Subdivision schemes for ruled surfaces and channel surfaces

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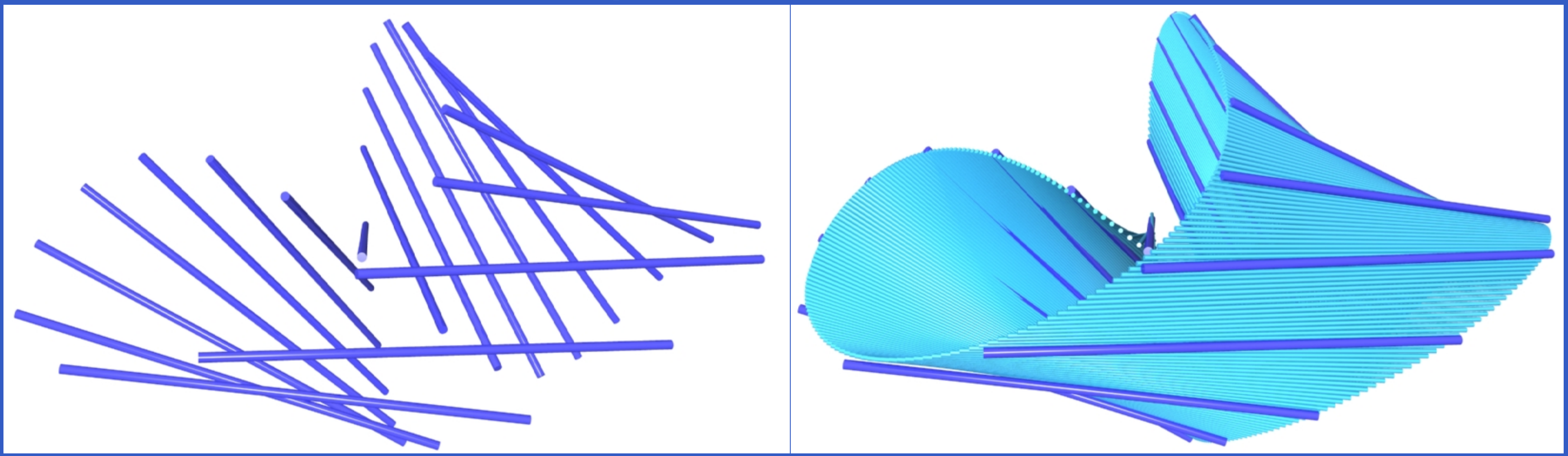
Aims & Motivation

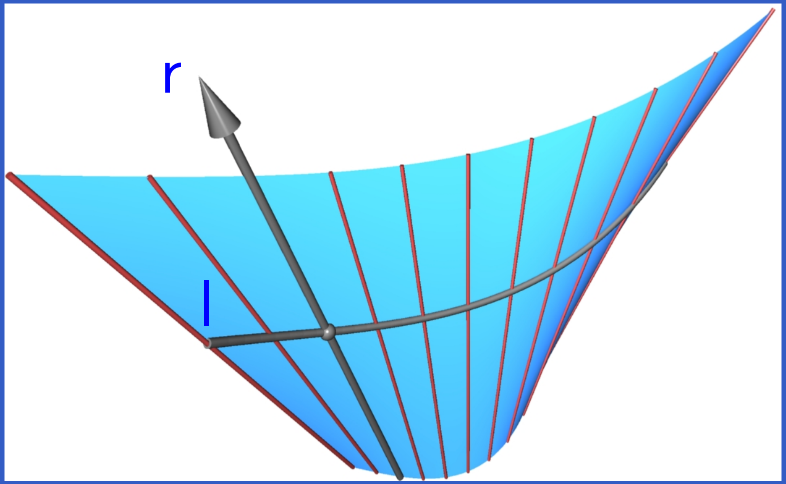
- approximation of ruled / channel surfaces by discrete models
- adapt known subdivision schemes for sets of lines / spheres
- use appropriate model spaces
- circumvent parameterizations

What are we going to do?

- given a coarse model of a ruled / channel surface = finite set of lines / spheres
- looking for a finer model (insert new lines / spheres)
- should lead to a pleasing (smooth) limit



Ruled surfaces

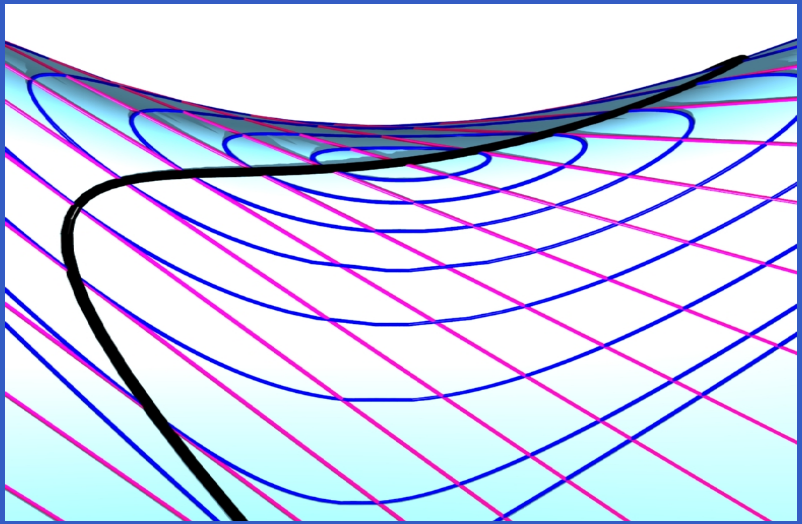


- smooth / discrete 1-param. family of lines (curve)
- directrix l, unit VF r parallel to rulings

- central curve $\mathbf{c}$:

$$c = l - \langle \dot{l}, \dot{r} \rangle / \langle \dot{r}, \dot{r} \rangle r$$

- locus of maximum Gaussian curvature



Klein model of line space

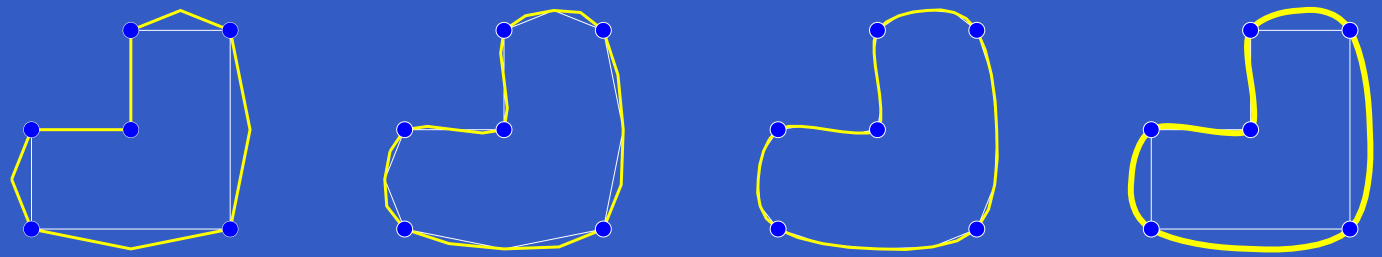
- lines $l + v \cdot r \to (r, l \times r) = (g, \overline{g})$
- Plücker coordinates $(g, \overline{g}) \in \mathbb{R}^{3+3}$ satisfy

$$M^4 : \ \langle g, g \rangle = 1, \quad \langle g, \overline{g} \rangle = 0$$

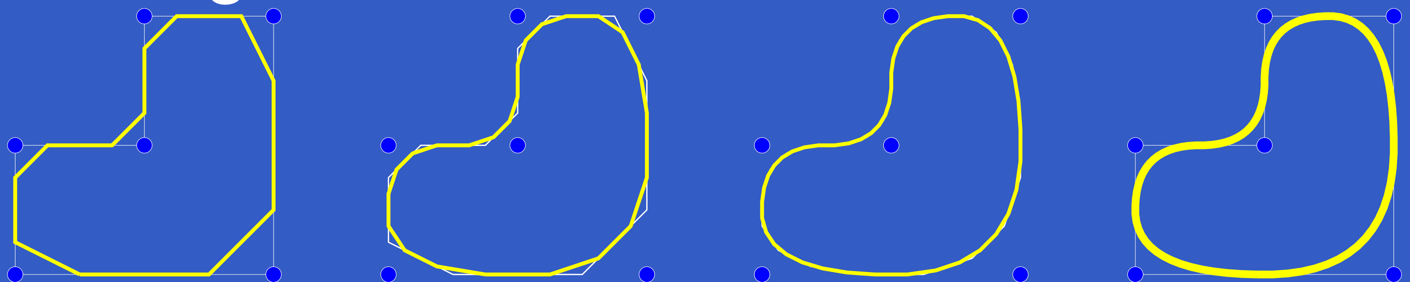
- ruled surfaces are curves in M^4

Subdivision schemes for curves

- interpolating scheme by DLG



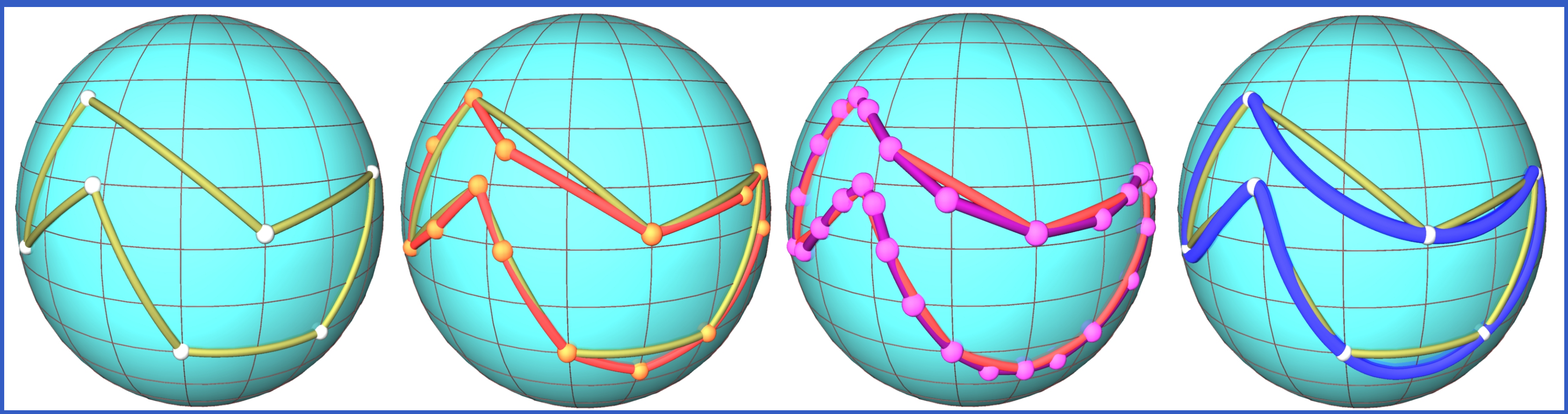
- approximating scheme: Chaikin's corner cutting



- limits and generalizations are known: different mascs, ternary schemes, . . .

Subdivision schemes for curves

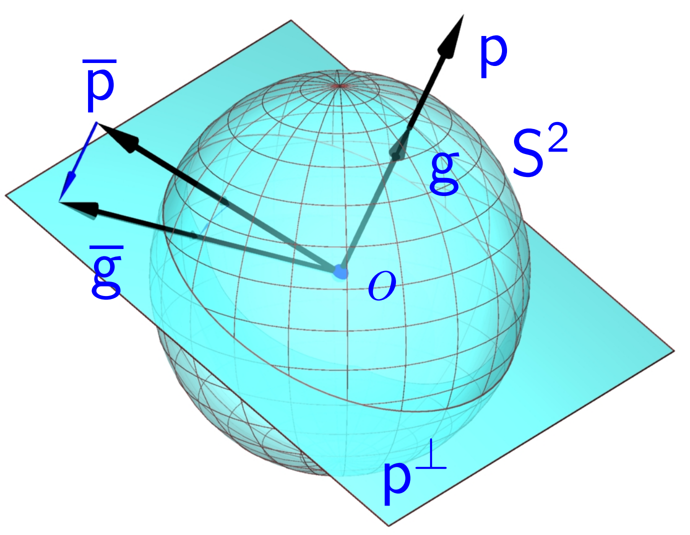
- originally for data in affine space
- generalized to data from arbitrary manifolds:
  - geodesic subdivision



  - subdivison + projection

Algorithm 1: subdivision + projection

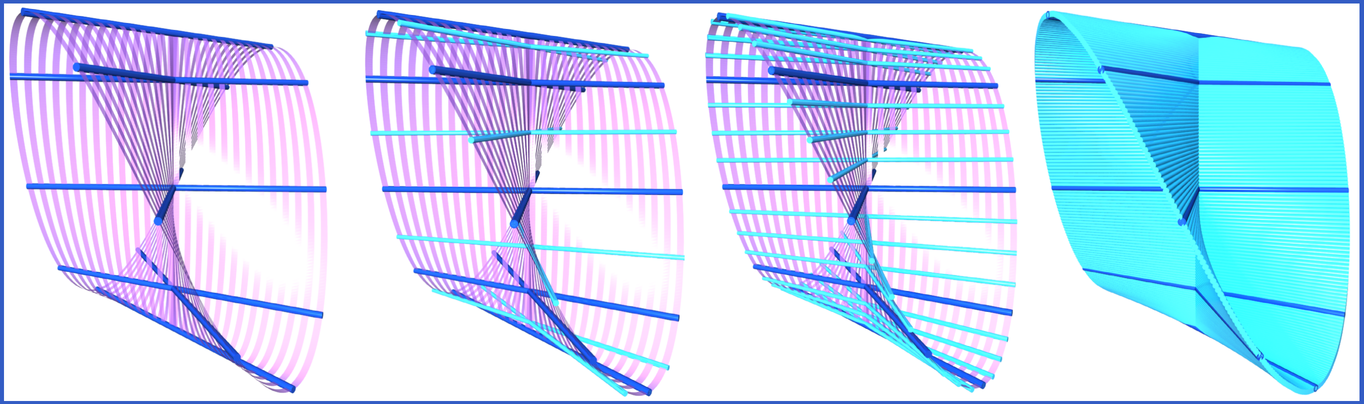
- discrete ruled surface = polygon in M^4 (vertices only)
- apply subdivision scheme to data
- project new vertices into M^4



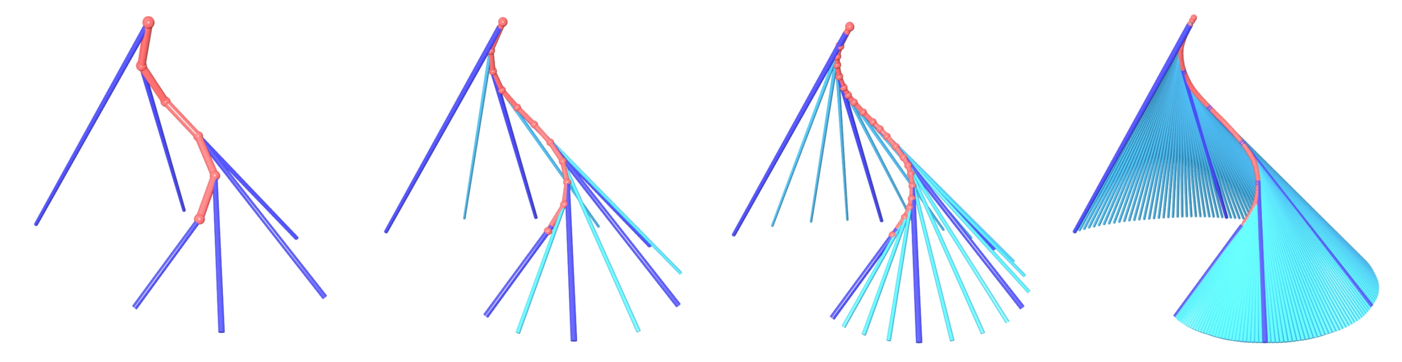
$$(g, \overline{g}) = \left(\frac{p}{\|p\|}, \frac{\overline{p}}{\|p\|} - \frac{p\langle p, \overline{p}\rangle}{\|p\|^3} \right)$$

Algorithm 1: examples

- reproducing Plücker's conoid

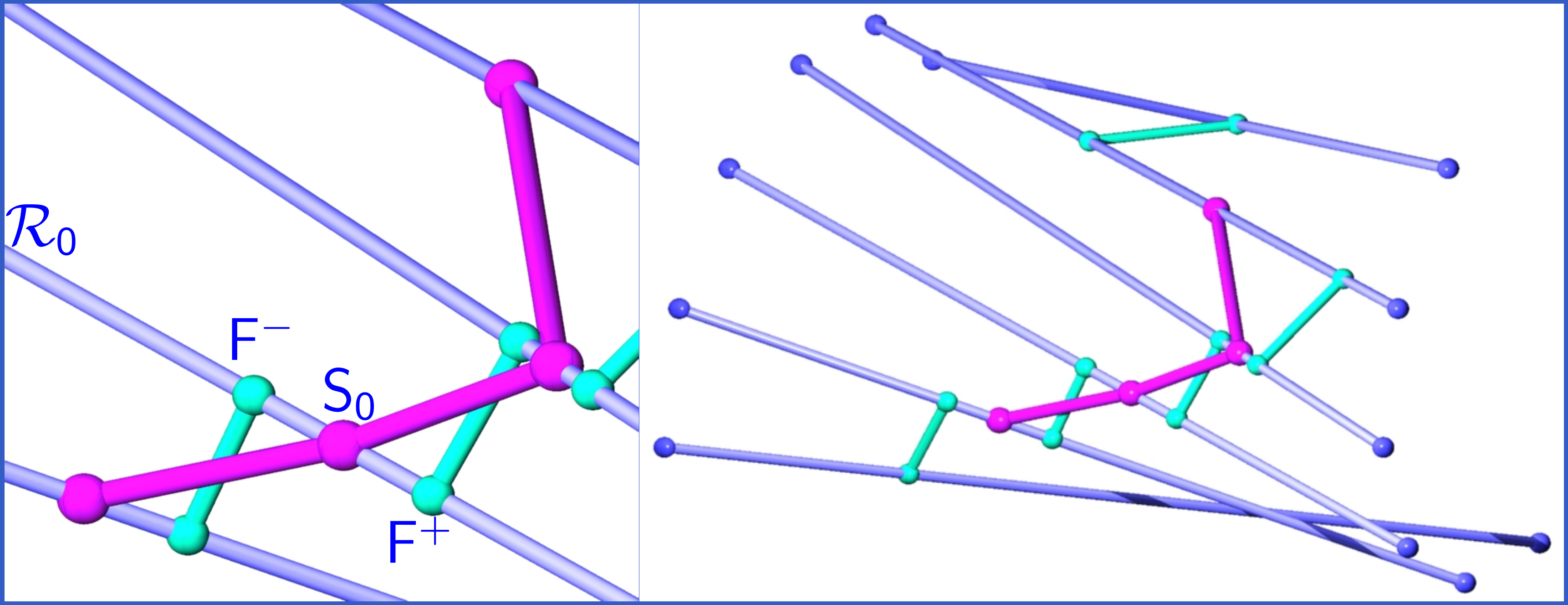


- algorithm handles torsal ruled surfaces properly



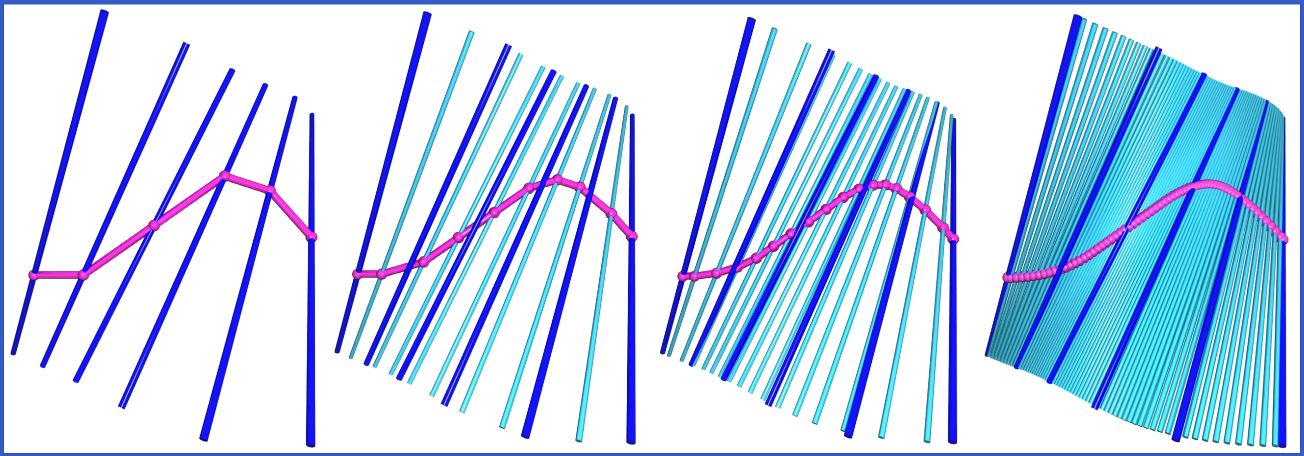
Algorithm 2: central curve + spherical image

- compute a discrete version of the central curve
- refine the discrete version of the central curve
- refine discrete spherical image of the rulings

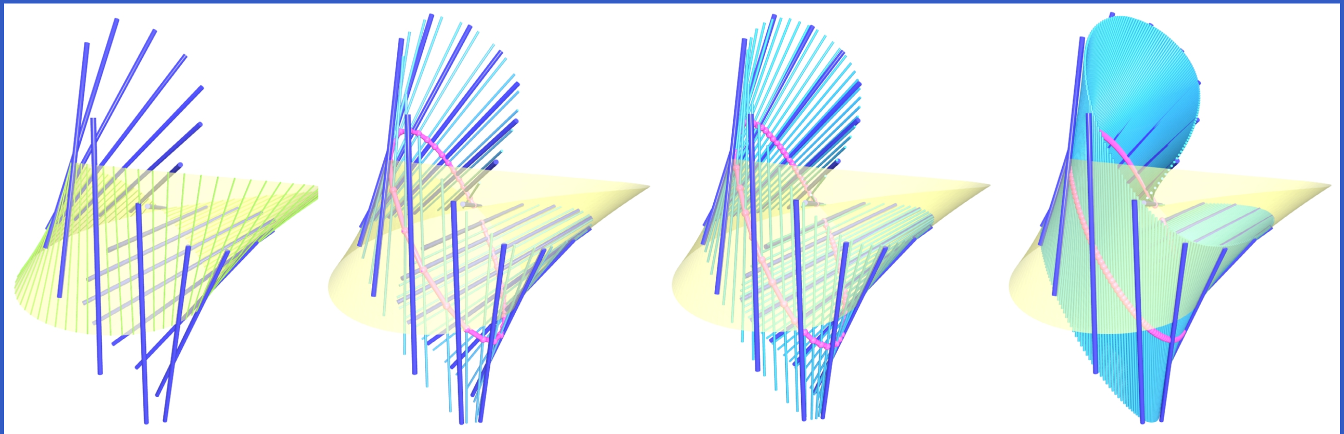


Algorithm 2: examples

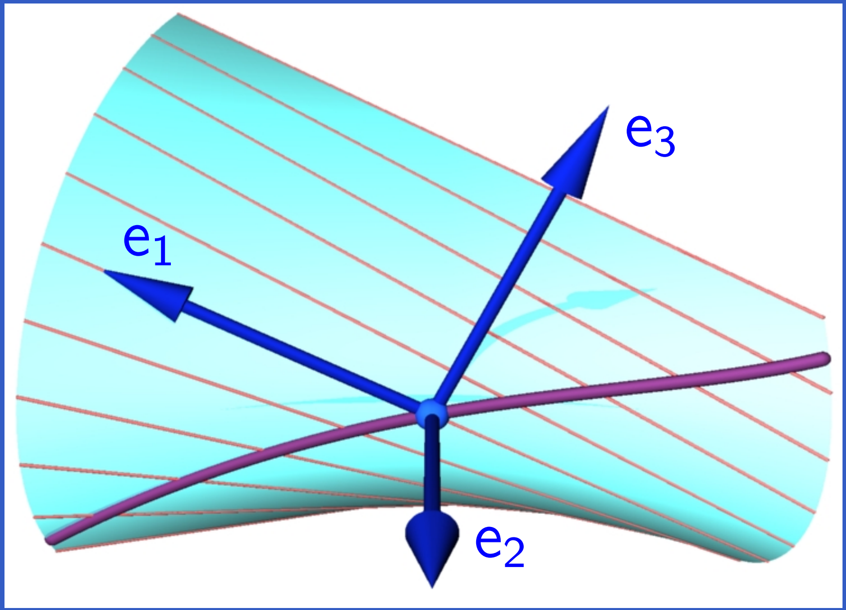
- approximating six arbitrarily given lines



- surface of Möbius type

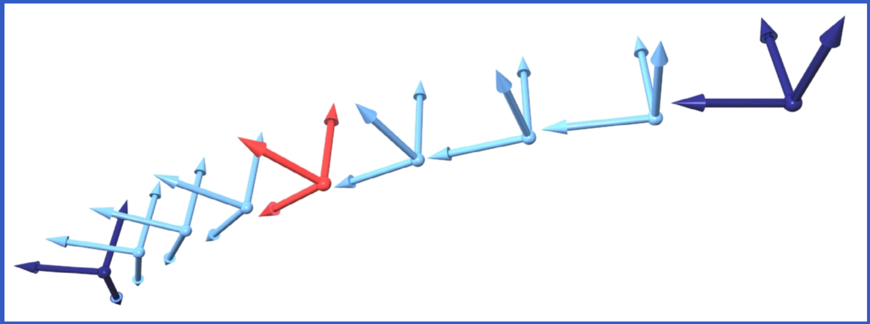


Algorithm 3: motion of the Sannia frame



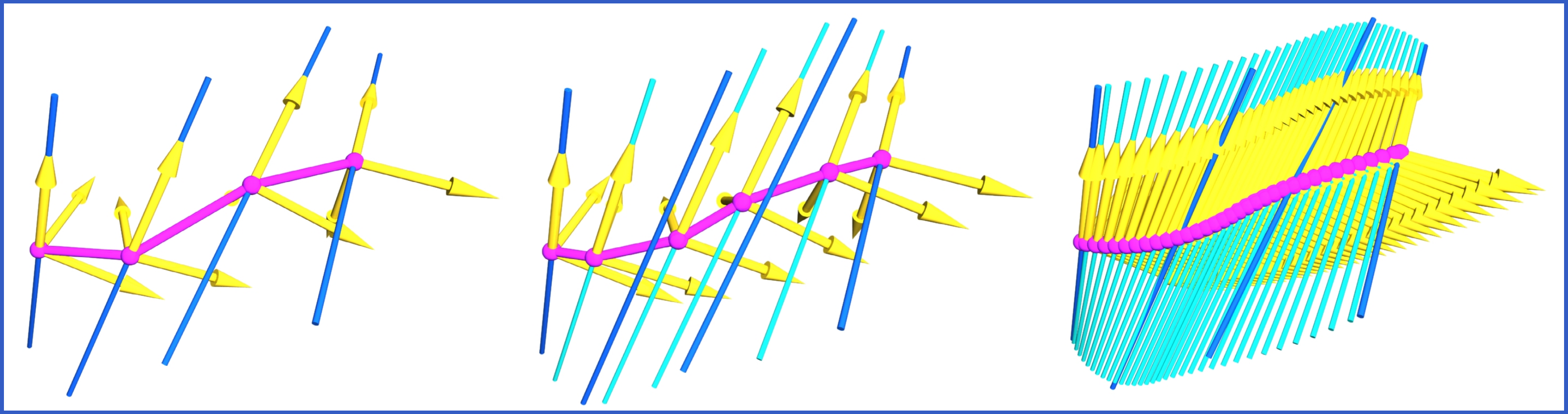
- Sannia frames define discrete 1-pm. motion
- refine Sannia motion by means of geodesic subdivision

- geodesics = helical motions
- computing intermediate positions

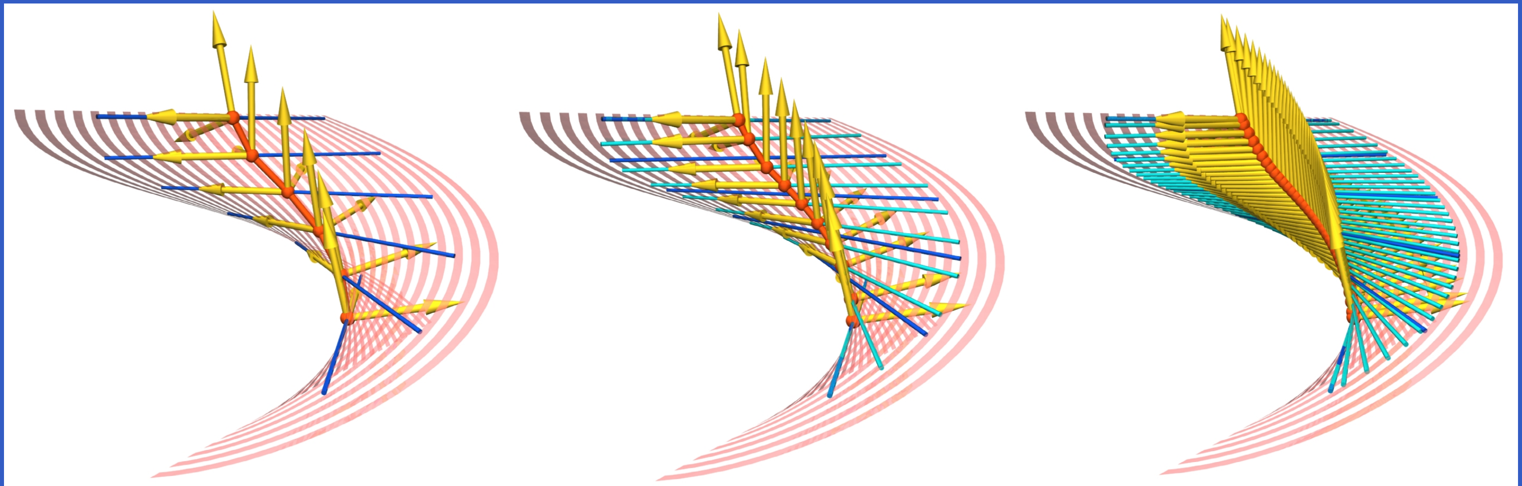


Algorithm 3: examples

- approximating four arbitrarily given lines



- data from a helical surface

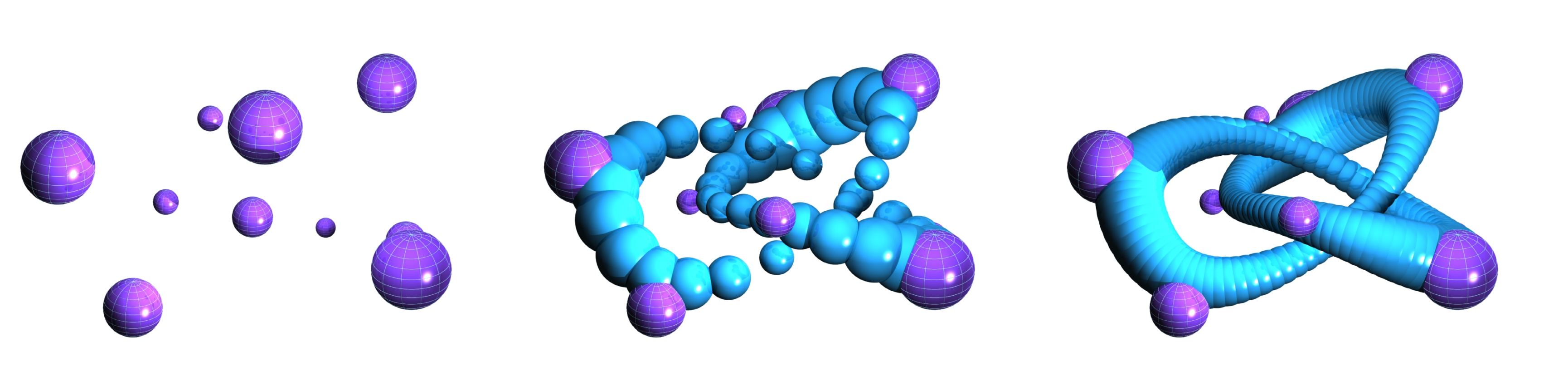


Channel surfaces

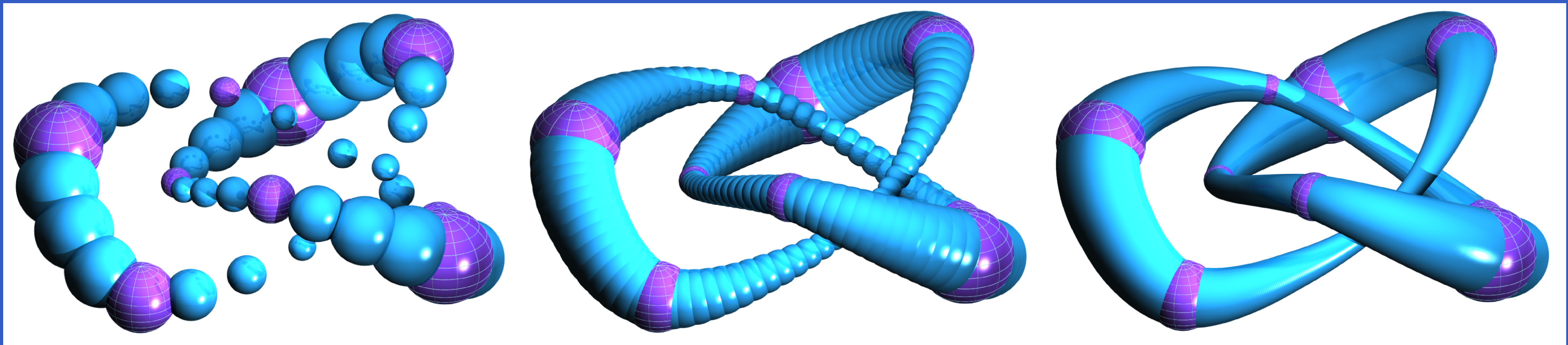
- use the cyclographic model space
- sphere $\Sigma(c, r)$ with center c and radius r is represented by a point $(c_1, c_2, c_3, r) \in \mathbb{R}^4$
- contact of spheres defines a metric
- model space is pseudo-Euclidean (Minkowskian) $\mathbb{R}^{3,1}$
- curves in $\mathbb{R}^{3,1} \leftrightarrow$ channel surfaces in $\mathbb{R}^3$

Approximation of channel surfaces: examples

- Chaikin’s scheme in Minkowskian space $\mathbb{R}^{3,1}$



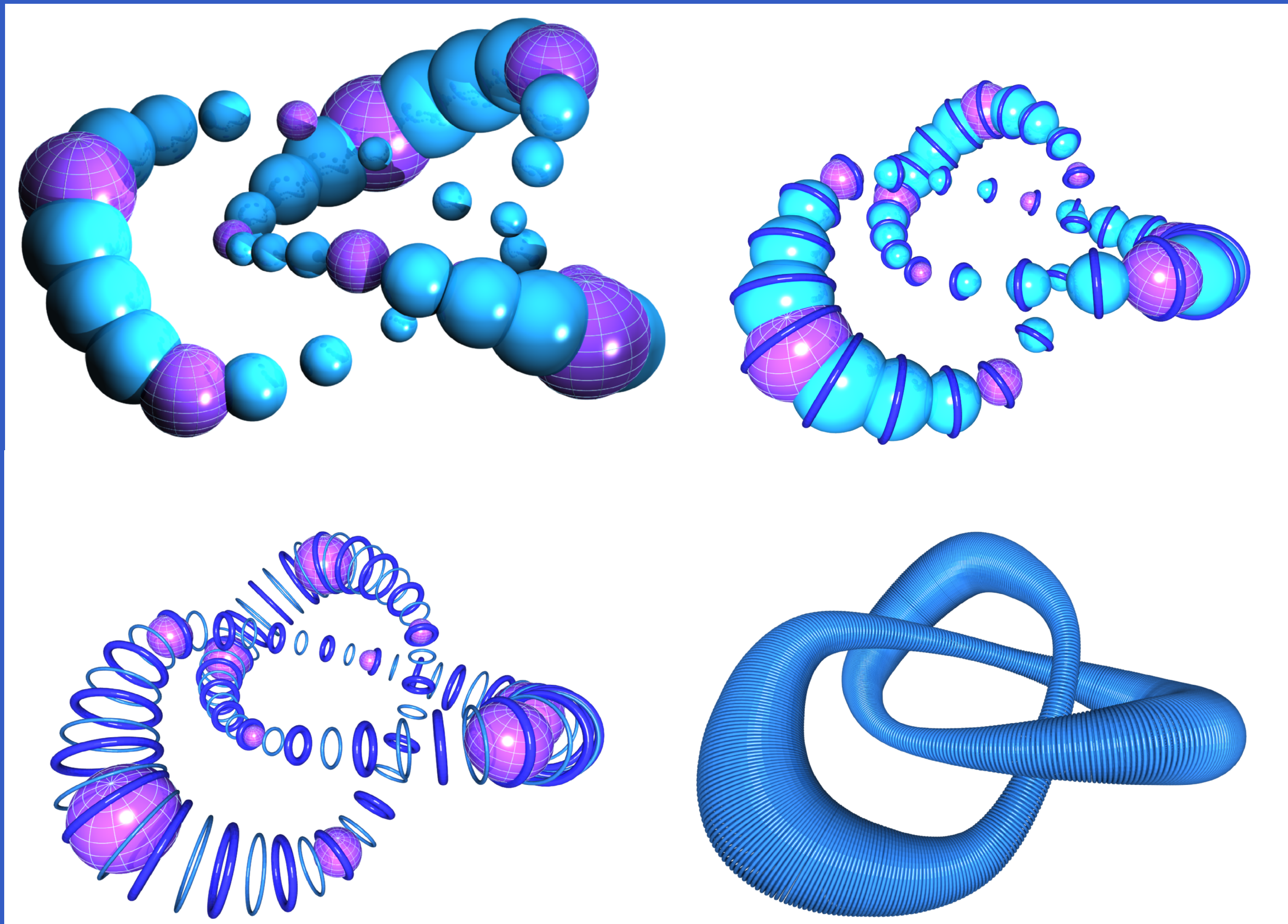
- Dyn’s scheme for the same input



characteristic circles

- use a point model for circles in Euclidean space
- circle $\mathcal{C}(A, c, r)$ (axis, center, radius) represented by $(a, \overline{a}, \alpha, r)$ with $\langle a, \overline{a} \rangle = 0$ and $c = a \times \overline{a} + \alpha a$
- circles correspond to points in a 6-dim MF in $\mathbb{R}^8$
- subdivison + projection allows to refine an initial set of characteristic circles

refining an initial set of circles



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Thank you for your attention!