Examples of isoptic ruled surfaces

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rough sketch of the talk

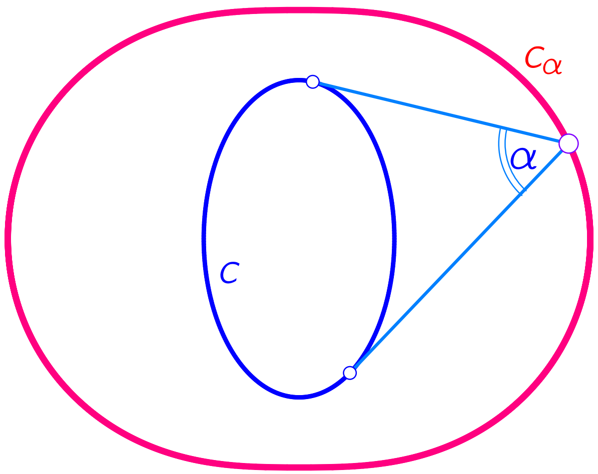
in the plane	isoptics, autoisoptics, equioptics
in three space	isoptics of polyhedra, spatial Thaloids
definition of isoptic surfaces	only for developable surfaces
helical and spiral surfaces	invariant surfaces
algebraic developables	computational problems
monomial curves, local expansions	projective generation of orthoptics
examples	cubics, quartics

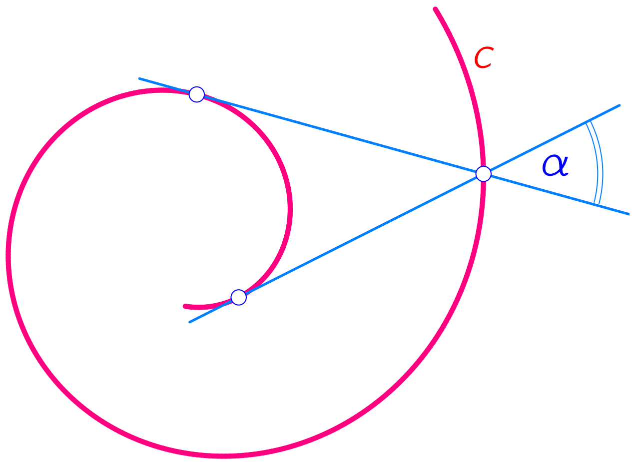
in the plane: isoptic, equioptic, autoisoptic

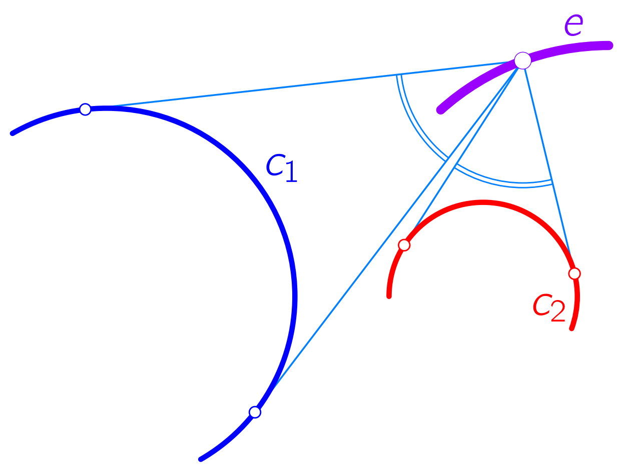
isoptic curve c_α:
locus of points from which a curve c is seen at the constant angle $0 < \alpha < \pi$. Tangents of c meet at constant angle α on c_α. [2,8,12,13,14]

autoisoptic curve:
The curve c is its own isoptic to some angle.
Such curves can be determined only approximately and numerically. [11,15]

equioptic curve:
The curve e is the locus of points from which two curves c_1 and c_2 are seen at the same (not necessarily constant) angle. [10]

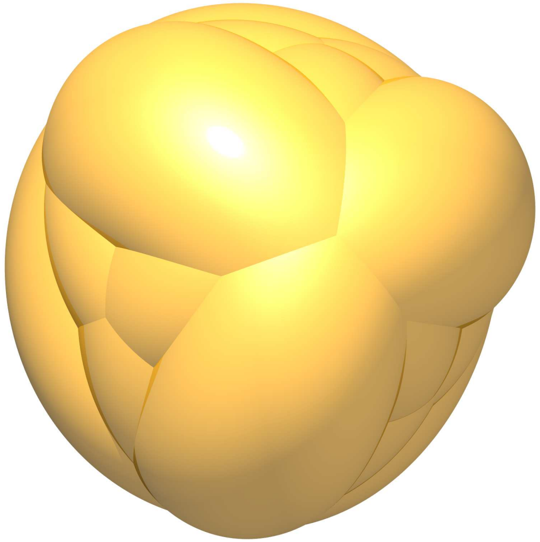






None of these can be found in three dimensional space, since arbitrarily chosen pairs of tangents of a space curve are skew (besides maybe countably many exceptions).

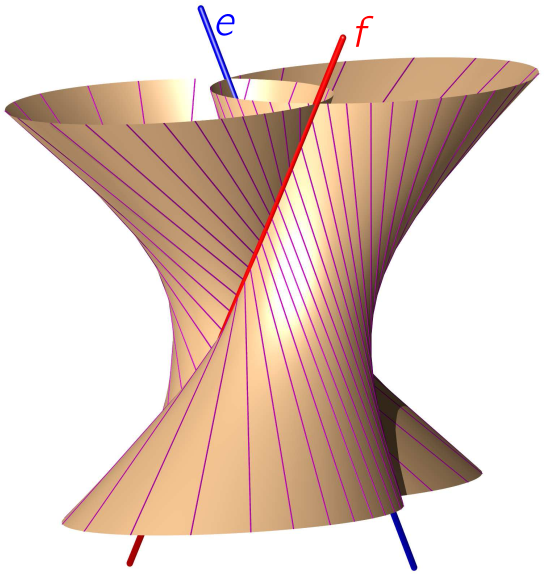
in three-space: isoptic surfaces



Spatial angle measures (surfaces on the Euclidean unit sphere) can be used to define an isoptic surface of a polyhedron. [1]

Disadvantages:

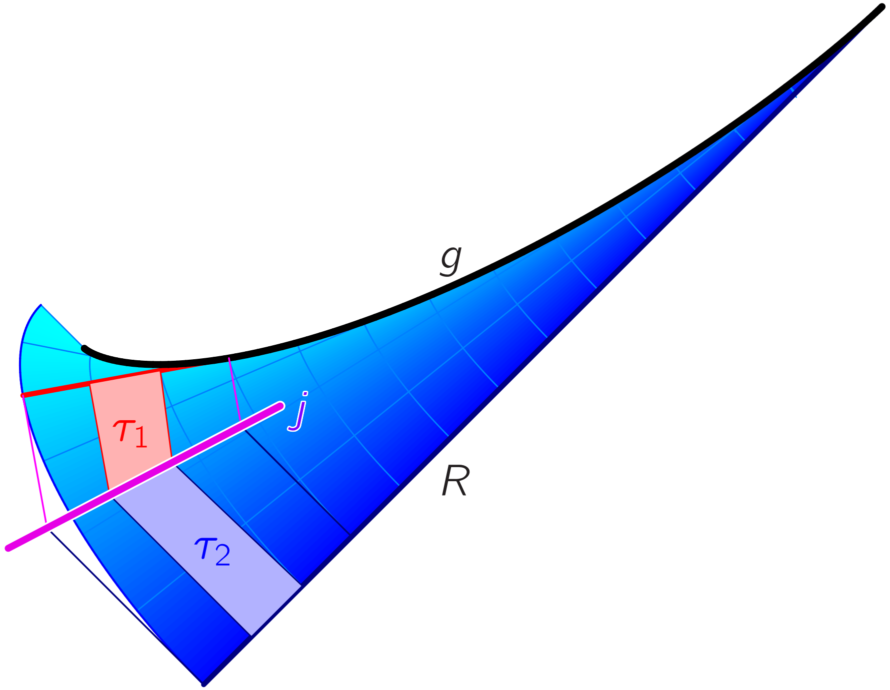
These isoptic surfaces can only be determined pointwise and numerically. An analytic representation (parametrization, equation) is missing.



Generalized Thaloids:

A spatial analog of the Theorem of the Angle of Circumference: Choose those planes from the pencils about (skew) lines e and f that enclose a fixed angle and intersect them. The intersections are the rulings of quadrics and quartics, the isoptic ruled surfaces of the pairs (e, f) of lines. [4]

in three-space: dual curves, developables



The one-parameter family of osculating planes of a space curve g is a dual curve and envelops a developable (ruled surface) R.

The curve g (as the curve of regression) and R define each other mutually.

Definition:

The isoptic ruled surface J_α of a developable (ruled surface) R is the locus of intersection lines j of pairs (τ_1, τ_2) of tangent planes of R that enclose the angle α.

cones and cylinders

Theorem:

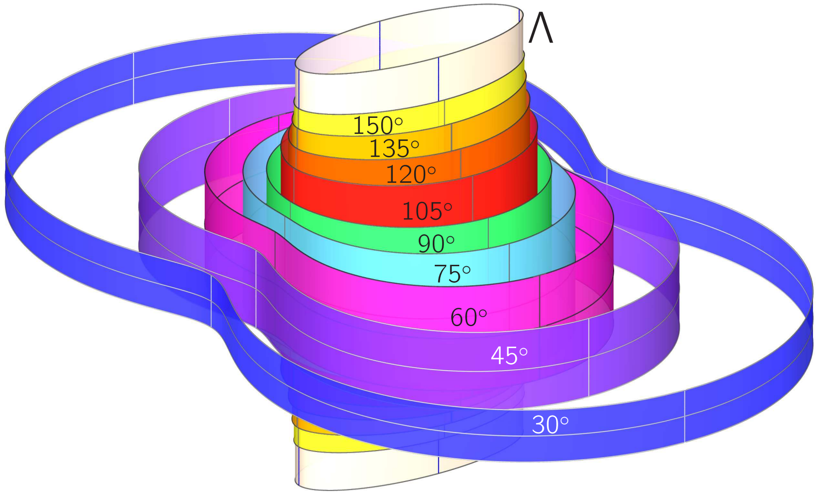
1. Isoptic ruled surfaces of cylinders are cylinders.
2. Isoptic ruled surfaces of cones are cones.

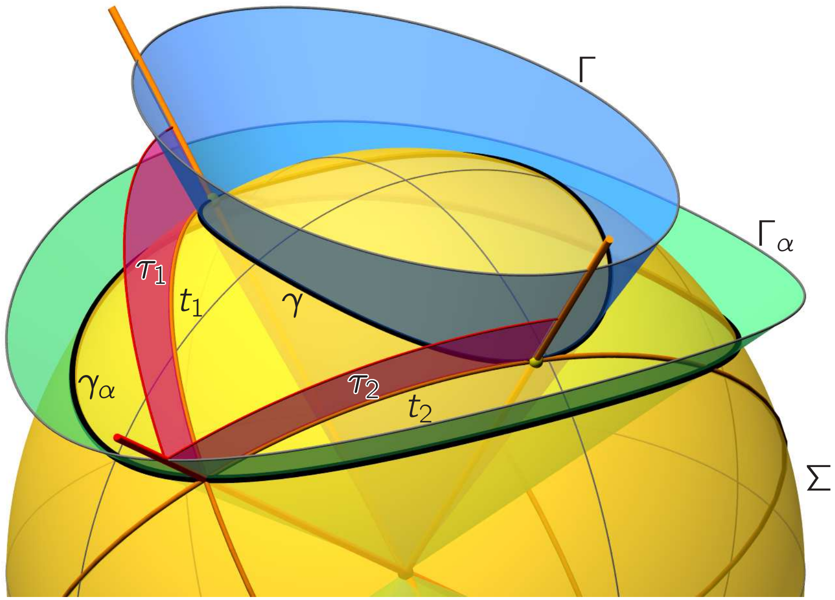
Sketch of a proof:

1. Tangent planes of a cylinder Λ are parallel to a certain direction. Isoptics of a cylinder are cylinders erected over planar isoptics of an orthogonal cross section of Λ.

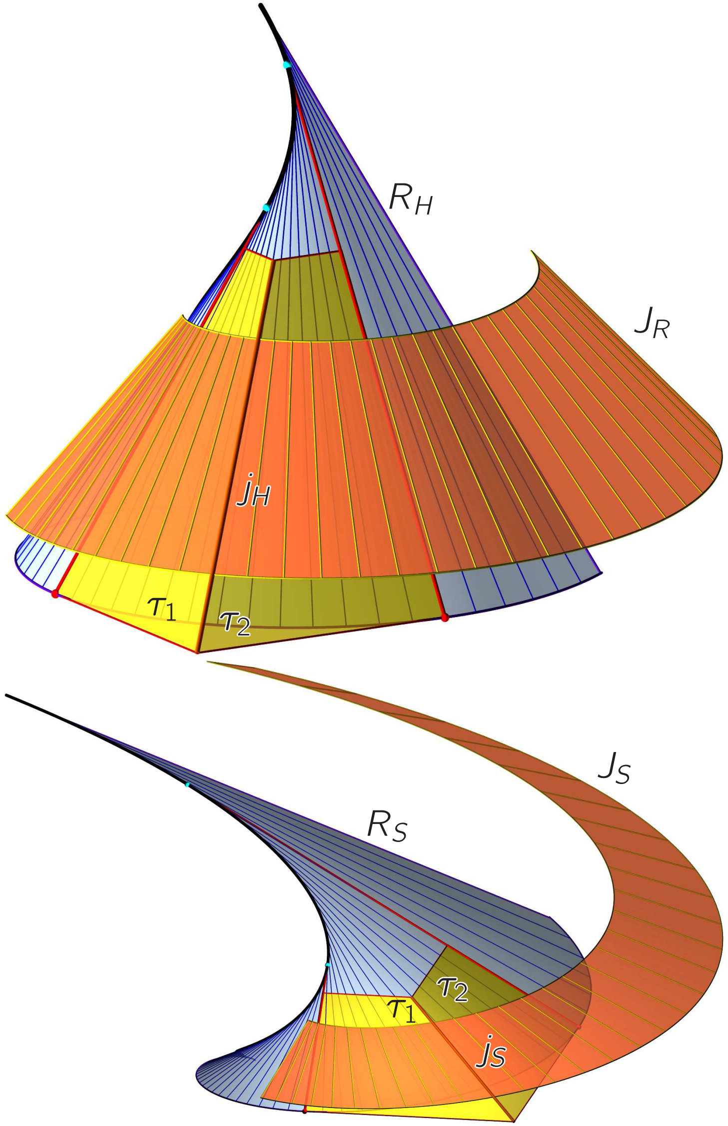
2. Tangent planes of a cone Γ pass through a common point. Tangent planes of Γ that enclose a fixed angle also pass through this point.

Isoptic cones Γ_α of Γ intersect concentric spheres Σ along spherical isoptics γ_α of $\gamma = \Gamma \cap \Sigma$.





helical and spiral developables



Theorem:
The isoptic ruled surfaces of helical/spiral developables are helical/spiral ruled surfaces.

Sketch of a proof:
Helical/spiral surface are invariant under a generating one-parameter subgroup of the Euclidean/equiform group of motions. So are the terms tangent plane and angle. [9]

Among the isoptic ruled surfaces there also developable surfaces.

Cylinders and cones are also invariant surfaces.

algebraic developables: how to compute isoptics?

The curve g of regression determines the developable R.

Tangent planes of R / osculating planes of g envelop R.

optic angle α between two tangent planes $\tau_1 = \sigma(u)$, $\tau_2 = \sigma(v)$

A pair of tangent planes that enclose the angle α ...

$\mathbf{g}(t) : I \subset \mathbb{R} \to \mathbb{R}^3$

parametrization over some interval I

$\sigma \ : \ \langle \mathbf{g}_3(t), \mathbf{x} - \mathbf{g} \rangle \ = \ 0$ ... with (unit) binormal vector $\mathbf{g}_3$ (of g)

$$\cos\alpha = \langle \mathbf{g}_3(u), \mathbf{g}_3(v) \rangle \qquad (\star)$$

for fixed α a curve p in the $[u, v]$-plane

... corresponds to a point (u_0, v_0) of the curve p.

Crucial point: Solving $(\star)$ explicitly for u or v is in general impossible.

Most cases: The curve c is of high genus and misses a useful parametrization.

Therefore: $\mathbf{j}(u, v, w) = \tau_1(u) \cap \tau_2(v)$ is a parametrization of chords of g.

The implicitization of $\mathbf{j} = \mathbf{x}$ by eliminating u or v with $(\star)$ yields an equation of J_α.

local expansions, monomial parametrizations – only orthoptics

Local expansions of g correspond to local expansions of R. Why not try:

$$\mathbf{g}(t) = \left(at^i, bt^j, ct^k\right), \quad \text{with} \;\; t \in \mathbb{R}, \;\; 1 \leq i < j < k \in \mathbb{N}, \;\; a, b, c \in \mathbb{R}.$$

For orthoptics, the normalization of $\mathbf{g}_3$ is obsolete:

$$\mathbf{g}_3(t) = \left(bcjk(k-j)t^{k-i}, caki(i-k)t^{k-j}, abij(j-i)\right).$$

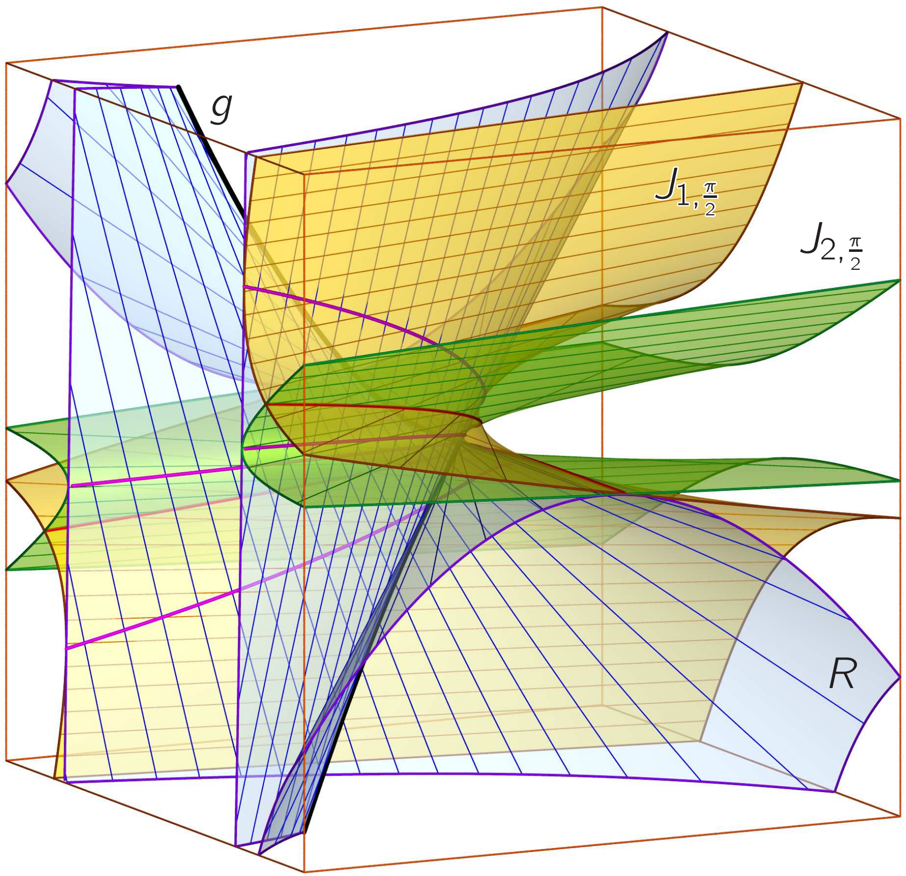
factor t^{i+j-3} dispensable, since $i+j-3$ is at least 0 $\Longrightarrow$ **($\star$)** becomes the equation of a degree-2(k-i)-curve

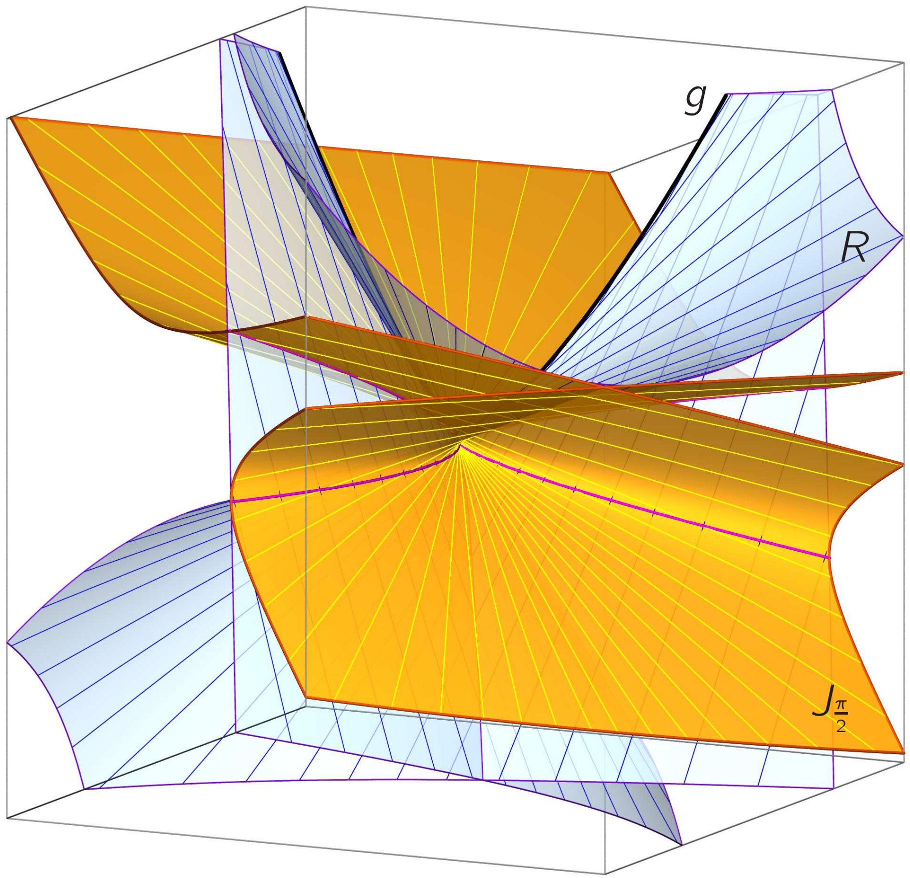
$$p: \;(bcjk)^2(k-j)^2(uv)^{k-i}+(caki)^2(i-k)^2(uv)^{k-j}+(abij)^2(j-i)^2=0. \quad (\star\star)$$

p is always singular, splits into at most $k-i$ equilateral hyperbolae (in the $[u, v]$-plane). equations of hyperbolae $uv = z_\mu$, were z_μ is a root of **($\star\star$)** with $z = uv$.

Theorem: The orthoptic ruled surfaces of the developables with monomial curves of regression consist of at most (not necessarily real) $k-i$ components. Each component is generated by a projective mapping $u \mapsto v(u)$ acting on the curve of regression and assigning to each point (and osculating plane) the corresponding point with the orthogonal osculating plane. The rulings of the orthoptic surface are the intersection lines of corresponding planes.

examples





$$\mathbf{g}(t) = \left(At, Bt^2, \frac{1}{3AC}(B^4 + C^2)t^3\right)$$

cubic parabola with a pair of orthoptic hyperbolic paraboloids

$$\mathbf{g}(t) = \left(\frac{1}{2}At^2, \frac{1}{3}Bt^3, \frac{B^2(1+C^2)}{8A(1-C^2)}t^4\right)$$

cusped quartic of the 2nd kind with a cubic isoptic of multiplicity two [3]

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Thank You For Your Attention!