On the geometry of spherical trochoids

Walther Jank, Georg Glaeser, Boris Odehnal

University of Applied Arts Vienna

overview

- Walther Jank
- spherical trochoids, various generations
- a result from planar kinematics
- kinematic generation of (top) views
- curves of constant slope
- special shapes
- curves of constant width

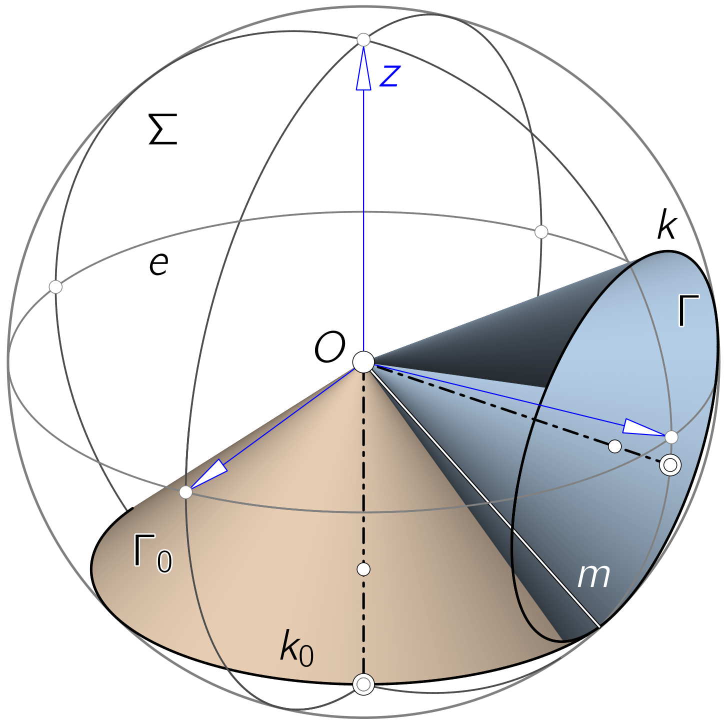
Walther Jank

born on June 19th, 1939 in Vienna
Mag. rer. nat., Vienna Univ. of Technology, 1965
Dr. phil. in Mathematics, Univ. of Vienna, 1971
Assistant, 2nd Inst. of Geometry,
Vienna Univ. of Technology, 1967–1978
Habilitation in Constructive Geometry and Kinematics,
Vienna Univ. of Technology, 1978
Senior Assistant 1978–1988
Assistant Prof., 1988–1997; Associate Prof., 1997–2003
at the Vienna Univ. of Technology
retired in 2004, died on April 9th, 2016 in Vienna

teaching & research:

Advanced Descriptive Geometry, Kinematics, Wallpaper and Crystallographic Groups; Classical Geometry and its Applications

spherical trochoids - generation



spherical trochoid motion is one of the following:

- rolling of a circle k on a circle k_0 (both in a sphere Σ)
- rolling of two concentric (not coaxial) cones of revolution
- rolling of a sphere on two coaxial circles

Γ_0, Γ . . . axodes

$m = \Gamma \cap \Gamma_0$ . . . instantaneous axis

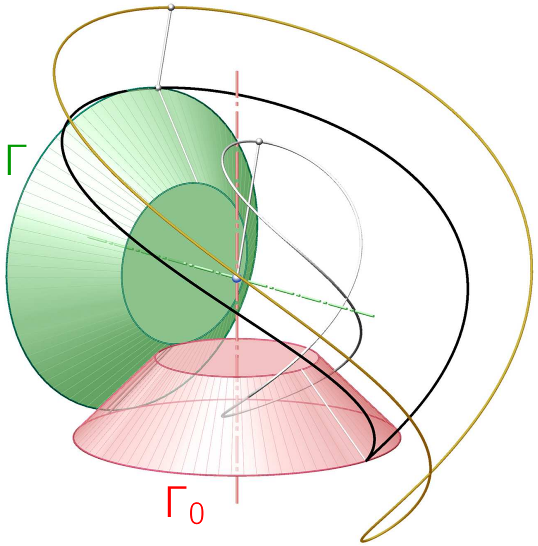
e . . . equator of Σ, henceforth considered horizontal

z . . . z-axis, lead direction,
direction of top-view projection

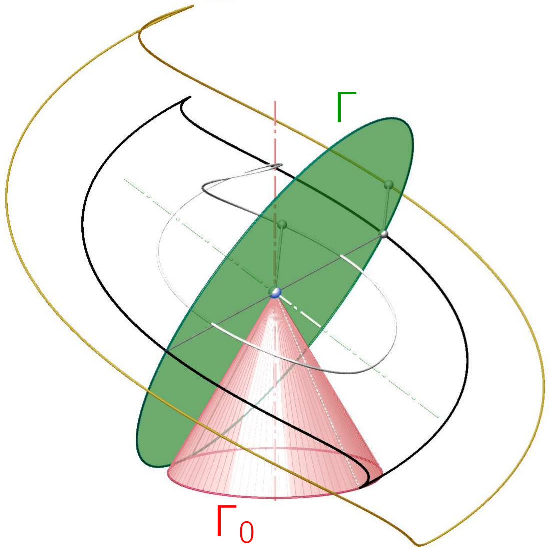
This holds true for all spheres concentric with Σ.

spherical trochoids - special cases

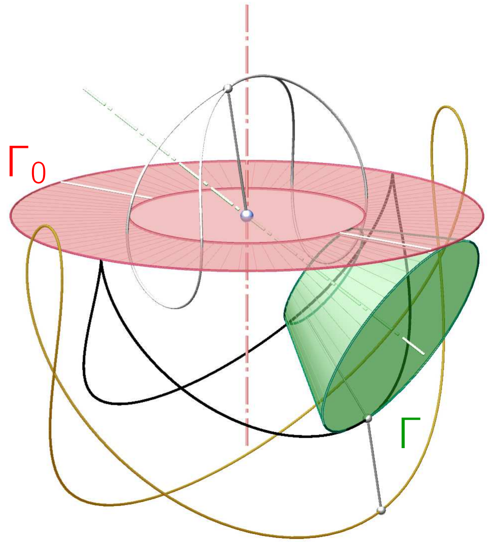
Special choices of the semi-apertures ω_0 / ω_1 of the polhodes Γ_0 / Γ cause special shapes of the spherical trochoids.



symmetric rolling
$\omega_0 = \omega_1$



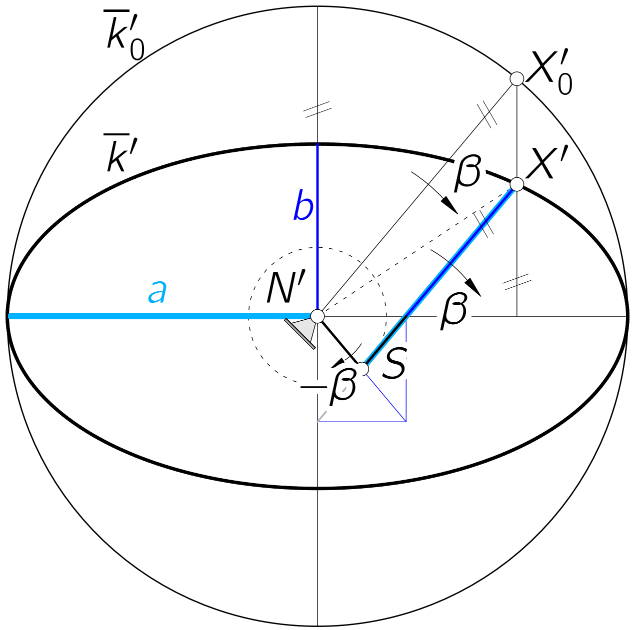
spherical involute
$\omega_1 = \frac{\pi}{2}$



spherical cycloid
$\omega_0 = \frac{\pi}{2}$

an important ingredient from planar kinematics

crank slider mechanism $N'SX'$, derived from the paper strip constr. of an ellipse $\overline{k}'$



$\overline{k}'$ . . . ellipse

center N', $\frac{1}{2}$-major axis length a, moving point $X' \in \overline{k}'$

crank slider $N'SX'$:

$-\beta$ / β = angular velocities of $N'S$ / SX' (w.r.t. $\overline{k}'$)

$\overline{k}'_0$ = circumcircle (affine image) of $\overline{k}'$

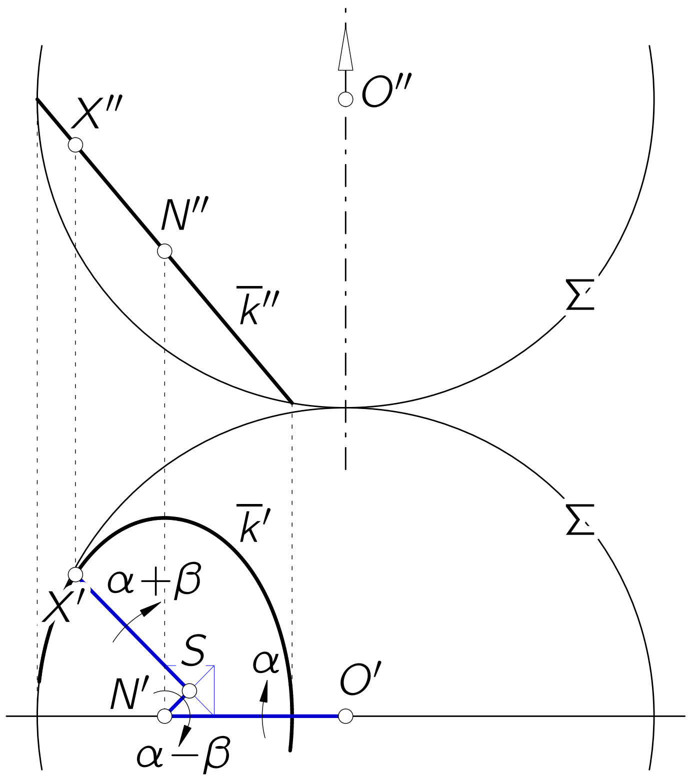
Thm.:

Any 2 out of the following 3 statements are equivalent:

- $\beta = \text{const.}$
- $N'X'_0$ rotates with constant angular, and therefore, also constant area velocity (with regard to $\overline{k'_0}$).
- $N'X'$ rotates with constant area velocity with respect to $\overline{k'}$.

top views of spherical trochoids

top view = orthogonal projection in the direction of the lead



Thm.:

The top-view l' of a spherical trochoid l is (in general) a trochoid of order 3.

Proof: $\overline{k}'$ rotates with angular velocity α about O'.

$\alpha = \text{const.} \Longrightarrow NX$ rotates with const. angular and area velocity w.r.t. $\overline{k}$

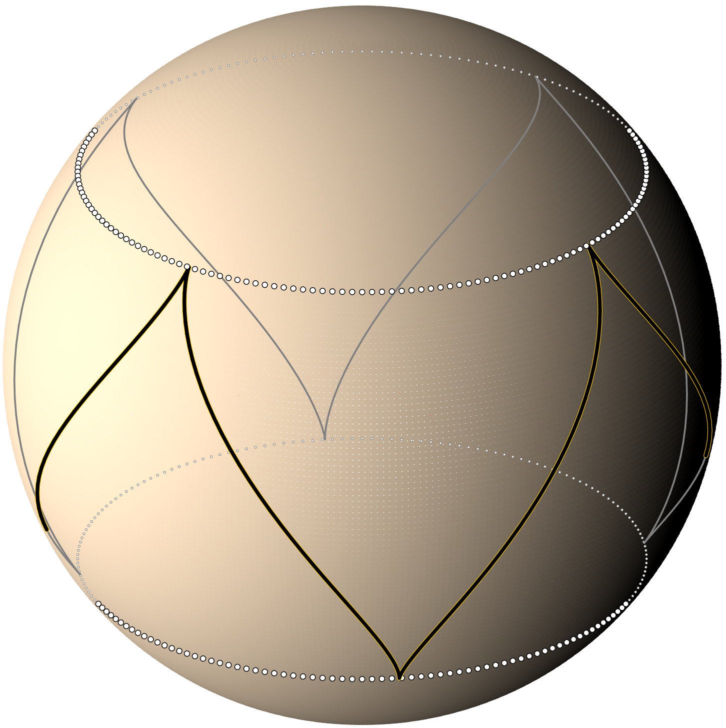
$\Longrightarrow N'X'$ rotates with constant area velocity (w.r.t. $\overline{k}'$).

$\exists$ the affine mapping between the ellipse and its circumcircle

$\Longrightarrow N'X'$ rotates with constant area velocity $-\beta$ w.r.t. $\overline{k}'$. $\Longrightarrow N'X'$ moves with constant and absolute angular velocity $\alpha - \beta(\alpha + \beta)$.

$\Longrightarrow$ in general: l' = trochoid of order three, characteristic $\alpha : \alpha - \beta : \alpha + \beta$

curves of constant slope on ellipsoids of revolution



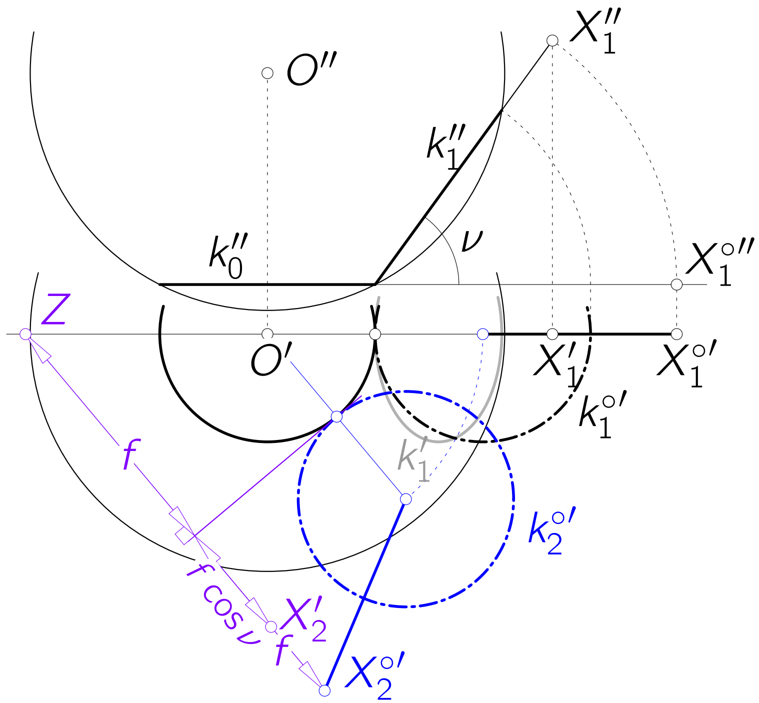
The synthetic approach towards spherical trochoids delivers a synthetic proof of Enneper's theorem:

Thm.:

The top views of the curves of constant slope on an ellipsoid of revolution with (its axis parallel to the lead) are epicycloids (cycloidal curves of order 2).

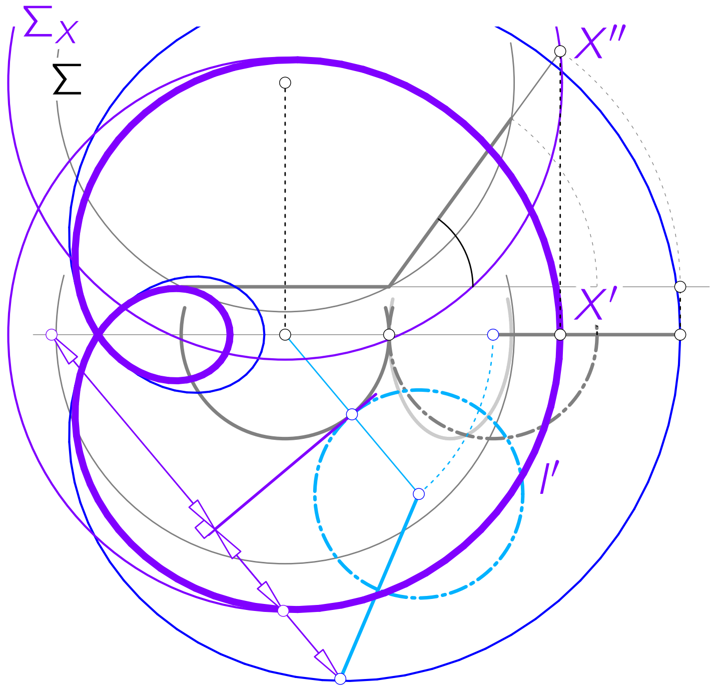
Here, $\omega_1 = \frac{\pi}{2} \Longrightarrow l$ is a spherical involute and l' is only formally of order 3.

special case – symmetric rolling



flipped top view: $\Longrightarrow$ symmetric rolling

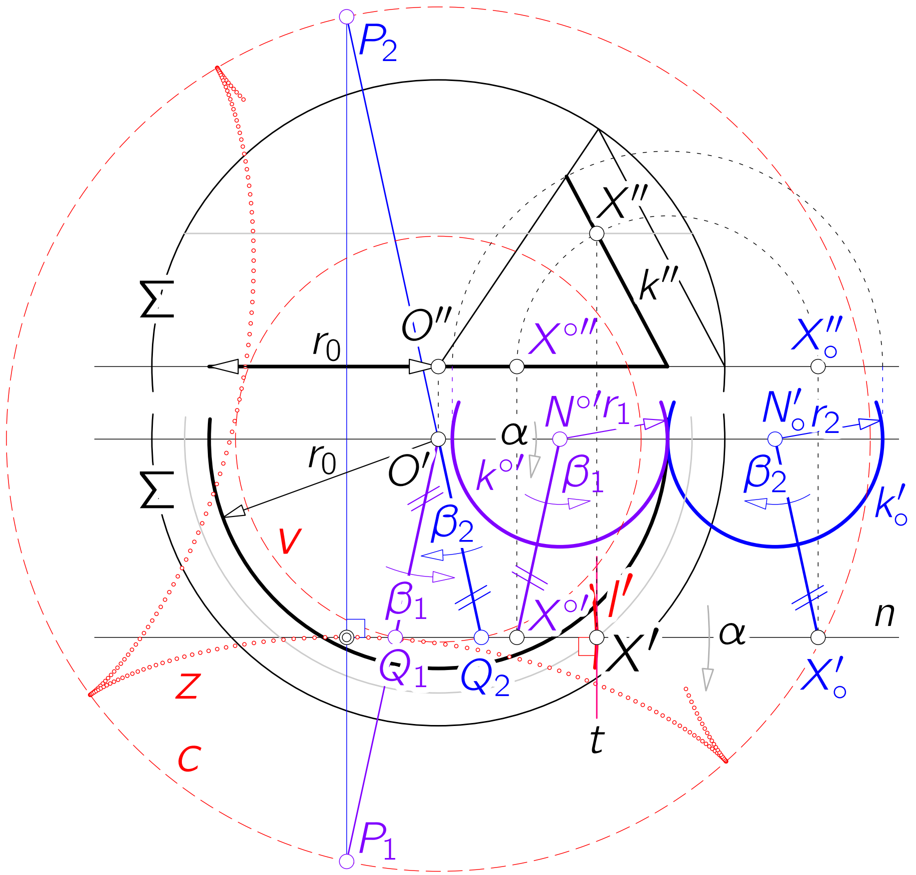
$l^{\circ\prime}$ is a central similar copy of l'



trace of $X^{\circ\prime}$, l' are both Pascal limaçons.

$X \notin k$ moves on Σ_X concentric with Σ.

Top views of spherical trochoids can be generated as involutes of cycloids or as offset of a similar involute.



flipping k to both sides: k_0 (outside), k^0 (inside)

parallelograms $O'N^{\circ\prime}X^{\circ\prime}Q_1$, $O'N'_\circ X'_\circ Q_2$

$r_0 > 0,\ 0 < r_1 < r_0,\ r_1 = -r_2$

$O'N^{\circ\prime}N'_\circ$ rotates with $\alpha =$ const.

$\Longrightarrow$ $O'Q_i$ rotates with $\beta_i =$ const. $\beta_i \in \{1, 2\}$

$\Longrightarrow$ double generation of a hypozycloid z

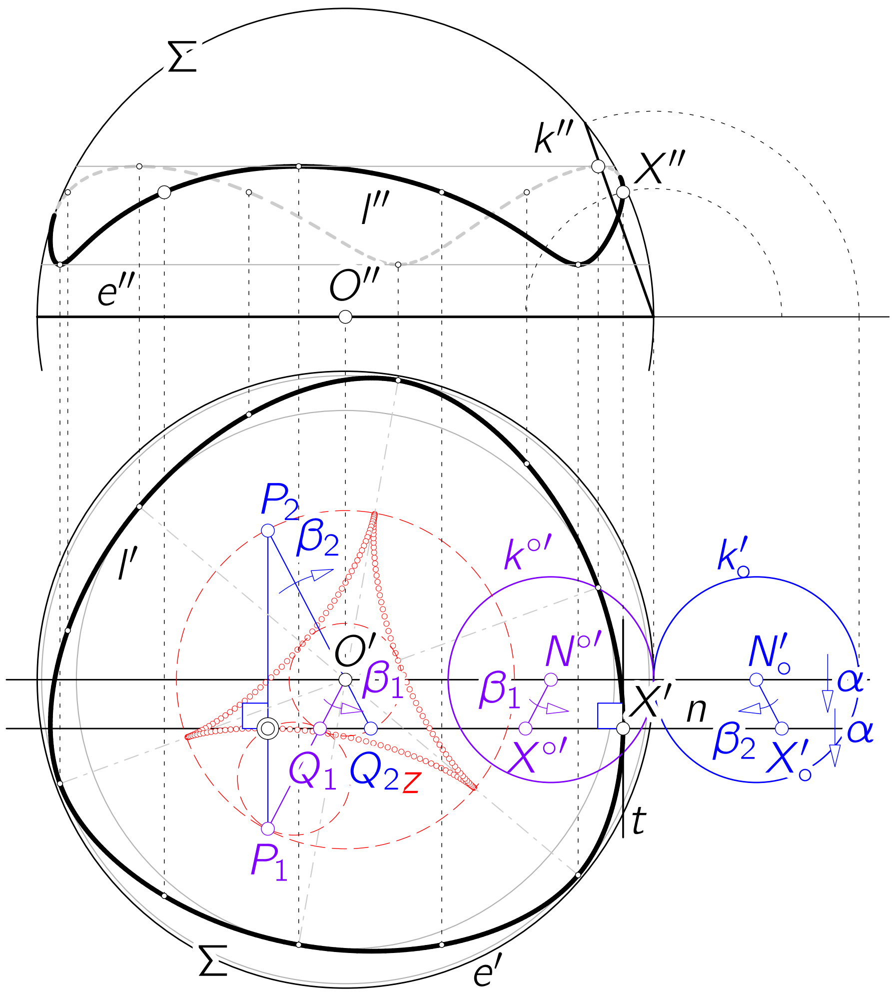
as the envelope of $n = [Q_1, Q_2 X^{\circ\prime} X' X'_\circ]$

acc. Thm. of $\perp$: top view $O'N^{\circ\prime}N'_\circ$ (inst. axis)

is orthogonal to l' at X'

$\Longrightarrow$ l' is an involute of z (or offset of a similar involute)

curves of constant width



three-fold symmetry $\Longrightarrow$ z is a Steiner cycloid

z lies completely in the interior of l' $\Longrightarrow$ l' is a closed convex (analytic, and even rational) curve of constant width (like all involutes of z).

J. W.'s question:

How to describe the totality of closed convex (analytic) curves of constant width?

possible (partial) answer:

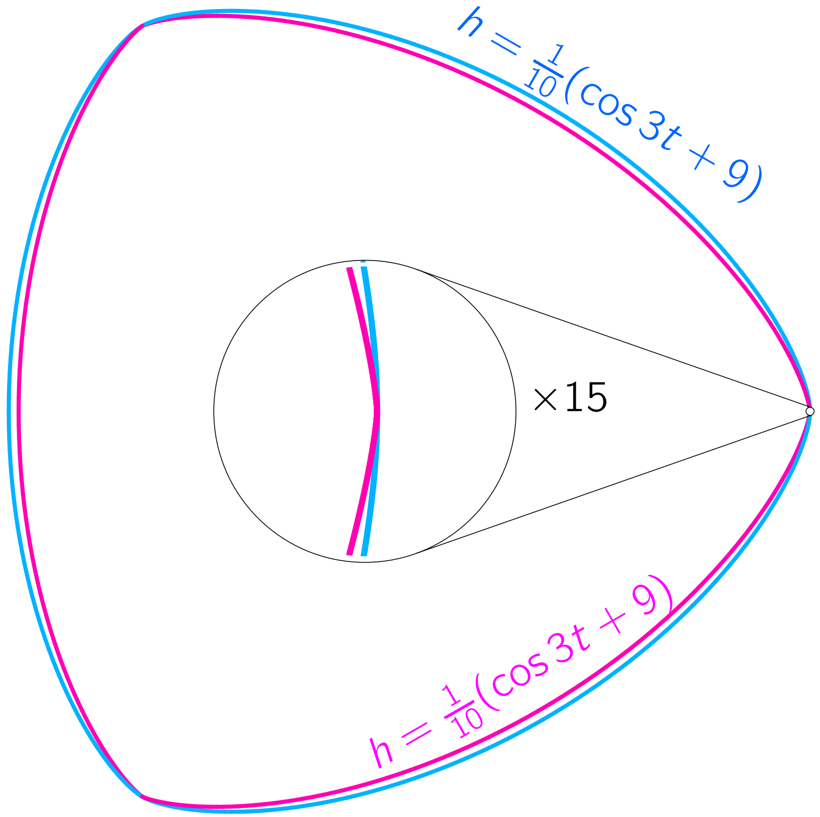
Describe the curve by the support function $h: \mathrm{S}^1 \to \mathbb{R}$ and solve

$$h(t) + h(t+\pi) = 0,$$

$$\dot{h}(t) + \dot{h}(t+\pi) = 0,$$

$$h(t) - h(t+2\pi) = 0.$$

curves of constant width



the other one = curve with $h = \frac{1}{9}(\cos 3t + 8)$
curve from [14]

Jank's top view = curve with $h = \frac{1}{10}(\cos 3t + 9)$
curve from [12] improving the
curve from [14], *i.e.*, moving
the 3 cusps of the third kind

$\Longrightarrow$ curve of constant width without visible real singularities on the real branch!

some references

[1] R. Bereis: *Über sphärische Radlinien.* Wiss. Z. Techn. Hochschule Dresden **7** (1957/58), 841–844.
[2] E. Brieskorn, H. Knörrer: *Ebene algebraische Kurven.* Birkhäuser, Basel-Boston-Stuttgart, 1981.
[3] J.L. Coolidge: A treatise on algebraic plane curves. Dover Publications, New York, 1959.
[4] K. Fladt: *Analytische Geometrie spezieller ebener Kurven.* Akademische Verlagsgesellschaft, Frankfurt am Main, 1962.
[5] L. Gao, Z. Zhang, F. Zhou: *An extension of Rabinowitz's polynomial representation for convex curves.* Beitr. Algebra Geom. **61**/3 (2020), 455–464.
[6] M. Husty, A. Karger, H. Sachs, W. Steinhilper: *Kinematik und Robotik.* Springer-Verlag, Berlin, 1997.
[7] H.M. Jeffery: *On spherical cycloidal and trochoidal curves.* Quart. J. Pure Applied Math. **XIX** (1883), 44–66.
[8] E. Kruppa: *Analytische und konstruktive Differentialgeometrie.* Springer, Wien, 1957.
[9] J.D. Lawrence *A catalog to plane algebraic curves.* Birkhäuser, Boston, 2000.
[10] G. Loria: *Spezielle algebraische und transzendente ebene Kurven.* B.G. Teubner, Leipzig-Berlin, 1911.
[11] B. Odehnal, H. Stachel, G. Glaeser: *The universe of quadrics.* Springer, Wien, 2020.
[12] C. Panraksa, L.C. Washington: *Real algebraic curves of constant width.* Period. Math. Hung. **74** (2017), 235–244.
[13] H. Pottmann: *Zum Satz von Holditch in der euklidischen Ebene.* Elem. Math. bf 41/1 (1986), 1–6.
[14] S. Rabinowitz: *A polynomial curve of constant width.* Missouri J. Math. Sci. **9**/1 (1997), 23–27.
[15] D. Rochera: *Algebraic equations for constant width curves and Zindler curves.* J. Symbolic Comp. **113**, Nov.-Dec. 2022, 139–147.
[16] W. Ströher: *Raumkinematik.* Unpublished manuscript, TU Wien, 1973.
[17] R.L. Tennison: *Smooth Curves of Constant Width.* Math. Gazette **60**/114 (1976), 270–272.
[18] H. Wieleitner: *Spezielle ebene Kurven.* Sammlung Schubert LVI, Göschen'sche Verlagshandlung, Leipzig, 1908.
[19] W. Wunderlich: *Höhere Radlinien.* Österr. Ingen. Archiv **1** (1947), 277–296.
[29] W. Wunderlich: *Ebene Kinematik.* Bibliographisches Institut, Mannheim, 1970.

Thank You For Your Attention!