Examples of autoisoptic curves

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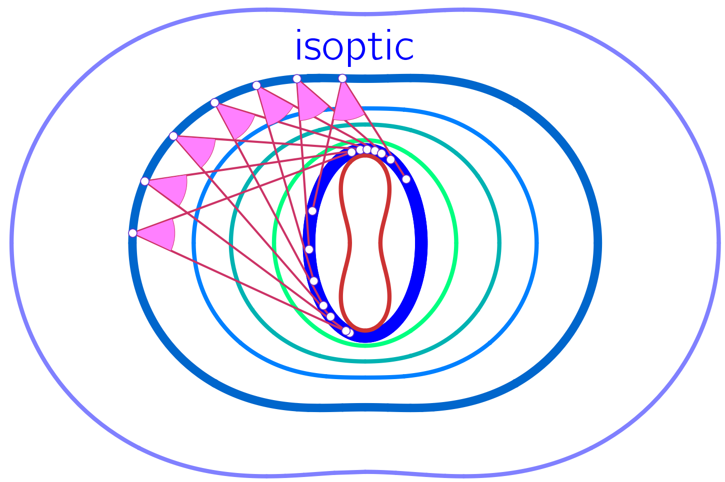
rough sketch of the talk

isoptics and autoisoptics	definitions and first examples
support function	dual representation of a curve
equations of autoisoptic curves	ordinary delay differential equations (ODDEs)
solutions of one ODDE	exact, autoevolutes, autoevolutoides
solutions of two ODDEs	generic autoisoptic curves, approximation, examples

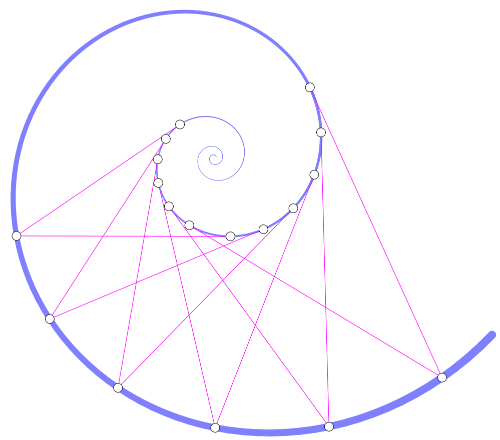
isoptic curve - autoisoptic curve

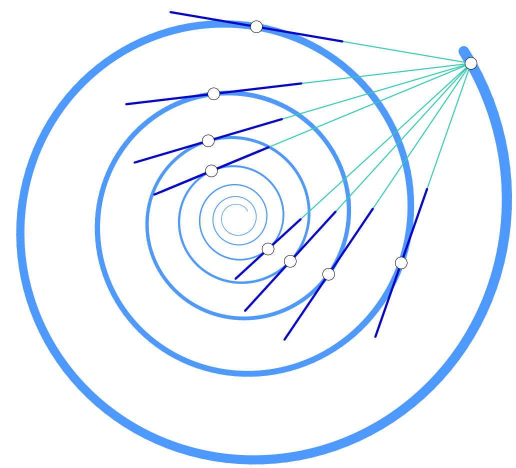
Definition. A planar curve k_α is the α-isoptic of a curve k (in the same plane) if k is seen at constant angle $\alpha \in]0, \pi[$ from all points of k_α.

Definition. A planar curve k is called an autoisoptic curve if it coincides with one of its isoptics.



Logarithmic spirals are autoisoptic curves to many different angles at the same time, due to the invariance under one-parameter subgroups of the equiform group.





$\Longrightarrow$ There are at least some solutions to our problem.

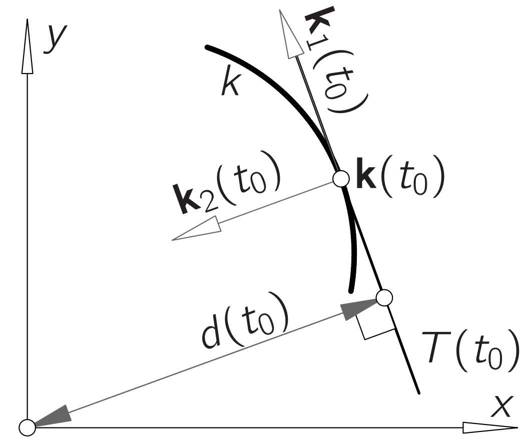
support function - dual representation of a curve

The support function

$$d(t) : I \subset \mathbb{R} \to \mathbb{R}$$

measures the oriented distance from the origin $(0,0)$ to the line

$T : -d(t) + x\cos t + y\sin t = 0.$



abbreviations: $c_\varphi := \cos\varphi$, $s_\varphi := \sin\varphi$

A parametrization $\mathbf{k}(t) : I \to \mathbb{R}$ of the envelope k of all T is obtained by intersecting T and $\dot{T} := \frac{\mathrm{d}}{\mathrm{d}t}T$:

$$\left.\begin{aligned} T &: -d + x c_t + y s_t = 0 \\ \dot{T} &: -\dot{d} - x s_t + y c_t = 0 \end{aligned}\right\} \qquad \Longrightarrow \mathbf{k}(t) = (d c_t - \dot{d} s_t,\, d s_t + \dot{d} c_t)$$

Note: $T \perp \dot{T}$, since $\mathbf{k}_2 = (c_t, s_t)$ and $\dot{\mathbf{k}}_2(t) = \mathbf{k}_1(t) = (-s_t, c_t)$

$(\mathbf{k}_1, \mathbf{k}_2)$ . . . Frenet frame of k

equations of autoisoptic curves

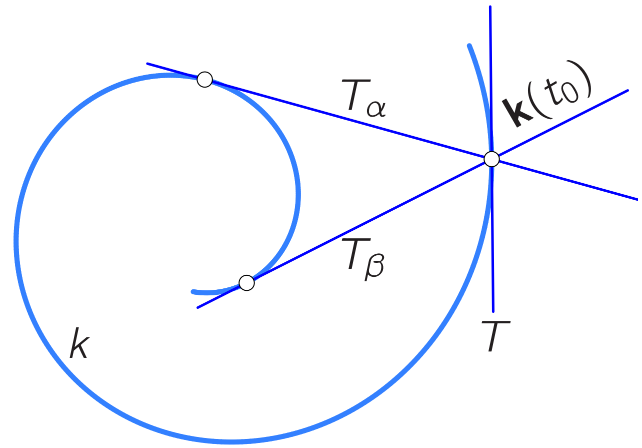
Now: α changes its meaning.

assume: $t+\alpha, t+\beta \in I$; $d_\alpha := d(t+\alpha)$

Two further tangents of k

$$T_\alpha : -d_\alpha + x c_{t+\alpha} + y s_{t+\alpha} = 0,$$
$$T_\beta : -d_\beta + x c_{t+\beta} + y s_{t+\beta} = 0$$

have to pass through $\mathbf{k}(t)$ (for all $t \in I$).



$\mathbf{k}(t) \in T_\alpha$ and $\mathbf{k}(t) \in T_\beta$ yield two ordinary delay differential equations (ODDEs):

$$d_\alpha - d c_\alpha - \dot{d} s_\alpha = 0 \quad \text{and} \quad d_\beta - d c_\beta - \dot{d} s_\beta = 0 \qquad (\bigstar).$$

Theorem. The support function $d: \ I \subset \mathbb{R} \to \mathbb{R}$ of an autoisoptic curve k_ω has to satisfy ($\bigstar$) and any function d solving ($\bigstar$) is the support function of an autoisoptic curve k_ω with optical angle $\omega = |\alpha - \beta|$.

Proof: Necessity: clear from deduction. Sufficiency: show that $T_\alpha \cap T_\beta = \mathbf{k}(t) \ \forall t \in I$.

exact solutions of ODDEs

consider just one equation

$$d_\alpha - d c_\alpha - \dot{d} s_\alpha = 0$$

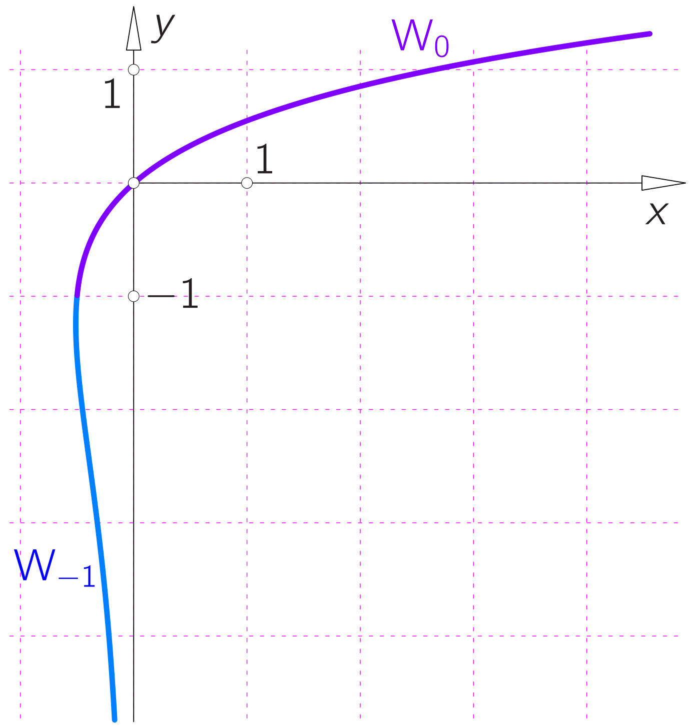
usually solved for $\dot{d}$:

$$\dot{d} = \frac{1}{s_\alpha} d_\alpha - \frac{c_\alpha}{s_\alpha} d$$

solutions in general are:

$$d(t) = \sum_{j=-\infty}^{\infty} q_j \mathrm{e}^{\mathrm{W}(j, a_1)\cdot t}$$

with $\mathrm{W}(j, a_1)$ being the j-th branch of the Lambert W function and $a_1 = 1/s_\alpha$



Note: Most of the (branches of the) Lambert W function evaluate to complex numbers.

$$\mathrm{W}(-j, x) = \overline{\mathrm{W}(j, x)} \quad \forall j \in \mathbb{Z} \quad \text{if } x > -\mathrm{e}^{-1},$$
$$\mathrm{W}(-j, x) = \overline{\mathrm{W}(j-1, x)} \quad \forall j \in \mathbb{Z} \quad \text{if } x \leq -\mathrm{e}^{-1}$$

elementary approach for one equation

assume $d(t) = \mathrm{e}^{p \cdot t}$ with $p \in \mathbb{C}$ and insert into $(\bigstar) \Longrightarrow$

$$\mathrm{e}^{p\alpha} - \mathrm{c}_\alpha - p\mathrm{s}_\alpha = 0 \qquad (\bigstar\bigstar)$$

$(\bigstar\bigstar)$ characteristic equation of $(\bigstar)$

$$p_j = -\operatorname{ctg}\alpha - \frac{1}{\alpha}\mathrm{W}\left(j, -\frac{\alpha}{\mathrm{s}_\alpha}\exp(-\alpha \cdot \operatorname{ctg}\alpha)\right) \quad j \in \mathbb{Z}$$

zeros of $(\bigstar\bigstar)$ ... eigenvalues of $(\bigstar)$

$\mathrm{e}^{p_j \cdot t}$ ... eigenvectors of $(\bigstar)$, elementary solutions

generic solution

$$d(t) = \sum_j \lambda_j \mathrm{e}^{p_j \cdot t}$$

$\Longrightarrow$ There exists an infinite dimensional vector space of solutions of an ODDE of that particular type.

We have assumed and will further assume that the delays are constant.

In general α and β can be real valued functions.

exact solutions of ODDEs - real solutions

The solutions of an ODDE form an infinite dimensional vector space.

Linear combinations of elementary solution of an ODDE also solve the ODDE.

Eigenvalues obey the rule $p_j = \overline{p_{-j}} = q_j + \mathrm{i} r_j$ (or the version with the shifted indices)

$$\Longrightarrow \quad \begin{aligned} a_j &= \tfrac{1}{2}\left(\mathrm{e}^{p_j \cdot t} + \mathrm{e}^{p_{-j} \cdot t}\right) = \cos(r_j \cdot t)\mathrm{e}^{q_j \cdot t}, \\ b_j &= \tfrac{1}{2\mathrm{i}}\left(\mathrm{e}^{p_j \cdot t} - \mathrm{e}^{p_{-j} \cdot t}\right) = \sin(r_j \cdot t)\mathrm{e}^{q_j \cdot t} \end{aligned}$$

are real elementary solutions and the generic real solution equals

$$d(t) = \sum_{j=0}^{N} \lambda_j a_j + \mu_j b_j$$

where N is called the order of the autoisoptic.

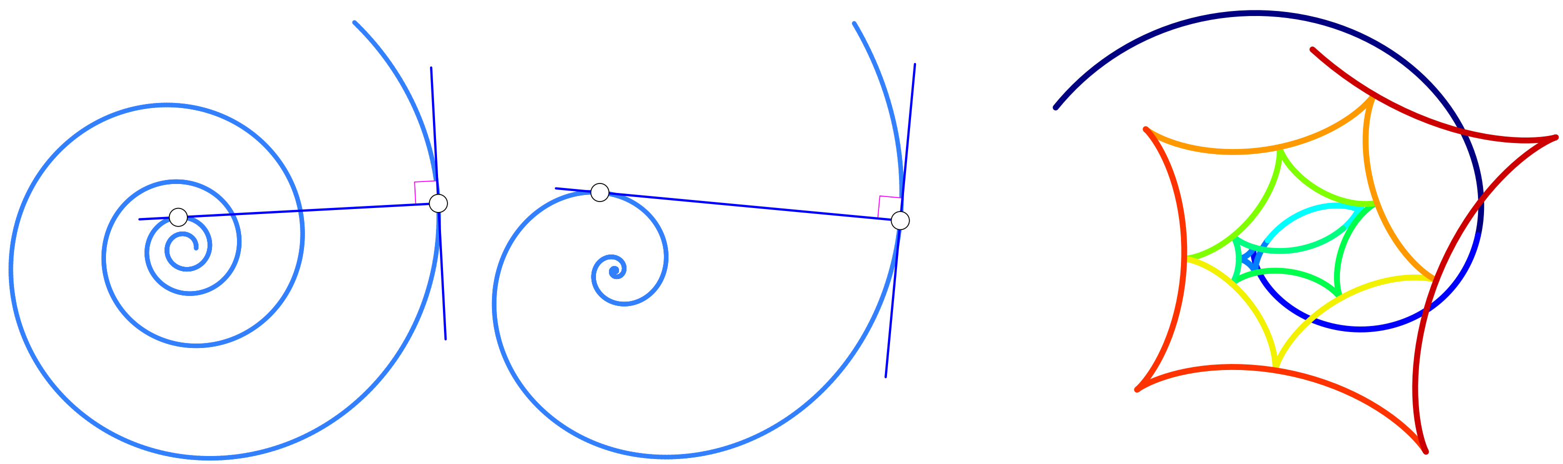
one equation - special case 1

$\beta = \alpha = \frac{k\pi}{2}$ or $\alpha = 0, \beta = \frac{k\pi}{2}$ or $\beta = 0, \alpha = \frac{k\pi}{2}$, $k \in \mathbb{Z}^\star$

$\Longrightarrow$ W. Wunderlich's autoevolutes and ($\bigstar$) changes to a single delay equation

$$d_{\frac{k\pi}{2}} - d = 0$$

$\Longrightarrow p_j = -\frac{2}{\pi} \mathrm{W}\left(j, -\frac{k\pi}{2}\right)$



Elementary solutions with real p_j correspond to logarithmic spirals.

Multiple linear combinations of complex conjugate solutions correspond to autoevolutes of higher order.

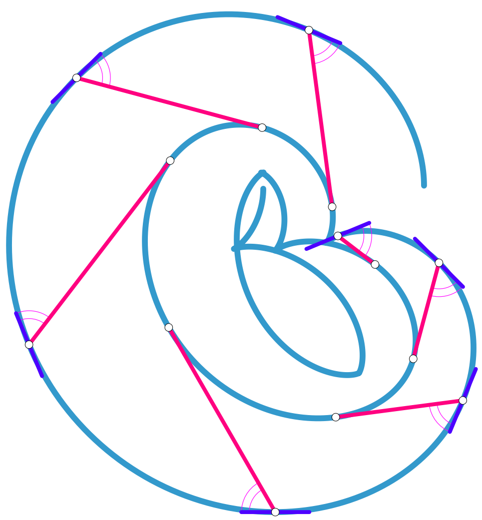
In any case, autoevolutes are spiraloid curves.

one equation - special case 2

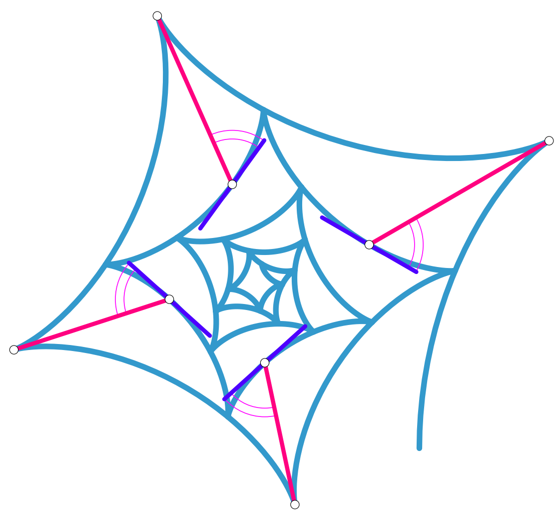
The evolutoid $k^\star_\alpha$ of a planar curve k is the envelope of a line m moving along k such that $\sphericalangle\ m, T_k = \alpha = \text{const.}\ \forall t \in I$.

$\beta = \frac{k\pi}{2}, \alpha \in \mathbb{R}$ or $\alpha = \frac{k\pi}{2}, \beta \in \mathbb{R}, \quad k \in \mathbb{Z}^\star$

$\Longrightarrow$ Autoevolutoides (apparently new) are generalizations of autoevolutes,



$\alpha = -\frac{8}{3}\pi$ and $t \in [-3\pi, 9\pi]$



$\alpha = -\frac{7}{3}\pi$ and $t \in [-8\pi, 4\pi]$

carry their own cusps, and

can be epi- or hypospiraloids.

sometimes: change from one type to the other

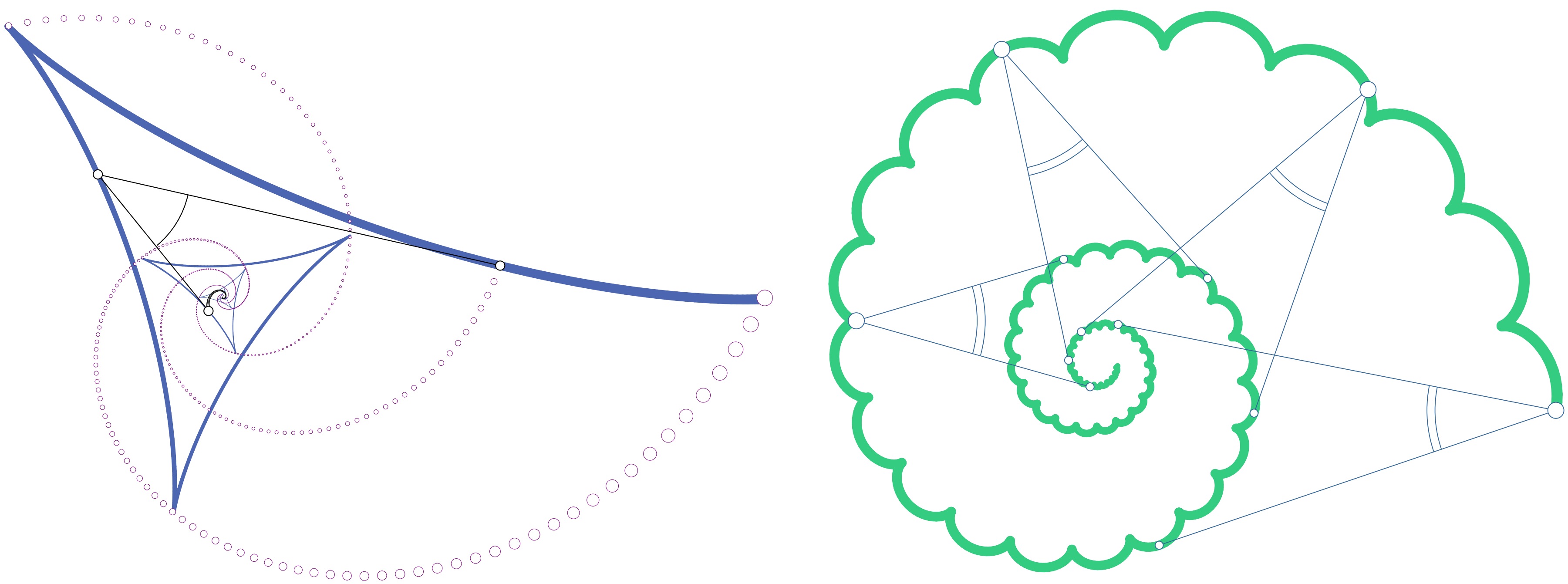
At a certain $t_0 \in I$, one exponential function dominates and the curve converges towards logarithmic spiral.

In any case, there exists an asymptotic spiral.

two equations - approximations and examples

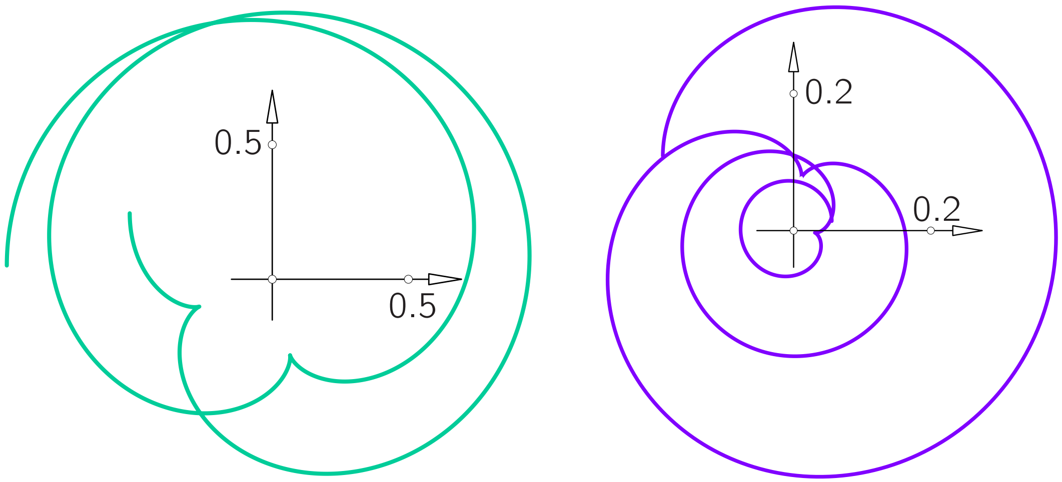
Assume $\alpha \neq \beta$ and $\alpha, \beta \neq 0, \frac{k\pi}{2}$ $(k \in \mathbb{Z})$

Now, d has to solve both equations in $(\bigstar)$. $\Longrightarrow$ impossible for a closed (exact) solution as long as $\alpha \neq \beta$ $\Longrightarrow$ In general, only approximate solutions exist!

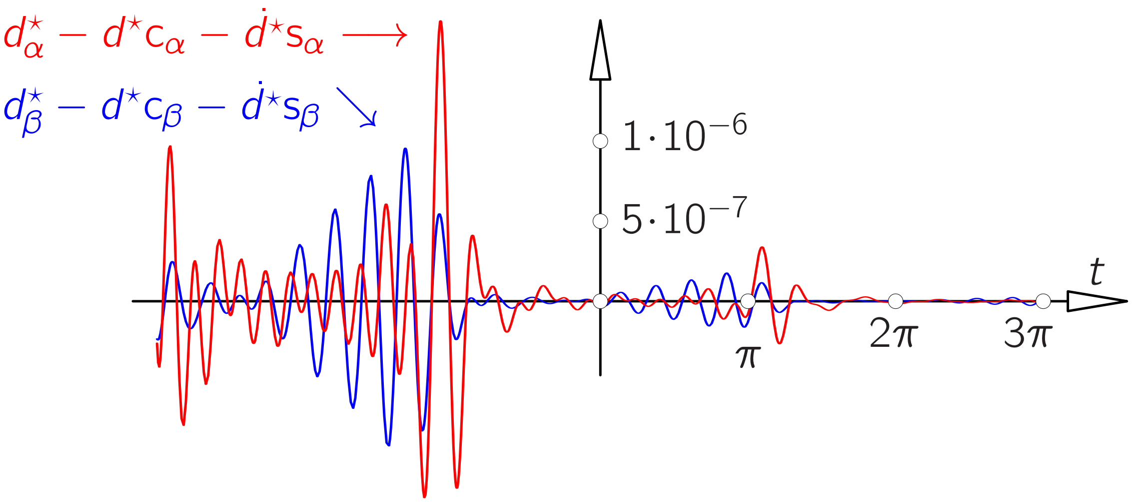


Hypospiraloid (left) and epispiraloid (right) as autoisoptic curves.

two equations - approximations and examples

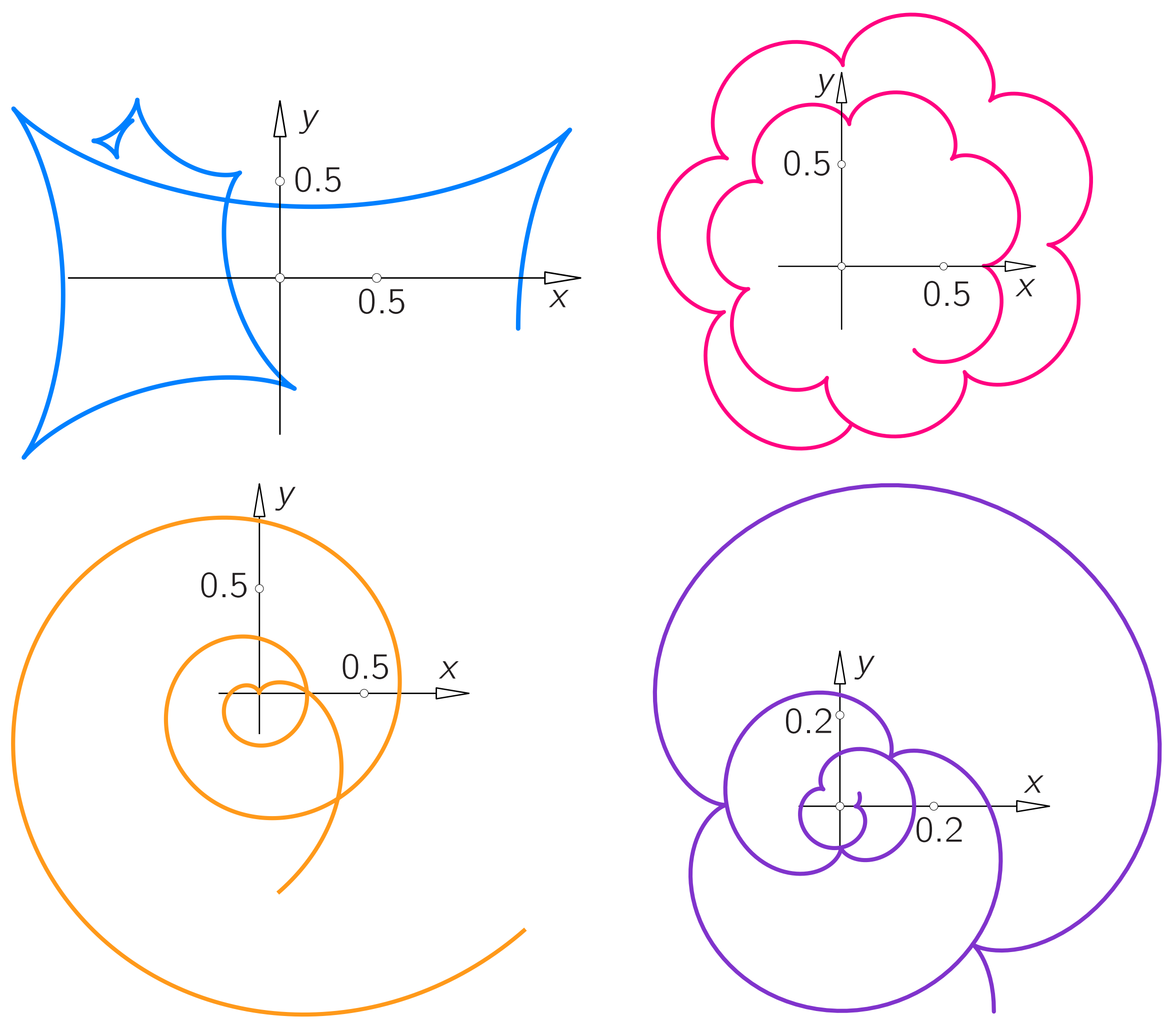


$\leftarrow$ two approximations in the space of basic elementary solutions



The support function $d^\star$ of the curve on the left ($\nwarrow$) is a very good approximation of the solutions of both ODDEs.

two equations - approximations and examples



⟵ hypospiraloid

⟵ epispiraloid

The polynomial approximations imitate the shape of the curves found by interpolation in the space of elementary exponential solutions up to a certain extent depending on the degree of the power series.

conclusion

The logarithmic spiral is by no means the only autoisoptic curve.

The logarithmic spiral is a multiple autoisoptic (due to the infintely many branches of the Lambert W function), *i.e.*,

the logarithmic spiral is an autoisoptic curve for infinitely many optical angles.

The fact that the logarithmic spiral is an autoisoptic curve in many ways allows us to conjecture that there may be more such curves.

Autoevolutoides and autoevolutes are bycatch.

We treated only the case of constant delays.

Variable delays . . . only numerical solutions? (if at all)

Literature

[1] L. Euler: *Investigatio curvarum quae evolutae sui similes producunt.* Comm. acad. scientiarum Petrop. **12** (1750), 3–52.

[2] L. Euler: *Investigatio curvarum quae similes sint suis evolutis vel primis vel secundis vel tertiis vel adeo ordinis cuiuscunque.* Nova acta acad. scientarum imp. Petrop. **1** (1787), 75–116.

[3] G. Glaeser, H. Stachel, B. Odehnal: *The Universe of Conics.* From the ancient Greeks to 21st century developments. Springer-Spektrum, Springer-Verlag, Heidelberg, 2016.

[4] J.K. Hale, S.M. Verduyn Lunel: *Introduction to Functional Differential Equations.* Applied Mathematical Sciences 99. Springer, 1993.

[5] J.M. Heffernan, R.M. Corless: *Solving some delay differential equations with computer algebra.* The Mathematical Scientist **31**/1 (2006), 21–34.

[6] R. Kunkli, I. Papp, M. Hoffmann: *Isoptics of Bézier curves.* Comp. Aided Geom. Design **30**/1 (2013), 78–84.

[7] G. Loria: *Spezielle algebraische und transzendente ebene Kurven I, II.* B.G. Teubner, Leipzig-Berlin, 1910.

[8] B. Odehnal: *Equioptic curves of conic sections.* J. Geom. Graph. **14**/1 (2010), 29–43.

[9] R.F. Puerta, M.A. Gómez-Elvira Gonzáles, Á. Miguel, C. San Antonio Gómez, E. Martín, M. Conejo: *The logarithmic spiral, autoisoptic curve.* An. de Ingeniería Gráfica **21**/2009, 41–44.

[10] A.F. Timan: *Theory of Approximation of Functions of a Real Variable.* Dover Publications, New York, 1994.

[11] H. Wieleitner: *Spezielle ebene Kurven.* G.J. Göschen'sche Verlagshandlung, Leipzig, 1908.

[12] E. Wölffing: *Ueber Pseudotrochoiden.* Zeitschr. Math. Phys. **44** (1897), 139–166.

[13] W. Wunderlich: *Autoevoluten.* Elem. Math. **XVII**/6 (1962), 121–128.

Thank You For Your Attention!