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Generalized Gergonne Points

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Aims and scope

present some generalizations (old and new ones)

collect some results

formulation in a more general setting (projective geometry)

References

[BS 1988] P. Baptist, E.M. Schröder: *Merkwürdige Punkte von Dreiecken in euklidischen und Minkowskischen Ebenen. (Remarkable points of triangles in Euclidean and Minkowskian planes.)* Mitt. Math. Ges. Ham. **11**/5 (1988), 591–616.

[DG 1903] F. Dingeldey: *Kegelschnitte und Kegelschnittsysteme.* In: Encykl. d. Math. Wiss. III, C1, 94–96.

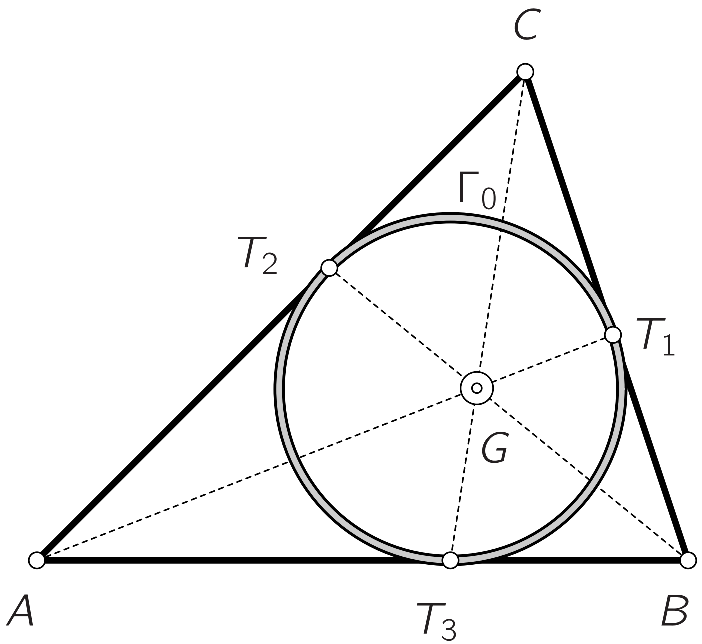
[HG 2008] M. Hoffmann, S. Gorjanc: *On the generalized Gergonne point and beyond.* Forum Geometricorum **8** (2008) 151–155.

[KM 1998] C. Kimberling: *Triangle centers and central triangles.* Congressus Numerantium, Vol. 129, Winniped, Canada, 1998.

[OD 2009] B. Odehnal: *Generalized Gergonne and Nagel points.* Geometry Preprint Series, Viennna University of Technology, Technical Report No. 197, June 2009.

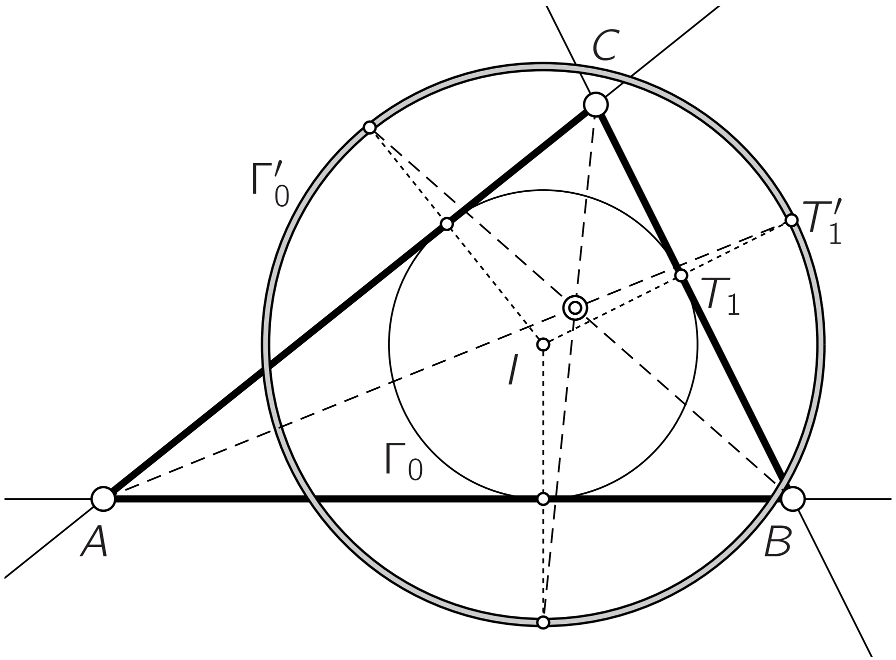
Gergonne point

$\Delta = \{A, B, C\}$ . . . triangle in Euclidean plane, Γ_0 . . . Δ's incircle



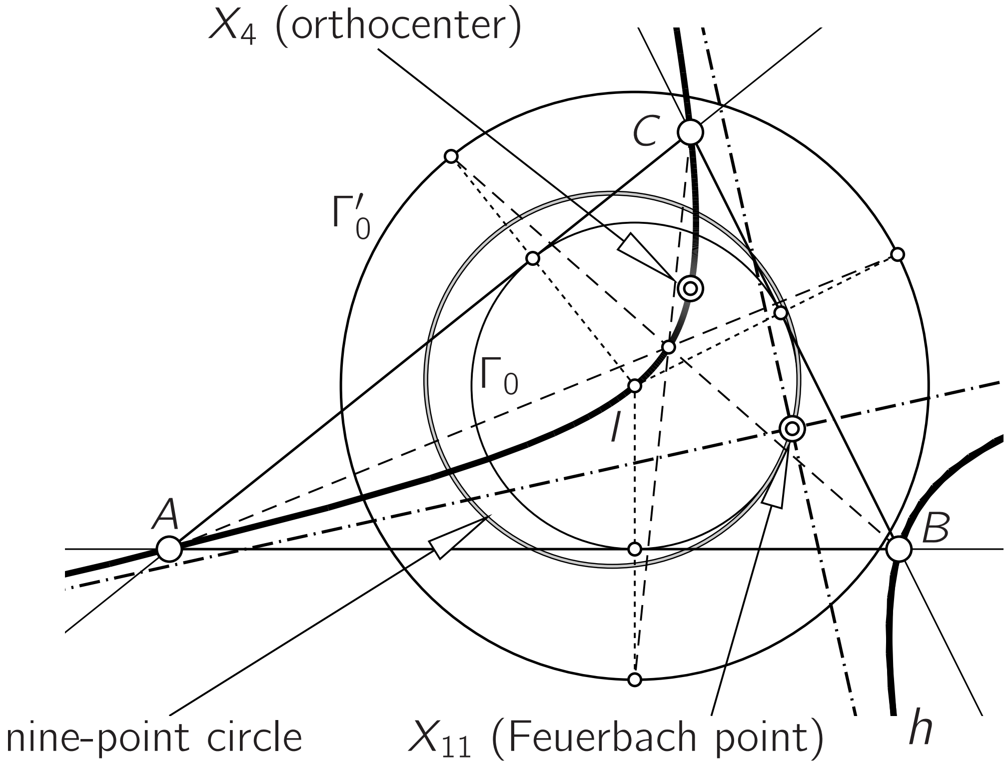
Cevians $[A, T_1]$, $[B, T_2]$, $[C, T_3]$ are concurrent in $G = X_7$.

Gergonne point (variation)



Replace incircle with a concentric one, apply scaling with center $I = X_1$ to contact points $T_i \Longrightarrow$ again concurrent cevians [HG 2008]

More Gergonne points

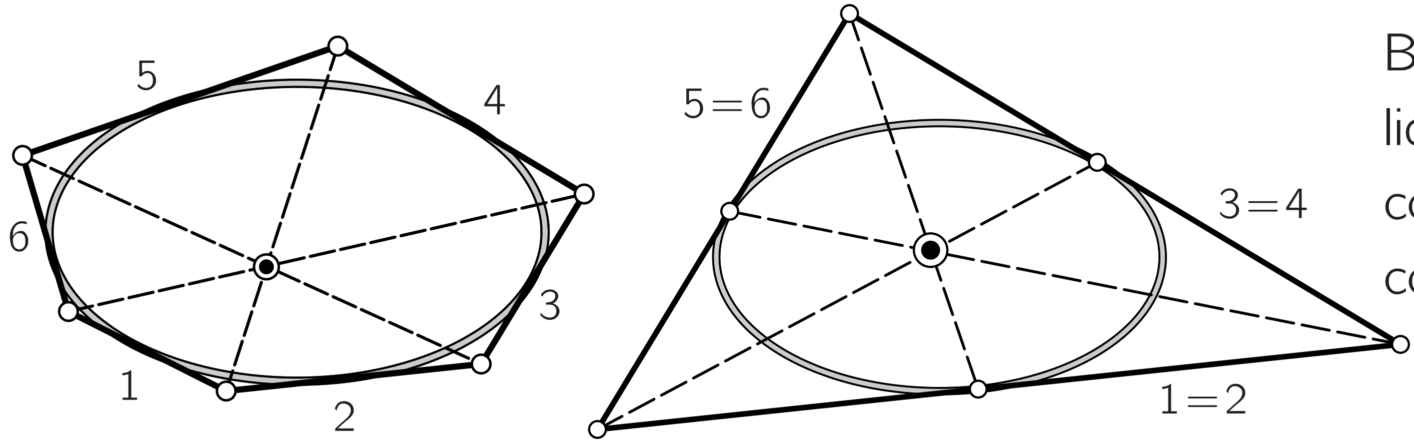


The generalized Gergonne points for any scaling factor trace an equilateral hyperbola h.

A, B, C, X_1, X_4, $\in h$

X_{11} . . . center of h

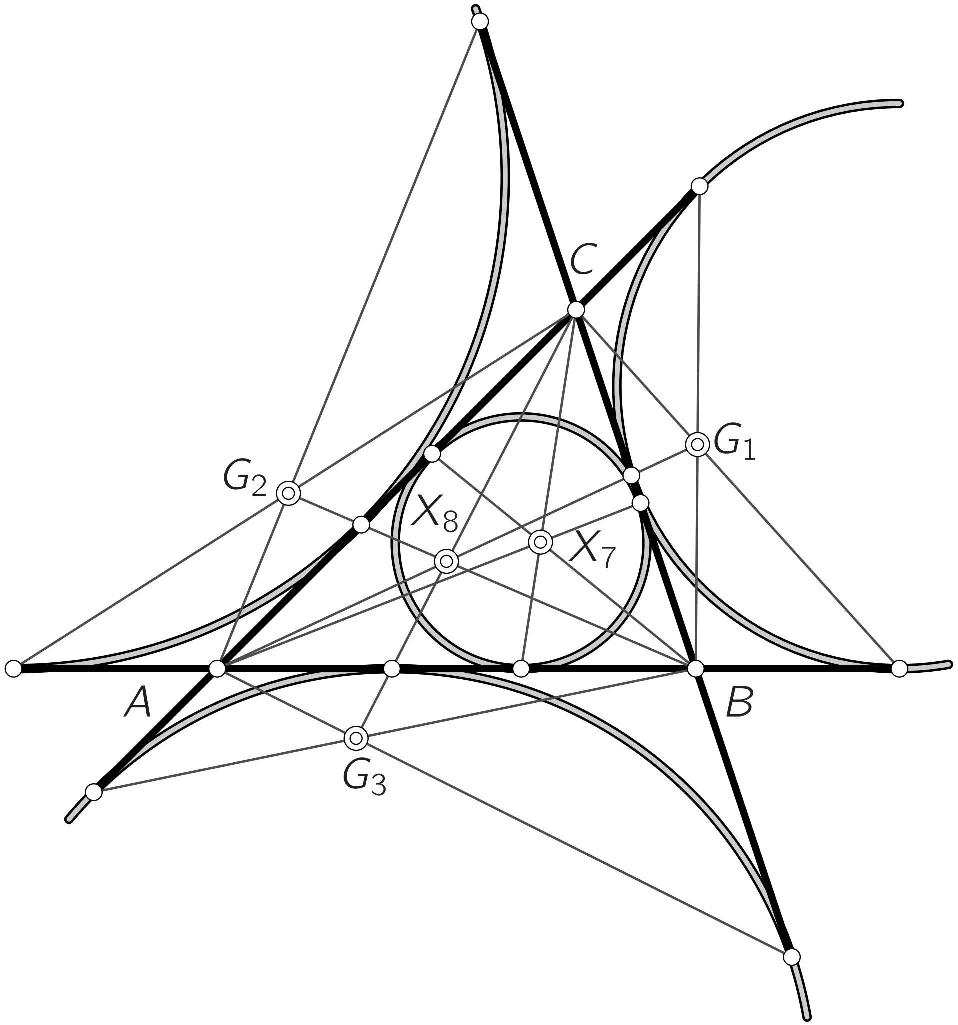
Gergonne points are Brianchon points



Brianchon's theorem is valid even for three pairs of coinciding tangents of a conic section.

$\Longrightarrow$ Any conic section tangent to the sides of Δ defines its own Gergonne point (Brianchon point).

Don't forget about the excircles!



Each excircle Γ_i determines a unique Gergonne point G_i.

X_8 (Nagel's point) = isotomic conjugate of $X_7 \Longrightarrow$ The lines

$$[A, G_1], \quad [B, G_2], \quad [C, G_3]$$

are concurrent in X_8.

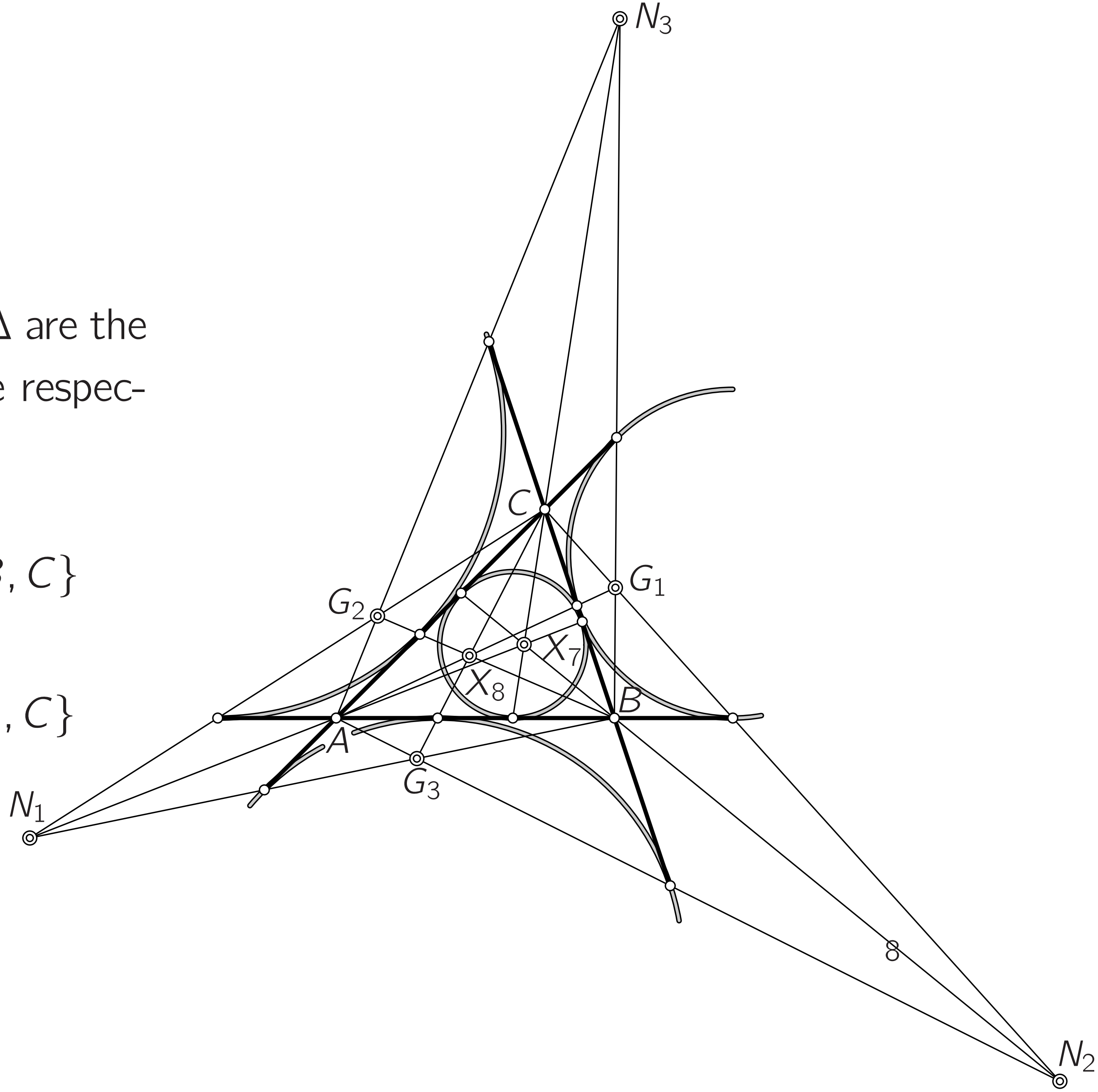
Any triangle has four Gergonne/Nagel points.

Generalized Nagel points

The **four** Nagel points of Δ are the isotomic conjugates of the respective Gergonne points.

$$\{N_1, N_2, N_3\} \overset{G}{\overline{\overline{\wedge}}} \{A, B, C\}$$

$$\{G_1, G_2, G_3\} \overset{N}{\overline{\overline{\wedge}}} \{A, B, C\}$$



More Gergonne and Nagel points

Construction of [HG 2008] applied to the excircles yields three equilateral hyperbolae h_i full of Gergonne points.

A, B, C, X_4, $I_i \in h_i$

The pencil of equilateral hyperbolae circumscribed to a triangle is well-known, see [DG 1903] (results by Poncelet, Gergonne, Steiner, etc. before 1830):

Centers are located on the nine-point circle, singular curves are (side of Δ) $\cup$ (respective altitude), . . .

Centers F_i of h_i are the vertices of the Feuerbach triangle Δ_F of Δ.

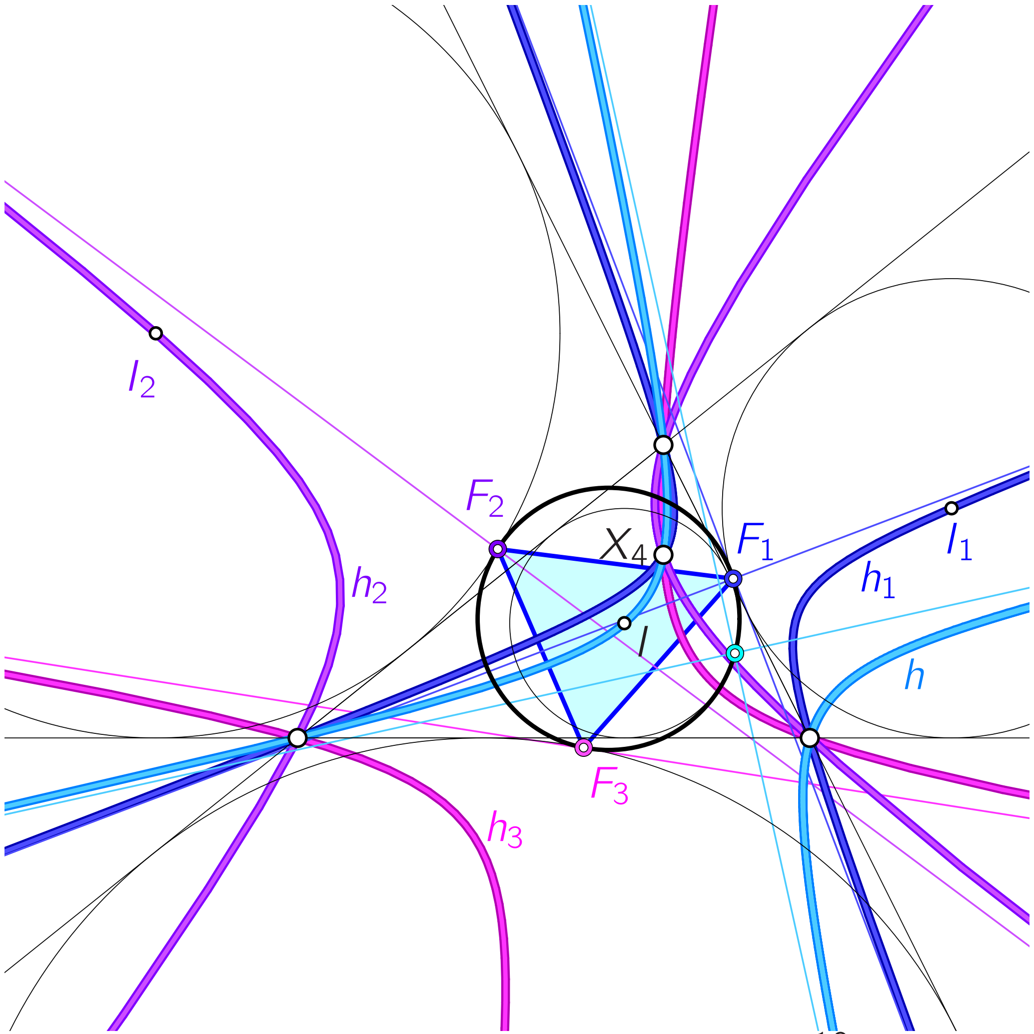
More Gergonne and Nagel points

Pencil of equilateral hyperbolae circumscribed to Δ

contains h, h_1, h_2, h_3

contains further the hyperbolae of Kiepert, Jeřabek, and Feuerbach.

Nagel points are located on lines concurrent in $\mathbf{X_{69}}$ = isotomic conjugate of $\mathbf{X_4}$ passing through $\mathbf{X_{75}}$, ...



What makes a circle?

Incircle and excircles are conic sections tangent to three lines and passing through two specific points (absolute points I, $\overline{I}$ of Euclidean geometry).

Replacing $(I, \overline{I})$ by any other pair $(U, \overline{U})$ of points (new absolute figure) Gergonne points are still well-defined (Brianchon points).

Projective version 1

Any pair of points $(U, \overline{U})$ (in admissable position) defines four Gergonne points with respect to a given triangle Δ.

admissable position: no point on any side of Δ, . . .

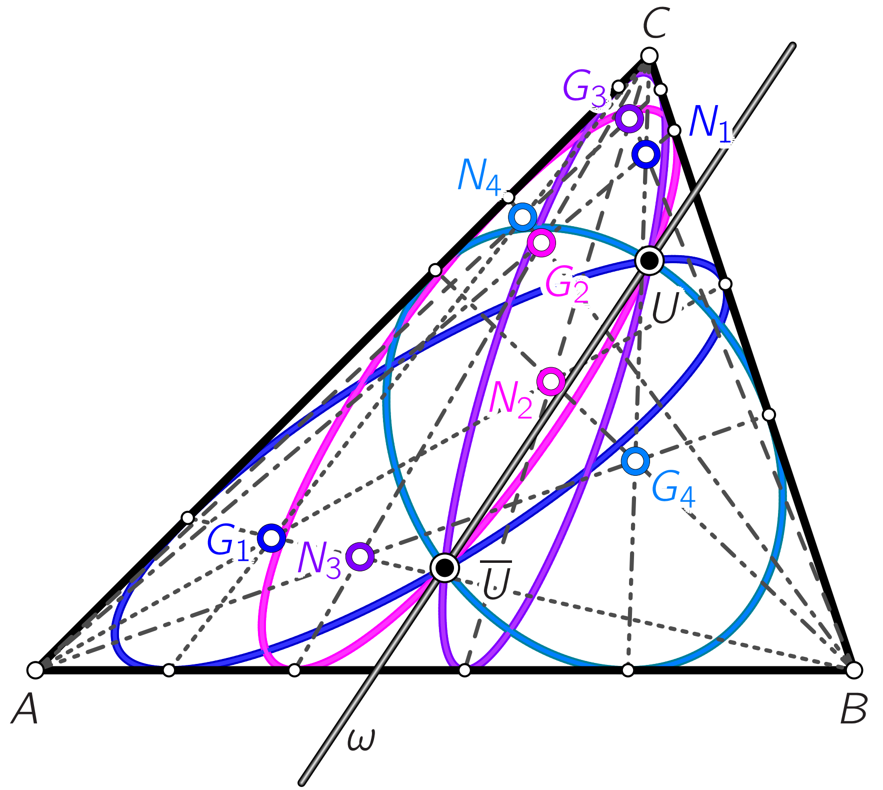
pair of real / conjugate complex points $(U, \overline{U}) \Longrightarrow$

pseudo-Euclidean (Minkowskian) / Euclidean geometry (ideal line $\omega := [U, \overline{U}]$)

isotomic mapping can be formulated in a projective invariant way $\Longrightarrow$

Any pair of points $(U, \overline{U})$ (in admissable position) defines four Nagel points with respect to a triangle Δ.

Gergonne and Nagel points together



Projective version 2

U, $\overline{U}$ are fixed points of an involutive mappipng $\iota:\ \omega \to \omega \Longrightarrow$

Any pair of points $(U, \overline{U})$ / any involutive mapping $\iota:\ \omega \to \omega$ defines four Gergonne and four Nagel points with respect to a triangle $\Delta = \{A, B, C\}$.

ι elliptic $\Longrightarrow$ Euclidean geometry

ι hyperbolic $\Longrightarrow$ pseudo-Euclidean (Minkowskian) geometry

admissable position:

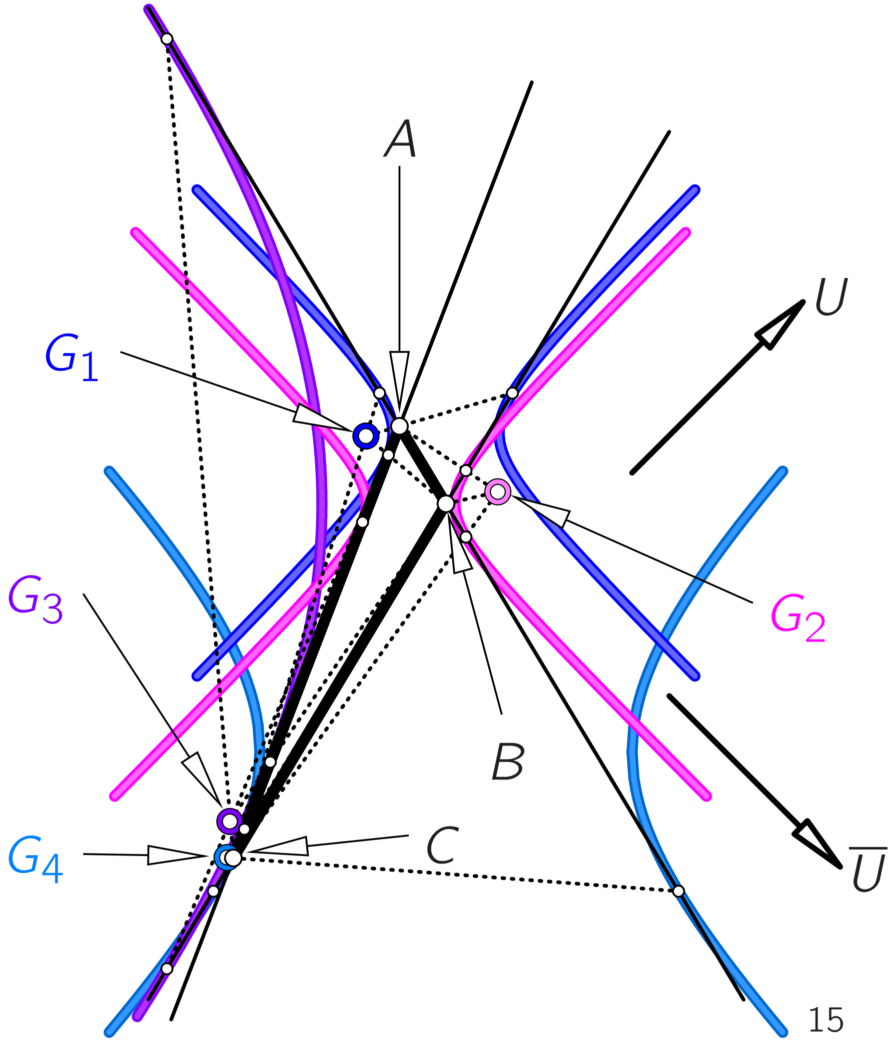
$\omega \neq [A, B], [B, C], [C, A]$,

ι's fixed points not on Δ's sides

Minkowskian version

No surprise: All *four* Gergonne points also appear in Minkowskian plane.

[BS 1988] used results from affine Geometry to find the Minkowskian version of (only one) Gergonne point.



Darboux's cubic and nine-point circle

P . . . Point $\notin [A, B] \cup [B, C] \cup [C, A]$

P_{AB} . . . P's orthogonal projection to $[A, B]$

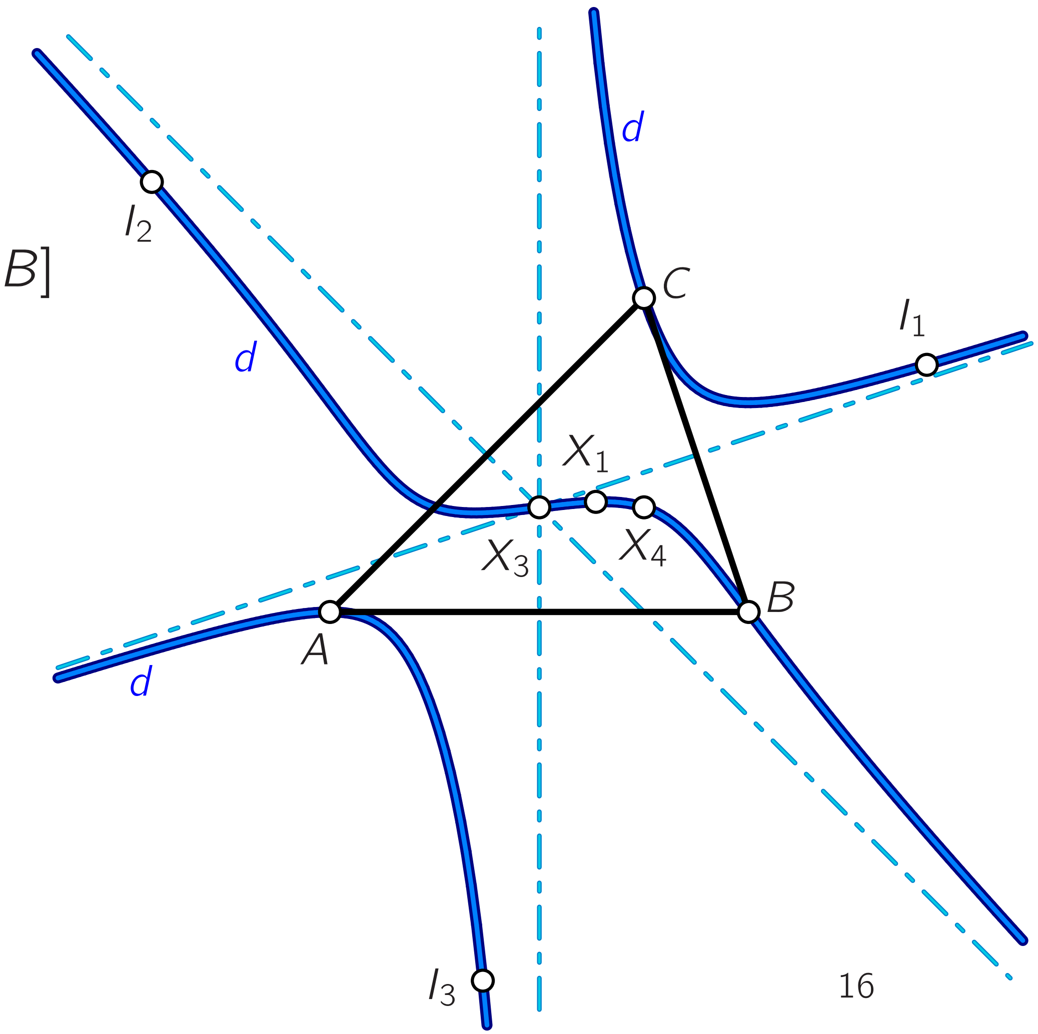
Darboux's cubic d

locus of points P such that

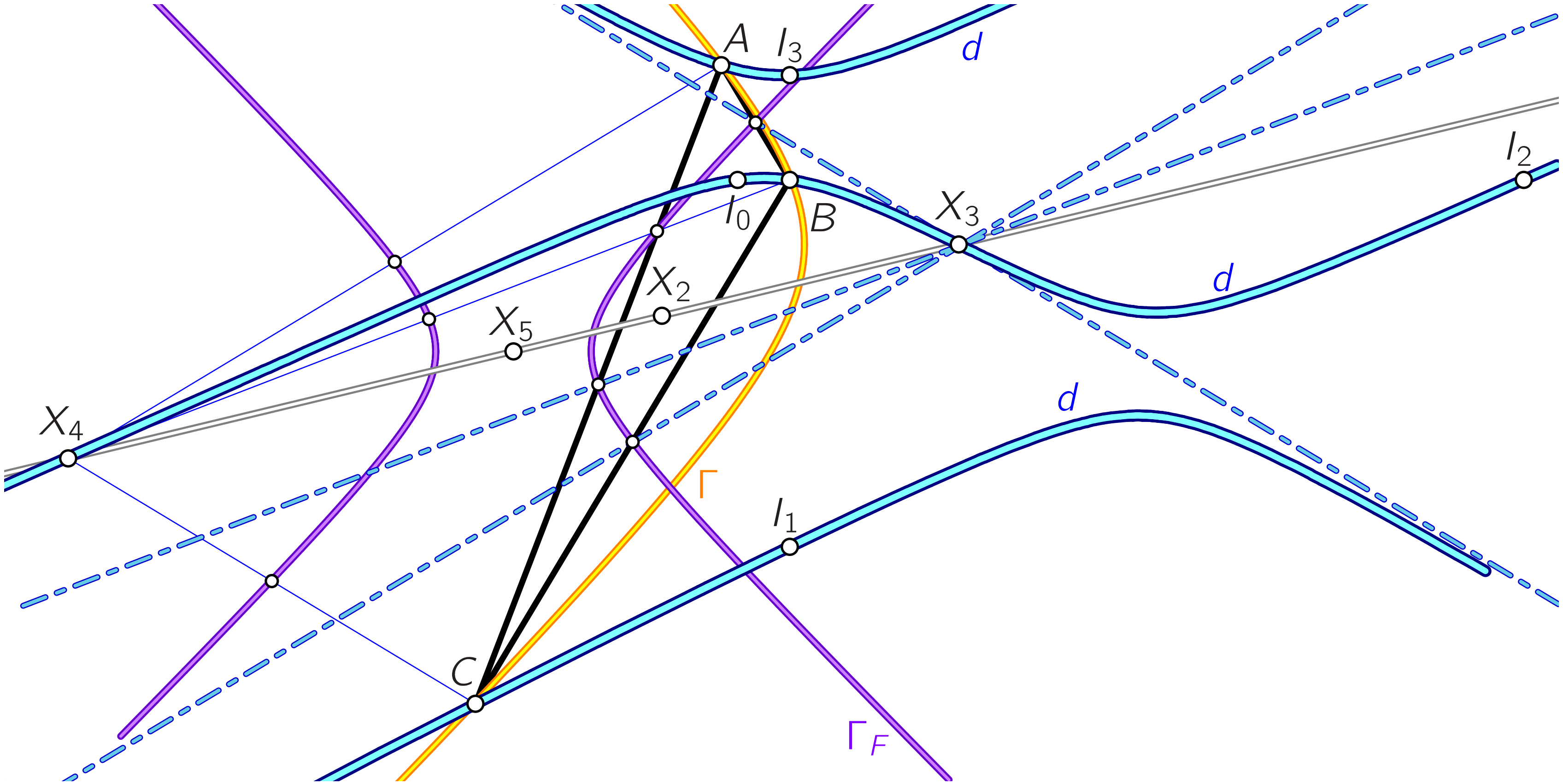
$$[C, P_{AB}], \quad [A, P_{BC}], \quad [B, P_{CA}]$$

are concurrent

$A, B, C, X_1, X_3, X_4, I_1, I_2, I_3,$
$AB^\perp, BC^\perp, CA^\perp \in d$



Darboux's cubic and nine-point circle



Thank You for your attention!

Who is who among triangle centers?

X_1	incenter
X_2	centroid
X_3	circumcenter
X_4	orthocenter
X_5	nine-point center
X_7	Gergonne point
X_8	Nagel point
X_{11}	Feuerbach point
X_{69}	isotomic conjugate of X_4, symmedian point of anticomplementary triangle
X_{75}	isotomic conjugate of X_1
X_{181}	perspector of Δ_F and Δ

labelling of triangle centers according to [KM 1998]