

A one-parameter family of triangle cubics

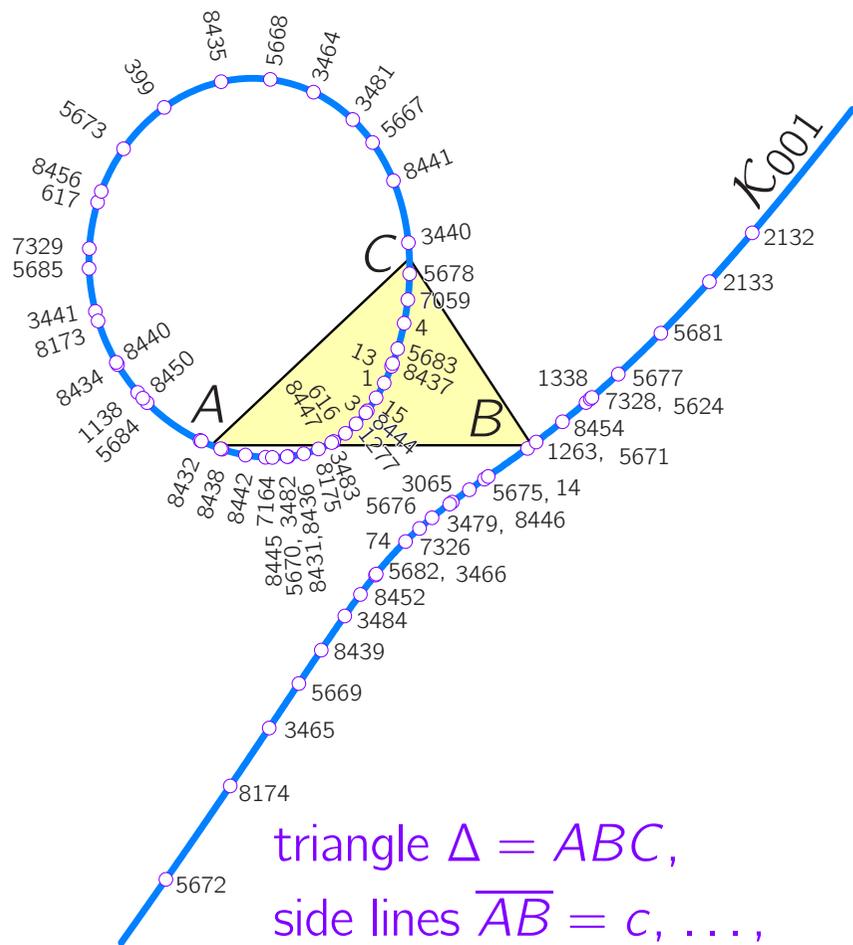
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rough sketch of the talk

triangle cubics	some of the known ones
constant product of distances	many of the new ones
properties and results	singular cubics, inflection points
one special cubic	that is already known
those related to triangle centers	centers with equal product of actual trilinear distances
how to find triangle centers on such cubics

known triangle cubics and many triangle centers

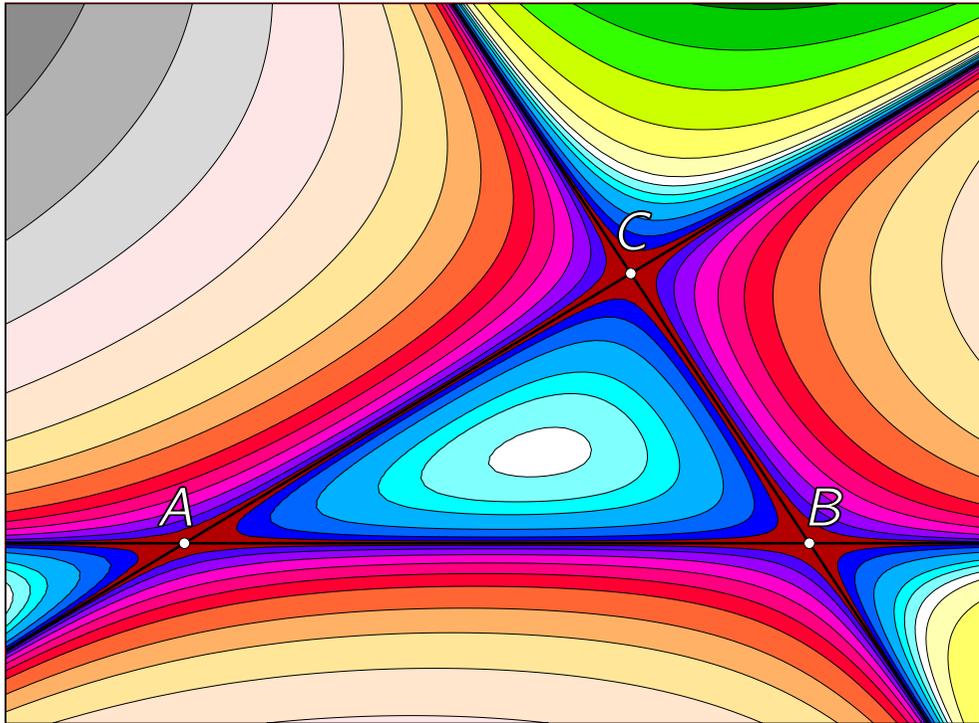


triangle $\Delta = ABC$,
 side lines $\overline{AB} = c, \dots$,
 coordinatization based on
 the frame $(A, B, C; X_1)$

The Neuberg cubic \mathcal{K}_{001}
 (notation \mathcal{K}_{001} according to Gibert's list [2])
 carries 177 triangle centers with Kimberling numbers
 1, 3, 4, 13 – 16, 30, 74, 370, 399, 484, 616, 617, 1138, 1157,
 1263, 1276, 1277, 1337, 1338, 2132, 2133, 3065, 3440, 3441,
 3464 – 3466, 3479 – 3484, 5623, 5624, 5667 – 5685, 7059,
 7060, 7164, 7165, 7325 – 7329, 8172 – 8175, 8431 – 8536,
 16882, 16883

(numbers according to Kimberling's list [4,5])
 \mathcal{K}_{001} passes through the triangle's vertices.
 \mathcal{K}_{001} admits approximately 20 geometric definitions.
 \mathcal{K}_{001} 's equation is cyclic symmetric in $(a, b, c, \xi, \eta, \zeta)$
 \implies a typical triangle cubic

product of trilinear distances



We describe points P by **trilinear coordinates**
 $\mathbf{x} = (\xi, \eta, \zeta)$, i.e., $\xi = \overline{P, [B, C]}$ (cyclic),

scale them to **actual trilinear coordinates**

$$\mathbf{x}^a = \frac{2F}{\langle \omega, \mathbf{x} \rangle} \mathbf{x} = (\xi^a, \eta^a, \zeta^a),$$

$F = \text{area}\Delta$, $\langle \cdot, \cdot \rangle \dots$ can. scalar prod.,

$\omega = [a, b, c] \dots$ trilinears o.t. ideal line.

Their **product** shall be **equal** to some **constant**

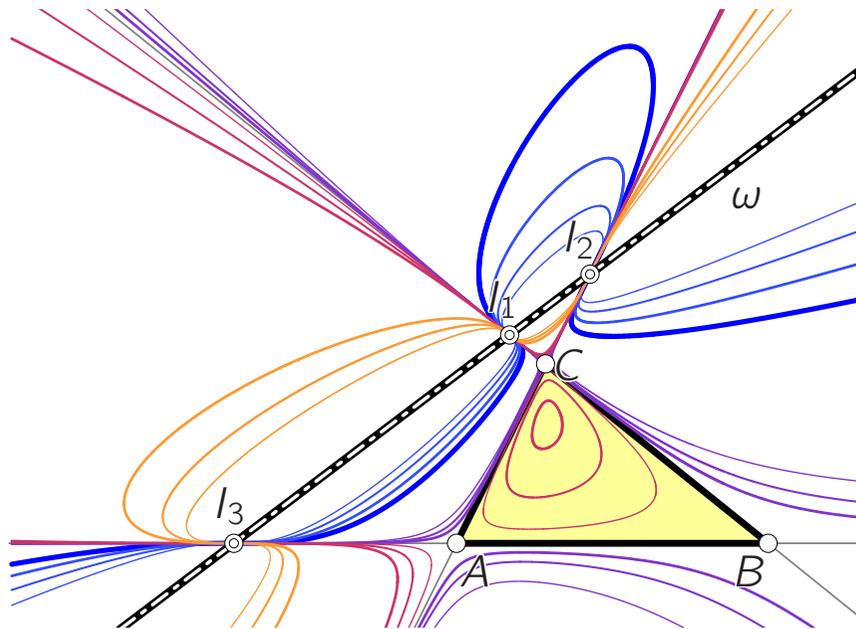
$$\delta \in \mathbb{R}: \quad \xi^a \eta^a \zeta^a = \delta \iff$$

$$k_\delta : 8F^3 \xi \eta \zeta - \delta (a\xi + b\eta + c\zeta)^3 = 0$$

Theorem:

The locus of points in the plane of a triangle that form a constant product of distances to the three side lines is a cubic curve.

a pencil of cubics



The cubics k_δ with $\delta \in \mathbb{R}$ from a pencil of cubics.
degenerate cubics in the pencil:

$$\xi\eta\zeta = 0 \dots \text{side lines of } \Delta$$

$$(a\xi + b\eta + c\zeta)^3 = 0 \dots \text{ideal line } \omega \\ \text{with multiplicity three}$$

\implies The three (real) ideal points of k_δ are the (real) inflection points I_1, I_2, I_3 of k_δ (independent of $\delta \in \mathbb{R} \setminus \{0\}, \delta \neq \infty$).

The harmonic polars (cf. [1]) of the three inflection points I_1, I_2, I_3 , are the medians, i.e., they concur in the centroid X_2 . \implies The centroid plays an outstanding role.

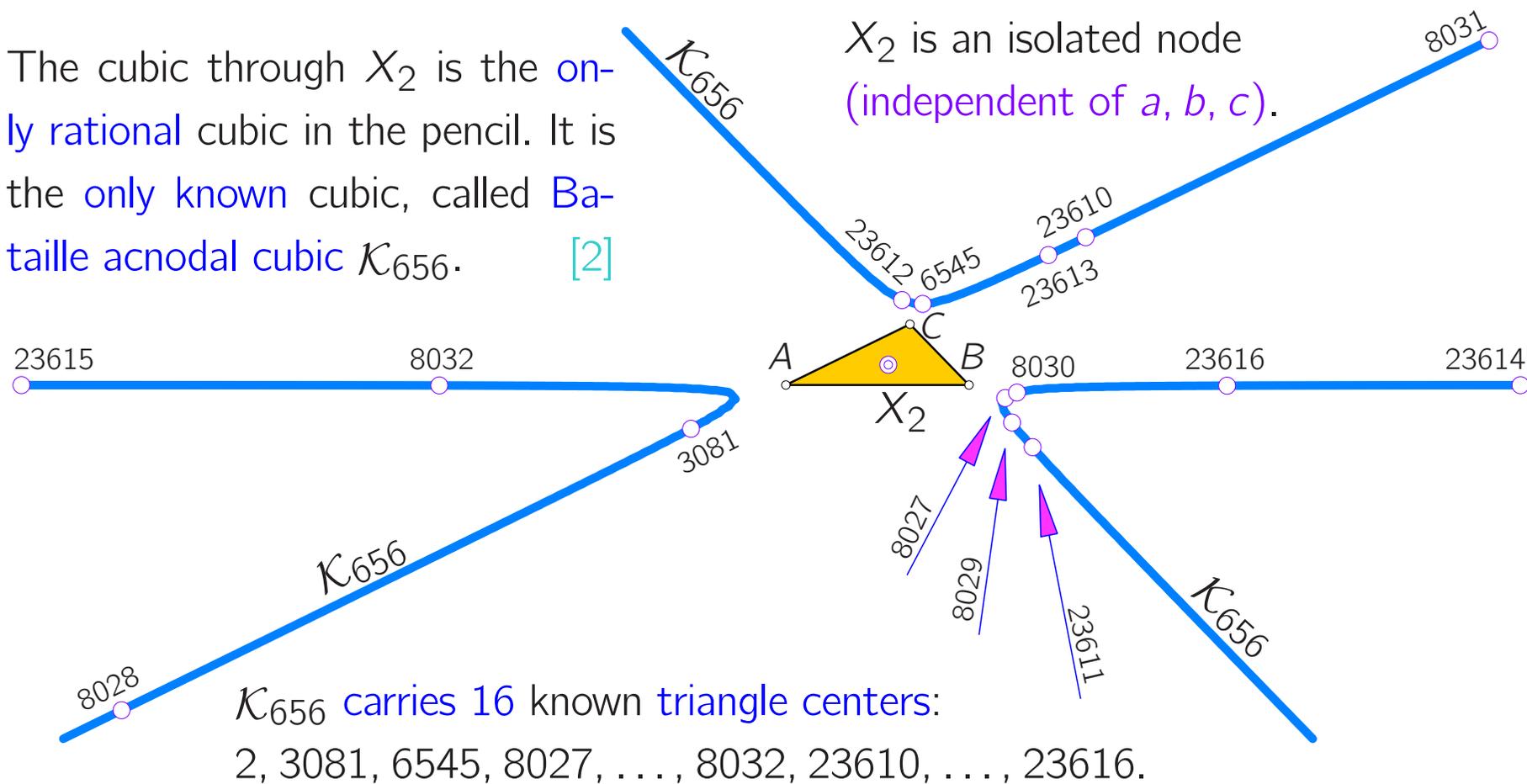
All k_δ are elliptic, provided that $\delta \neq 0, \infty, \frac{8F^3}{27abc}$.

The latter corresponds to X_2 .

cubic through the centroid - Bataille acnodal cubic

The cubic through X_2 is the **only rational** cubic in the pencil. It is the **only known** cubic, called **Bataille acnodal cubic** \mathcal{K}_{656} . [2]

X_2 is an isolated node
(independent of a, b, c).



cubics defined by triangle centers

The following groups of triangle centers share their distance product cubics:

largest group on \mathcal{K}_{656} (2,3081,6545,8027 – 8032, 23610 – 23616), 16 centers

only 3 triples (42,321,8034), (669,850,32320), (875,4375,4444)

and 57 pairs

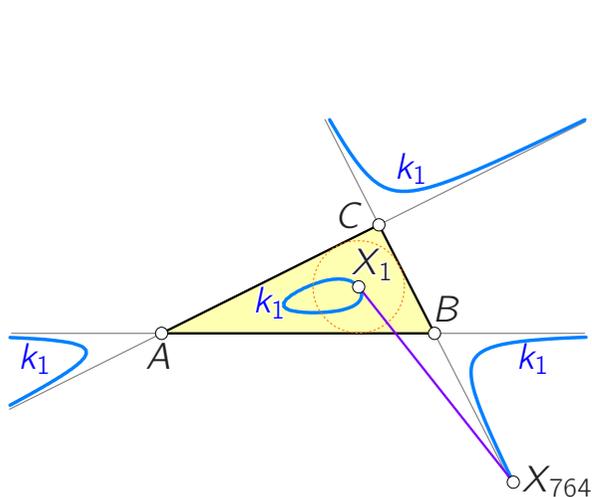
(1,764),	(99,14444),	(1026,3675),	(3233,12079),	(6544,6548),	(20058,40472),
(4,5489),	(145,23764),	(1422,40212),	(3234,15634),	(6557,15519),	(20696,20700),
(6,22260),	(324,418),	(1641,14423),	(3239,3676),	(8013,8025),	(21140,23354),
(8,21132),	(455,40144),	(1647,17780),	(3572,27855),	(8023,8039),	(21438,23655),
(25,394),	(459,3079),	(1648,5468),	(3733,4036),	(14163,14164),	(27919,40217),
(55,40166),	(649,693),	(1649,5466),	(4024,7192),	(14214,14215),	(36414,40146),
(57,200),	(671,14443),	(1650,4240),	(4358,8661),	(14401,34767),	(40149,40152),
(75,21143),	(756,8042),	(2501,3265),	(4500,4507),	(15630,15631),	
(76,23099),	(903,14442),	(3051,8024),	(6358,40213),	(15632,15635),	
(86,21131),	(1022,3251),	(3227,14441),	(6384,8026),	(16748,21820),	

Compared to all known triangle cubics (except \mathcal{K}_{656}), there is a rather small number of distance product cubics that carry two or more triangle centers.

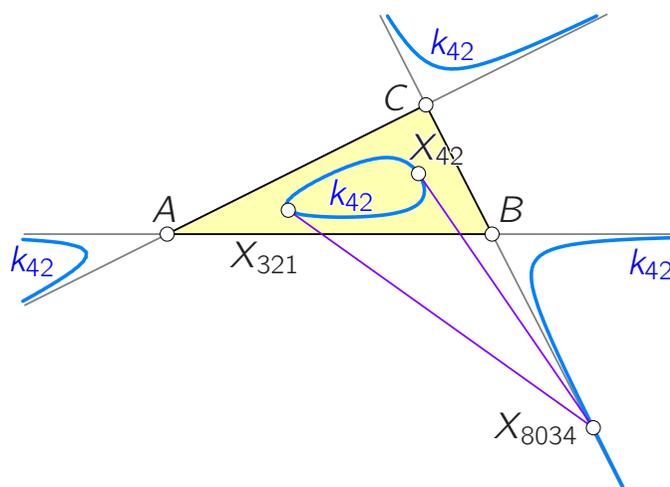
40672-16-3-3-57-2=40533 centers do not share **their** cubic with other centers.

how to find centers on cubics

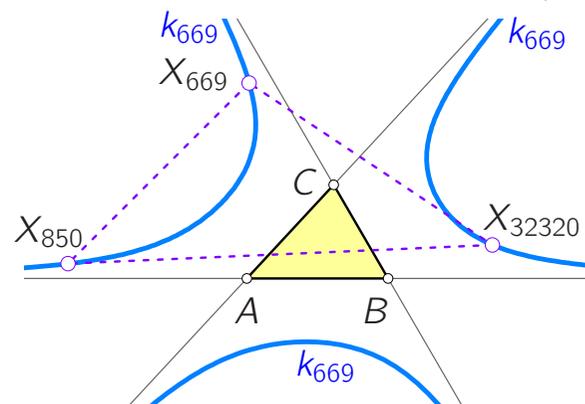
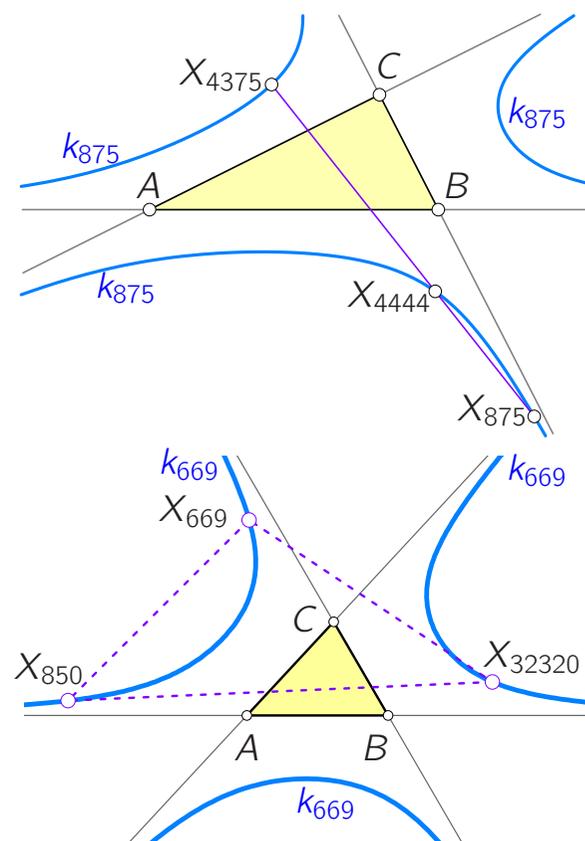
Once a center $X_j \in k_\delta$ is known, compute the tangential remainder $R_j = \{T_{X_j}k_\delta \cap k_\delta\} \setminus \{X_j\}$. That's another center (due to cyclic symmetry of involved functions).



blue and red group



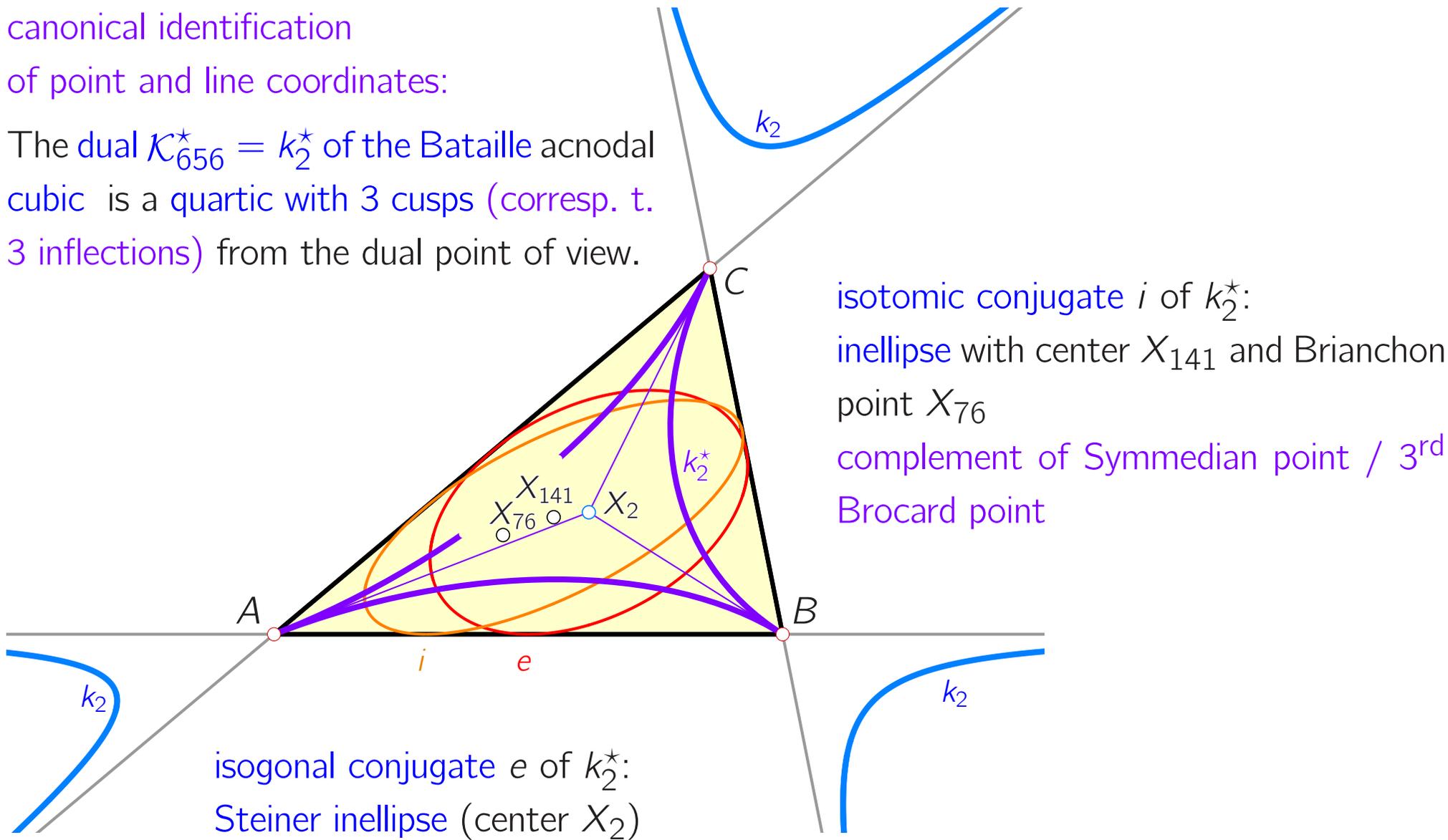
centers with common tangential remainder, violet group = remainder unknown



finally

canonical identification
of point and line coordinates:

The dual $\mathcal{K}_{656}^* = k_2^*$ of the Bataille acnodal cubic is a quartic with 3 cusps (corresp. t. 3 inflections) from the dual point of view.



isotomic conjugate i of k_2^* :
inellipse with center X_{141} and Brianchon point X_{76}
complement of Symmedian point / 3rd Brocard point

isogonal conjugate e of k_2^* :
Steiner inellipse (center X_2)

Literature

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Thank You For Your Attention!