A Miquel-Steiner Transformation

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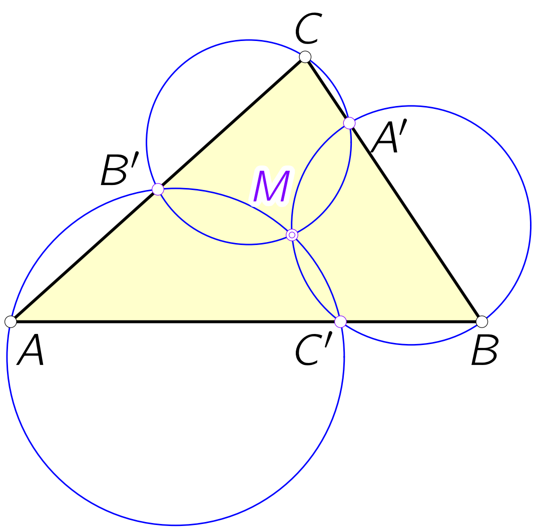
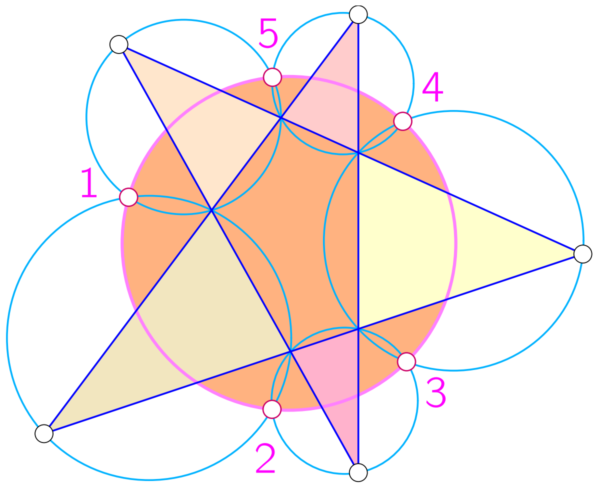
hopefully within 15 minutes

- recalling some Miquel theorems
- defining "the" Miquel-Steiner transformation μ
- properties, exceptional sets, inverse
- images of particular lines, concics, cubics under μ
- aiming at geometric meaning of some entries of Kimberling's *Encyclopedia of Triangle Centers* and Gibert's *Catalogue of Triangle Cubics*

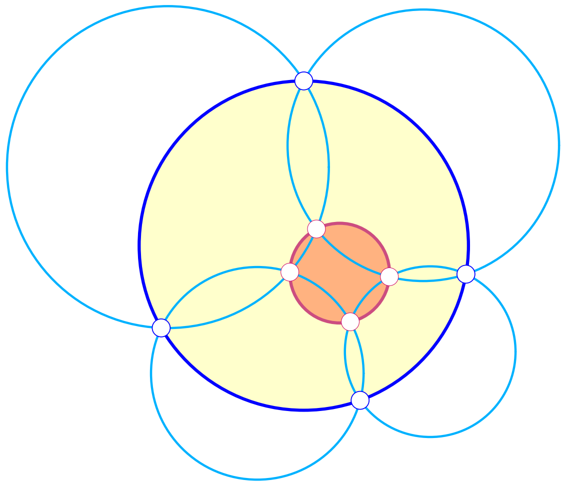
These are not the Miquel theorems I'm talking about!



Three circles meet in one point M independent of the choice of A', B', C'.

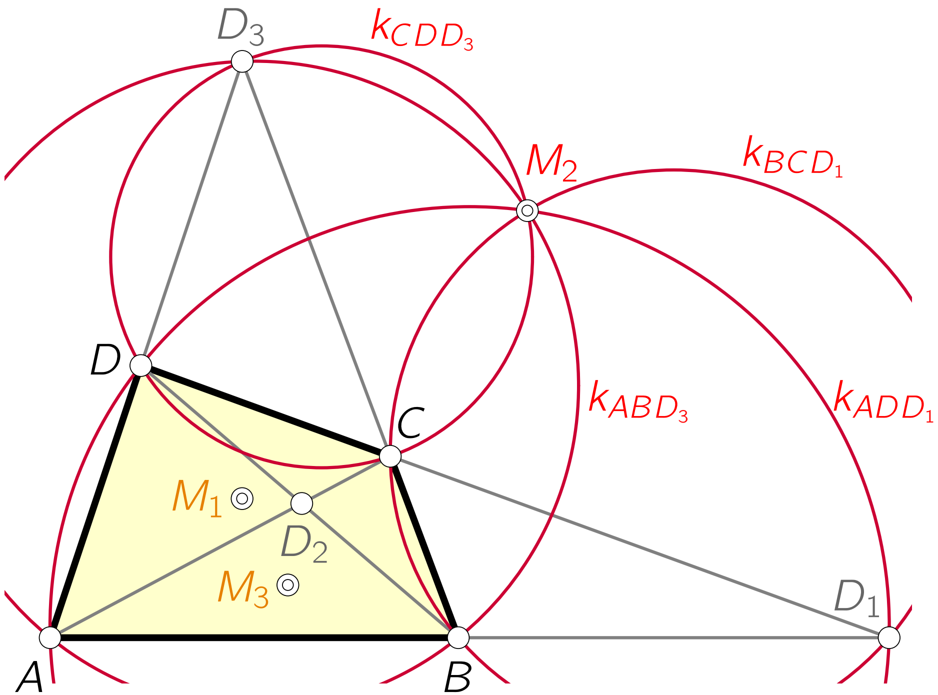


Miquel's *5 Circles Theorem*: five concyclic points: 1, . . . , 5



Miquel's *6 Circles Theorem*: four circles share triple points $\Longrightarrow$ remaining 4 intersections concyclic

Miquel-Steiner Theorem



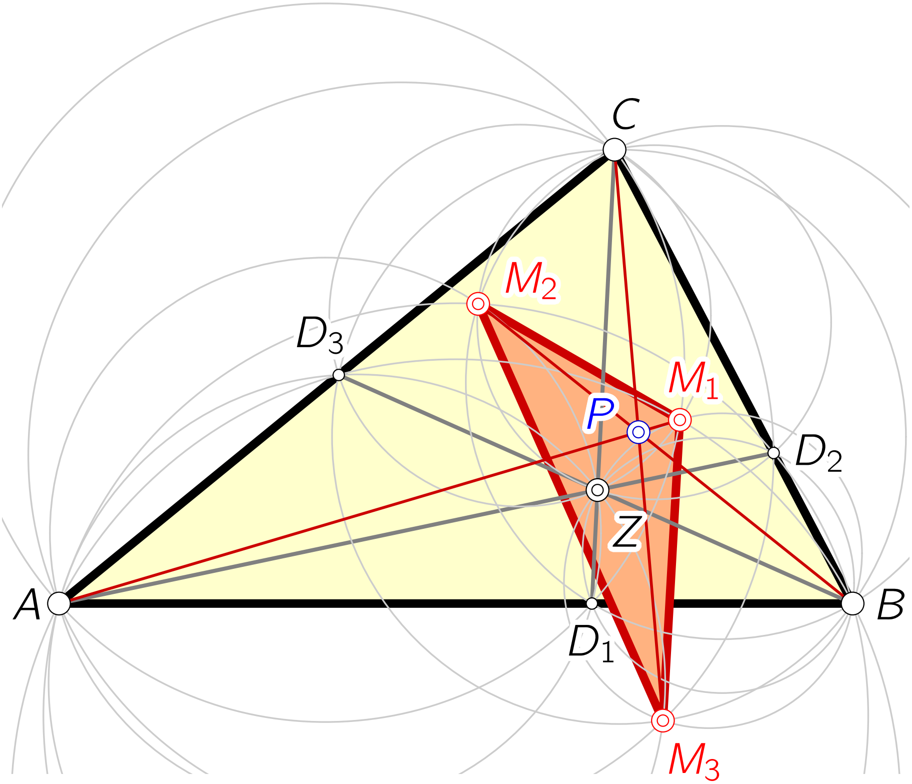
Each pair of diagonal points in a complete quadrilateral defines a Miquel point as the meet of four circumcircles of certain subtriangles:

e.g.: $M_2 \in k_{ABD_3}, k_{CDD_3}, k_{ADD_1}, k_{BCD_1}$.

Any triangle $\Delta = ABC$ together with any admissible point Z defines 3 Miquel points M_i.

$$M_1M_2M_3 =: \Delta_M \overset{P}{\doublebarwedge} \Delta$$

"the" Miquel-Steiner transformation



Miquel-Steiner transformation

$$\mu : \ Z \mapsto P$$

μ is a quadratic Cremona transformation:

$$\xi : \eta : \zeta \mapsto [bc(b\eta + c\zeta)]^{-1} : \ldots : \ldots$$

μ is not involutive!

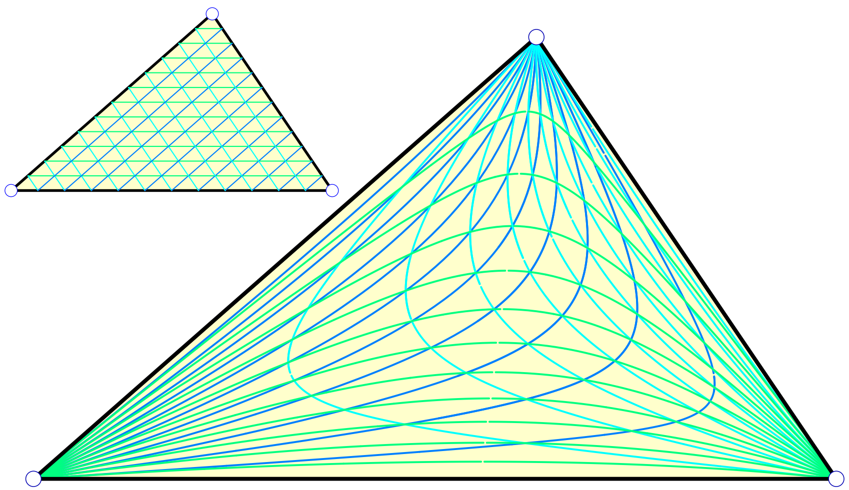
μ's exceptional triangle $= \Delta_a$

(anticomplementary / -medial triangle of Δ)

μ leaves X_4 (orthocenter of Δ) fixed.

μ maps triangle centers to triangle centers.

“the” Miquel-Steiner transformation



μ = composition of isogonal conjugation (after)
central similarity with center X_2 (centroid of Δ)
with scaling factor -2

$$\mu(\xi,\eta,\zeta) = \iota\left(\underbrace{\begin{pmatrix} 0 & b^2c & bc^2 \\ a^2c & 0 & a^2c \\ a^2b & ab^2 & 0 \end{pmatrix}}_{=:\mathbf{T}}\begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix}\right),$$

where $\iota(\xi',\eta',\zeta') = \xi'^{-1} : \eta'^{-1} : \zeta'^{-1}$

fixed point / center

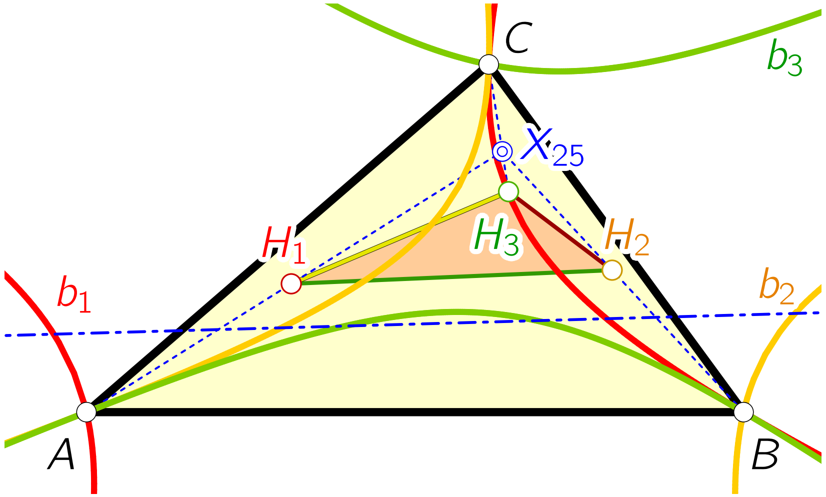
$$C = \mathbf{T}\cdot X_2 = 2abc\cdot X_2,$$

fixed line / axis = ideal line ω

$$\mathbf{T}\cdot\underbrace{\begin{pmatrix} b & c \\ -a & 0 \\ 0 & -a \end{pmatrix}}_{\ker(\ldots)=\omega} = -abc\underbrace{\begin{pmatrix} b & c \\ -a & 0 \\ 0 & -a \end{pmatrix}}_{\ker(\ldots)=\omega},$$

characteristic crossratio $= \mathrm{cr}(C,[C,P]\cap\omega,P,\mathbf{T}\cdot P) = -2 \quad \forall P \neq C,\ P \notin \omega$

Cremona transformation: An inverse must exist!



inverse μ^{-1}

$\xi:\eta:\zeta \mapsto bc(-a\eta\zeta + b\zeta\xi + c\xi\eta) : \ldots : \ldots$

μ^{-1}'s exceptional triangle = Δ

μ^{-1}'s leaves X_4 (orthocenter of Δ) fixed.

base conics (coordinate functions of μ^{-1})

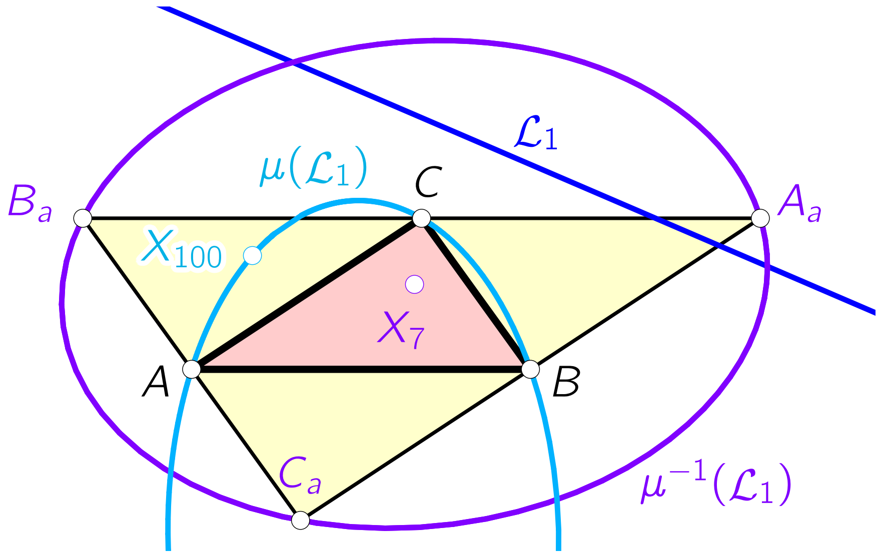
= hyperbolae h_i with centers H_i and

$$H_1H_2H_3 =: \Delta_H \overset{X_{25}}{\overline{\overline{\wedge}}} \Delta$$

X_{25} = homothetic center of Δ_o and Δ_t

action of μ and μ^{-1} – I

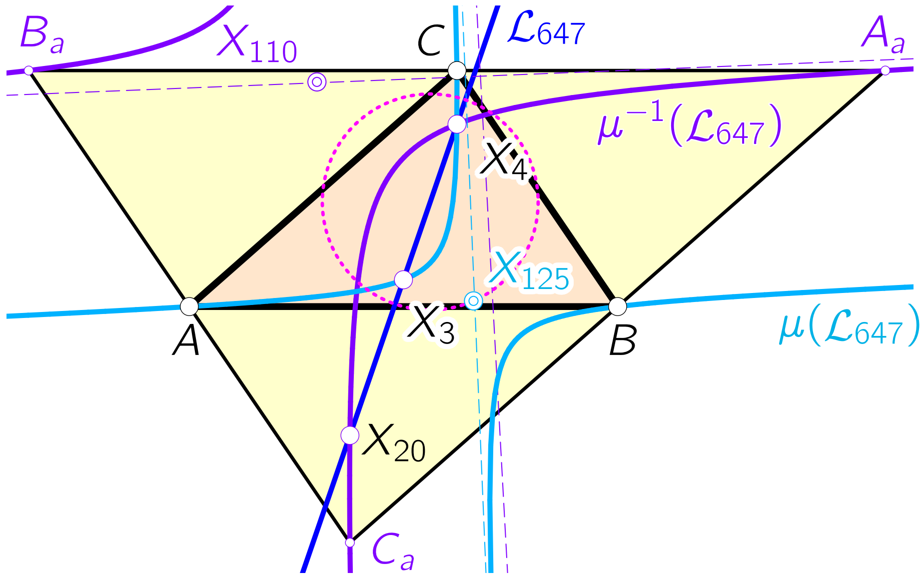
antiorthic axis $\mathcal{L}_1$



$\mu^{-1}(\mathcal{L}_1)$ is centered at X_7 (Gergonne point of Δ)

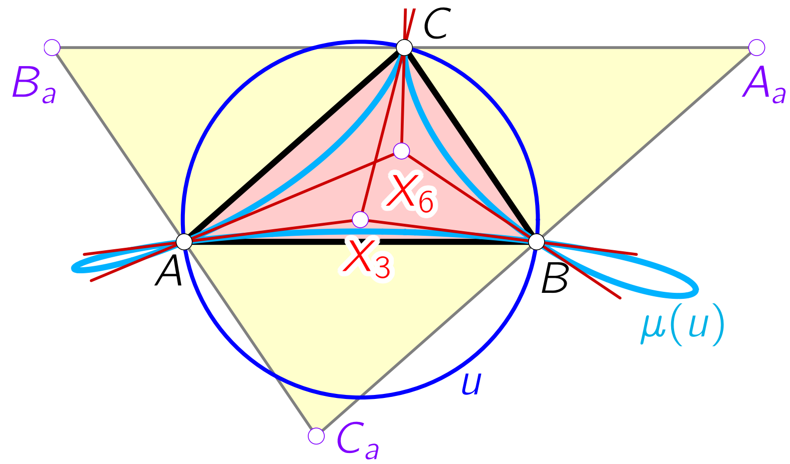
$\mu(\mathcal{L}_1)$ carries (among others) X_{100} (anti-complement of the Feuerbach point)

Euler line $\mathcal{L}_{647}$



$\mu^{-1}(\mathcal{L}_{647})$ and $\mu(\mathcal{L}_{647})$: equilateral hyperbolae and parallel asymptotes with centers X_{110} (focus of the Kiepert parabola) and X_{125} (center of the Jeřabek hyperbola)

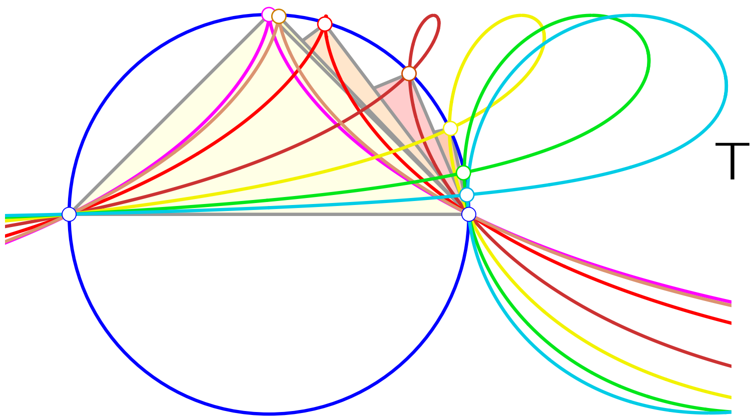
action of μ and μ^{-1} – II



circumcircle $u \mapsto$ quartic $\mu(u)$

with double points at A, B, C

and tangents through X_3 and X_6



$\mu(u)$ becomes cubic

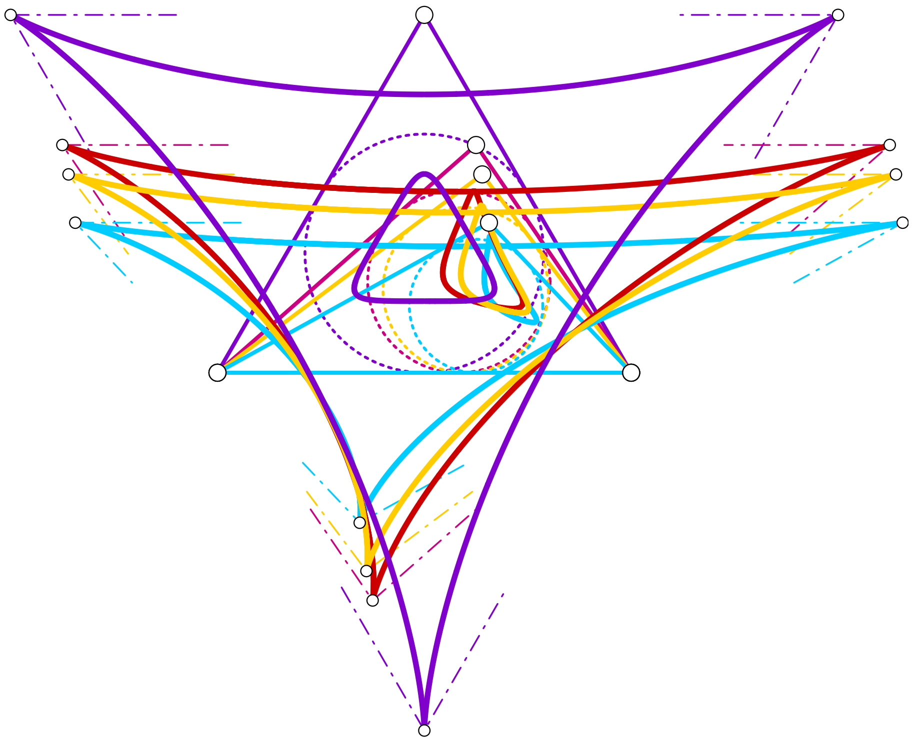
if Δ becomes right

The one (and only) double point becomes a cusp

if Δ becomes equilateral (and right).

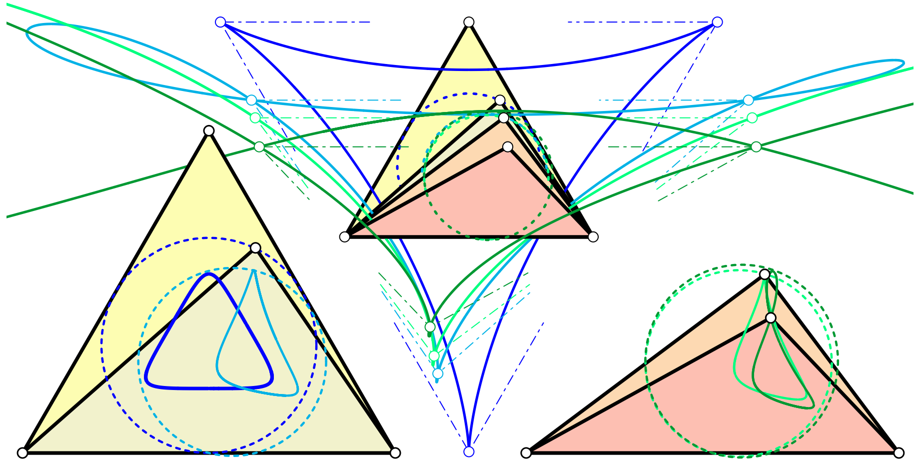
Line opposite to $\perp$ splits off.

action of μ and μ^{-1} – III

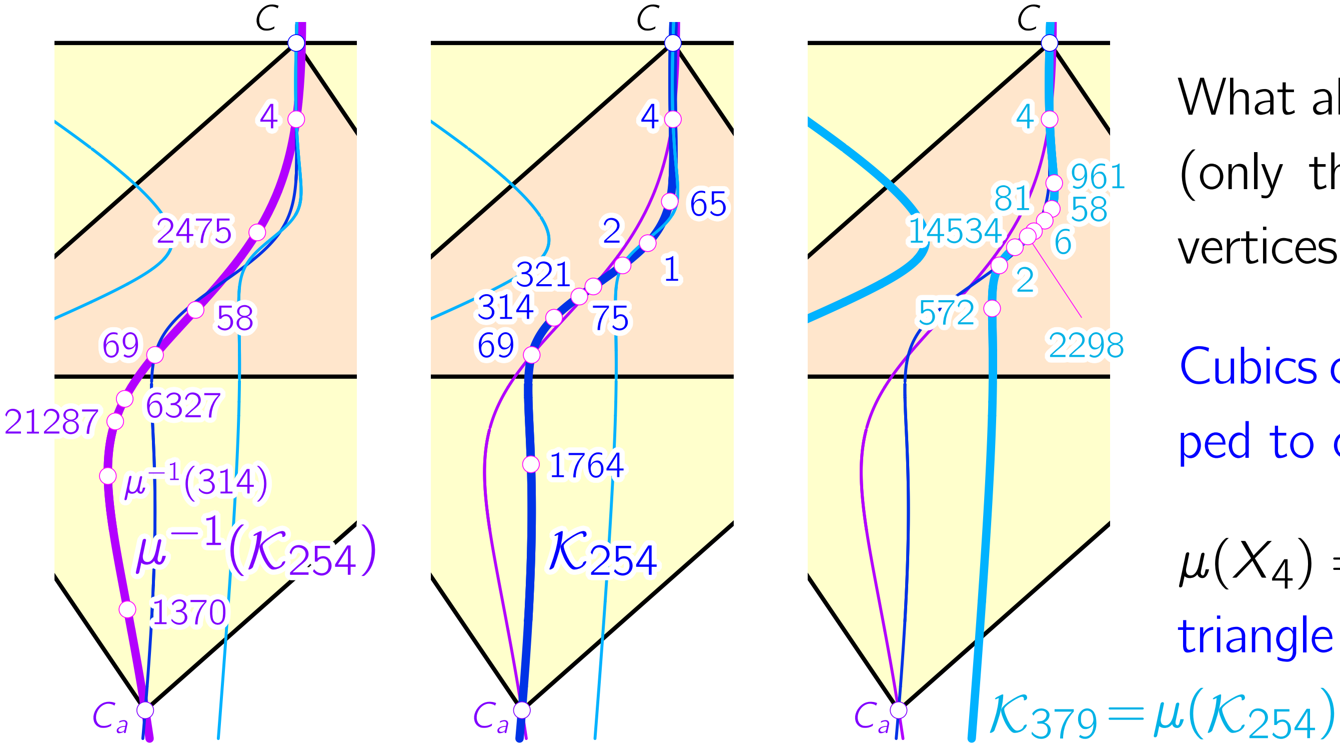


$\leftarrow$ incircles and their μ- and μ^{-1}-images

$\downarrow$ nine-point circles and their μ- and μ^{-1}-images



action of μ and μ^{-1} – IV



What about some (triangle) cubics? (only those that pass through the vertices of Δ_a)

Cubics circumscribed to Δ_a are mapped to cubics circumscribed to Δ.

$\mu(X_4) = X_4 \implies \exists$ 5-pm. family of triangle cubics with cubic images.

$\mathcal{X} = \mathcal{K}_{254}$	corresponding triangle centers X_i on these curves															
$\mu^{-1}(\mathcal{X})$	8	69	4	2475	1370	6327	75	?	?	?	?	?	?	?	?	?
$\mathcal{X}$	1	2	4	65	69	75	81	314	321	1764	2995	2869	16049	17751	20028	
$\mu(\mathcal{X}) = \mathcal{K}_{379}$	58	6	4	961	2	81	1169	14534	2298	572	19607	21	1798	1220		41364

action of μ and μ^{-1} – V

known cubics with
known cubic images

$\mathcal{K}_i$	$\mu(\mathcal{K}_i) = \mathcal{K}_j$
7	2
8	273
80	361
141	644
170	233
254	379
311	454
355	380
449	447
611	1172
617	28
753	73
1000	354
1002	135
1037	1013
1053a,b	1145a,b
1131	1134

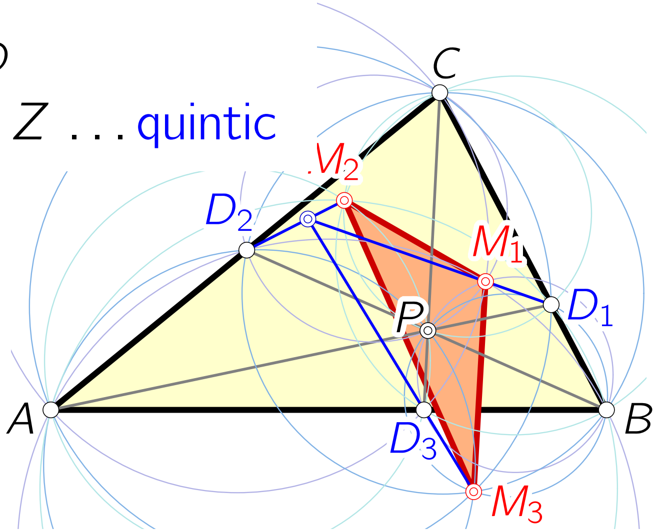
known cubics whose
images are only determined by triangle centers

$\mathcal{K}_i$	centers X_j on $\mu(\mathcal{K}_i)$
45	2, 4, 6, 54, 275, 1993, 8882, 34756
92	?
133	69, 6223, 21279, 52365
144	6, 54, 74
146	20, 69, 11442
154	69, 962, 3434, 3430, 52366
240	69, 316, 512, 3448, 14360, 53365
242	6, 110, 1174, 1379, 1380, 8115, 8116, 15460, 15461
279	2, 4, 6, 30, 323, 2986, 5504, 10419, 14910, 15262
347	6,
371	6
380	4, 6, 251, 1976, 2065
455	1, 6, 35, 37 , 57, 1126, 1171, 1255, 21353, 33635
548	76, 2896
605	6, 58, 63, 81, 284, 2287, 7123, 40403
659	6, 32, 83, 251, 51951
860	6, 15, 16, 74, 40384
985	6, 58, 81, 291, 1922, 2311, 7132, 24479, 38810, 38813
1004	6, 163, 2149,
1078	1, 6, 56, 57, 266, 289, 1743

This is by no means the end of the story!

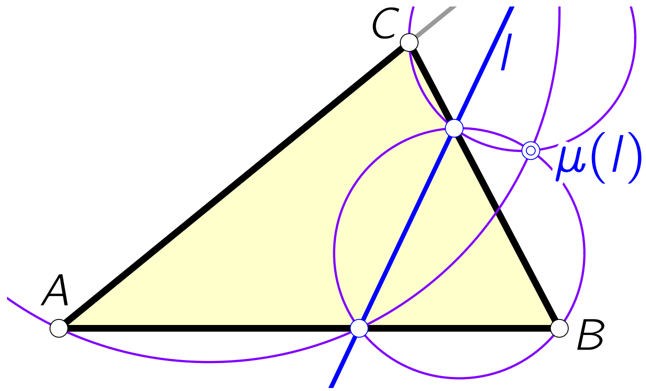
$\Delta_M \overset{Z}{\doublebarwedge} \Delta_D$

$\mu: \ P \mapsto Z \ldots$ quintic

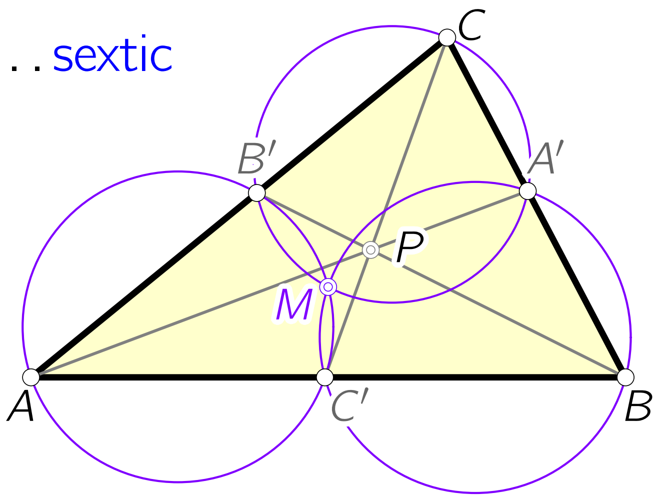


$l \mapsto \mu(l) \ldots$ quartic

$x : y : z \mapsto ayz(bx - ay)(cx - az) : \ldots : \ldots$



$P \mapsto M \ldots$ sextic



$\longleftarrow$ replace $\Delta_C(P) = A'B'C'$ by

$\Delta_\delta = A''B''C''$ with $\mathrm{cr}(A, B, C', C'') = \delta, \ldots$

$M^\delta = k_{AB''C''} \cap k_{A''BC''} \cap k_{A''B''C}$

$P \mapsto M^\delta \ldots$ sextic

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Thank You For Your Attention!