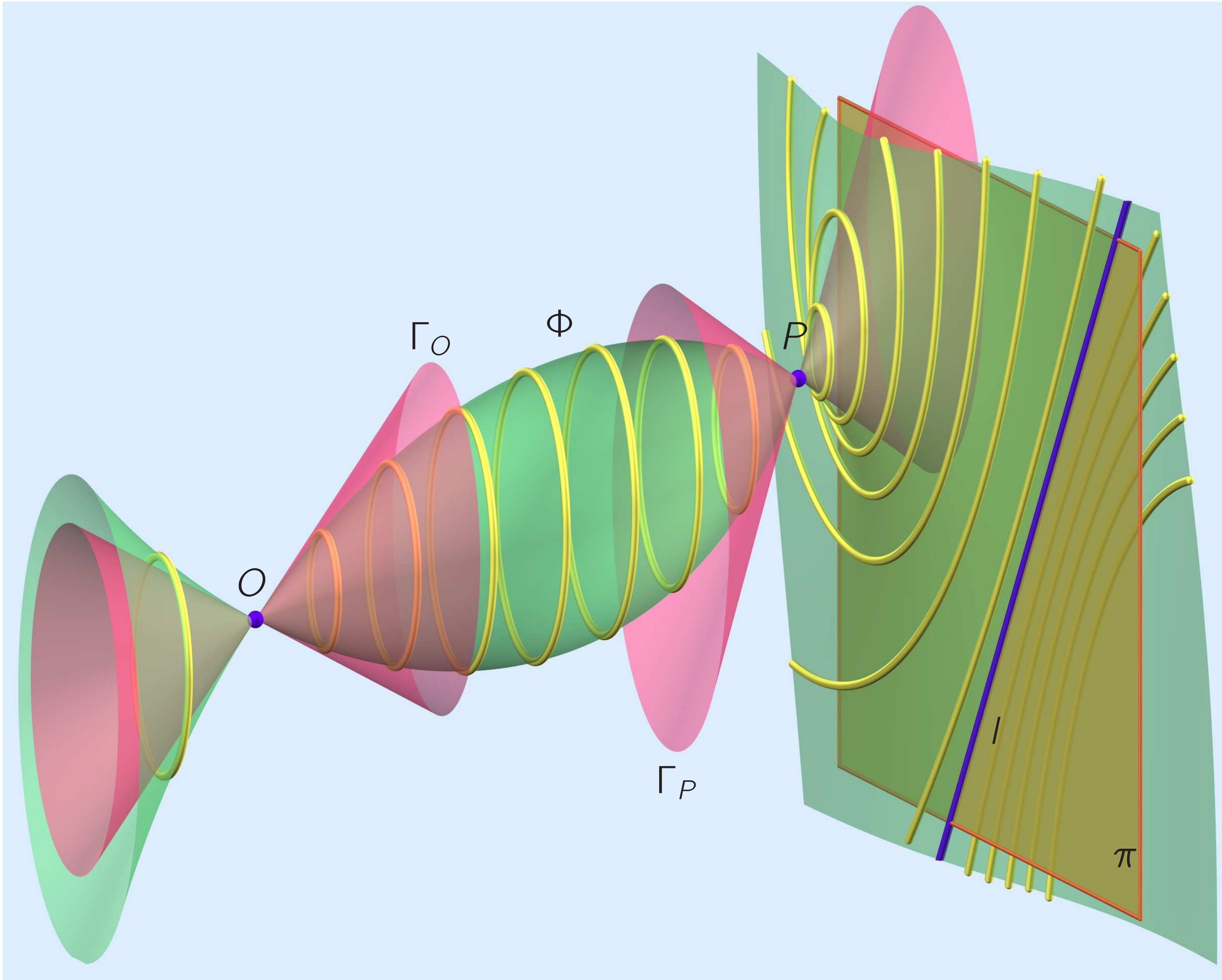
Γ_O / Γ_P ... quadratic cones, second order approximation of Φ at O / P

Computation of $\Gamma_\cdot$ similar to computation of tangents at multiple points on planar curves.

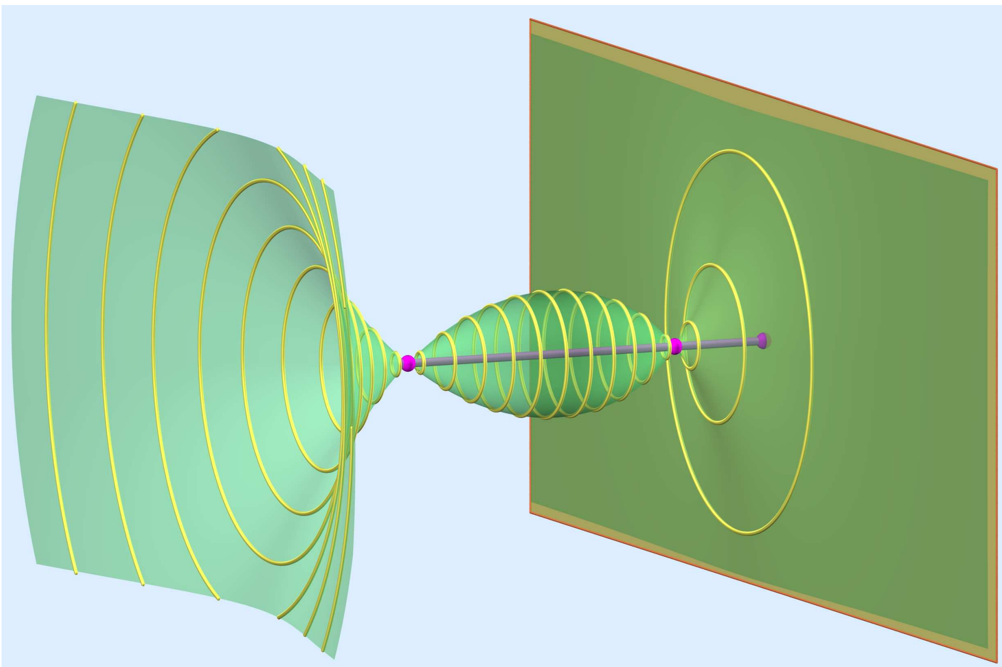
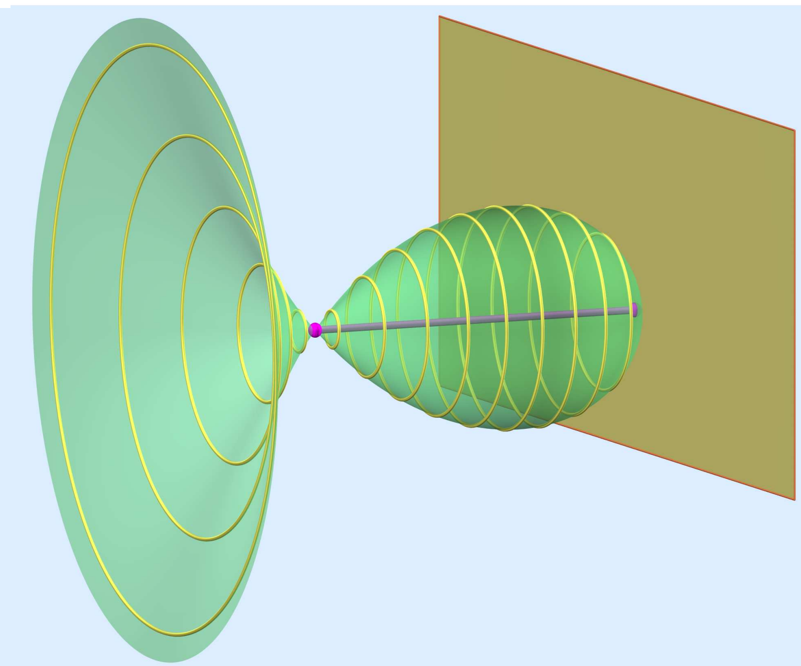
One family of circular sections of $\Gamma_\cdot$ lies in planes parallel to π.

Algebraic properties of Φ

The two singular points O and P are conical nodes: The second order approximation of Φ at O or P are quadratic cones. The circular sections of Φ lie in planes that meet the quadratic cones Γ_O and Γ_P along circles.



Special cases

Φ is a quartic surface of revolution if $P \in [O, H]$ and $P \neq O, H$.

Φ is the union of π and a cubic surface of revolution touching π at H if $P = H$.

Thank You For Your Attention!

references

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