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Hermite interpolation of ruled and channel surfaces

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rough sketch of the talk

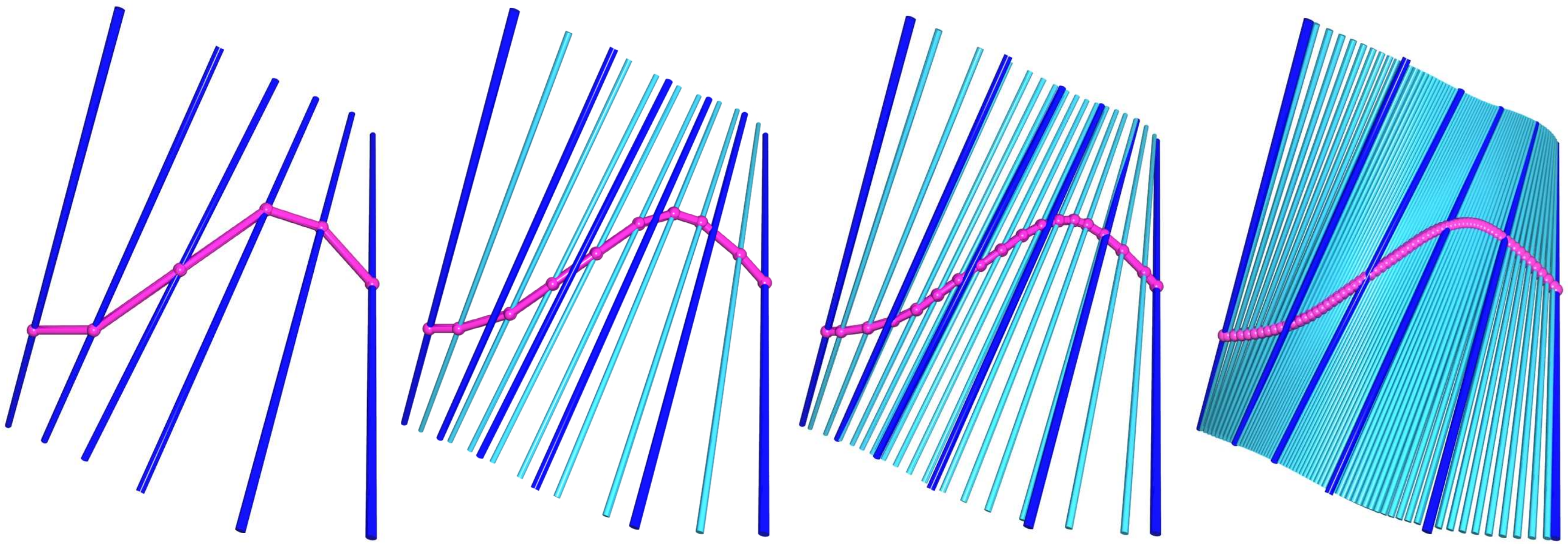
ruled surfaces / channel surfaces	|	What is done so far?
line and sphere geometry	|	basic facts, Plücker's and Lie's quadric
Bézier curves within quadrics	|	why Bézier curves?
techniques and aims	|	low degree interpolants
computations, algorithms, results	|	and some examples
discussion	|	problems

What is done so far?

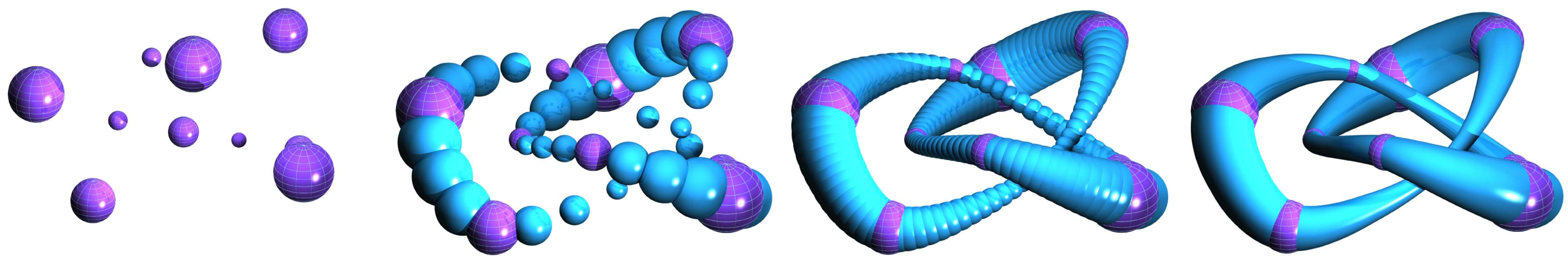
for ruled surfaces - for channel surfaces - numerically - algebraically

ruled surfaces / channel surfaces – What is done so far?

(1) interpolatory (and approximating) subdivision schemes for ruled surfaces [11]



(2) interpolatory (and approximating) subdivision schemes for channel surfaces [12]



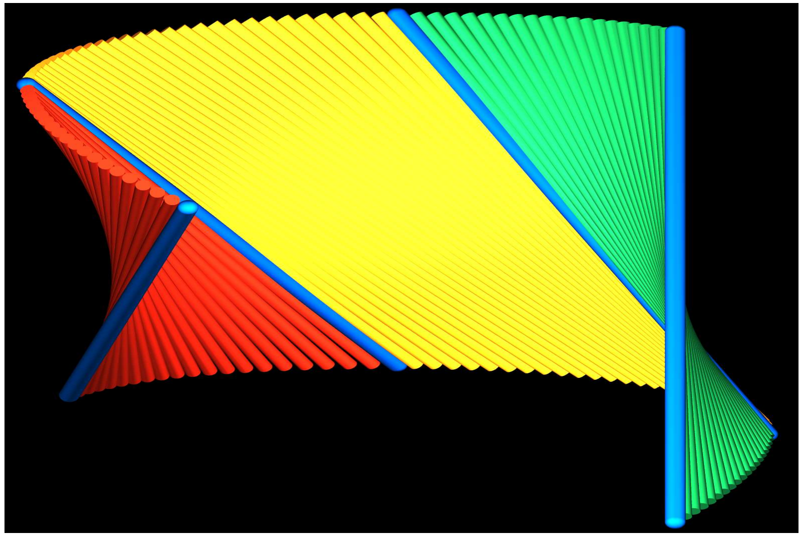
advantages: fast, sufficient for computer graphics and animation
disadvantages: no exakt parametrization (although convergent)

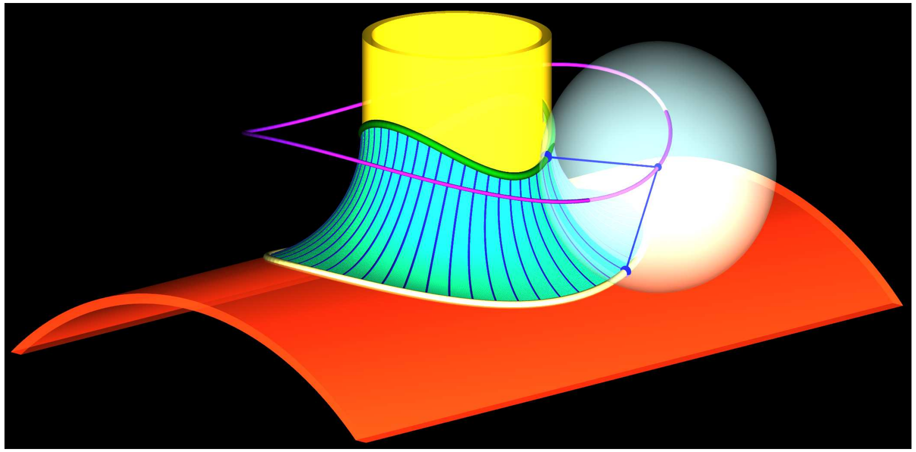
ruled surfaces / channel surfaces – What is done so far?

(3) G^1 Hermite interpolation with quadric bi-arcs = interpolation on Plücker's quadric with conic bi-arcs [17]

advantages: simple construction / computation, exact parametrization

disadvantages: restricted to degree 2, no torsal / inflection rulings, lines have to be inserted ...





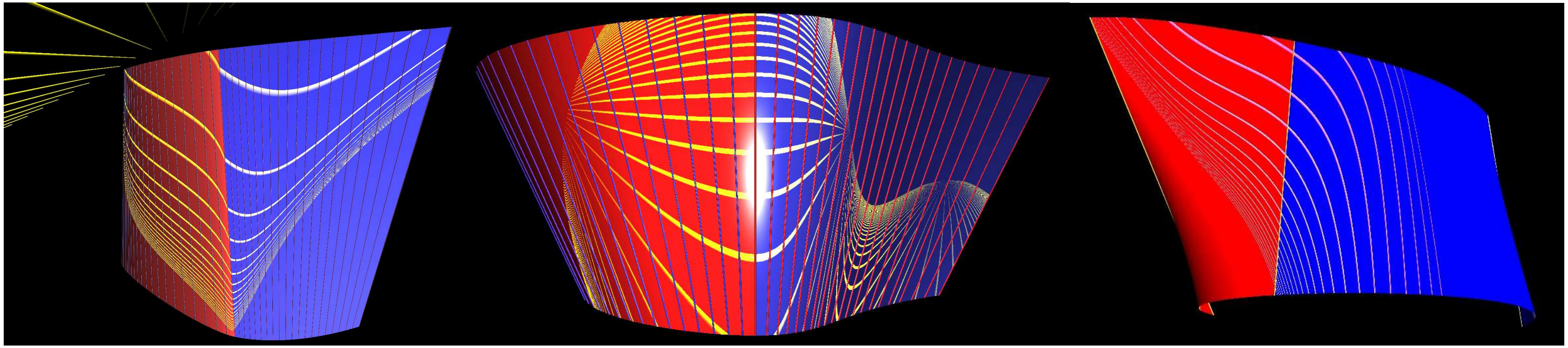
(4) exact rational parametrization of minimal degree of channel surfaces

advantages: rational low degree parametrizations, exact

disadvantages: complicated computation, real time? [7,8,14]

(5) ...and many algorithms for developable (ruled) surfaces ... [15,16,18]

Why interpolation with higher order continuity?



Isophotes (especially shadow boundaries) and reflection lines show a G^{k-1} transition at the G^k join of two surfaces.

Gliding or rolling along / on such surfaces becomes more convenient if the joins are of higher order continuity.

line geometry, sphere geometry

point models - Plücker's quadric - Lie's quadric - dictionaries

line geometry

[4,5,10,19,21,22]

$L = [P, Q]$ ... line L spanned by points $P \neq Q$ in some three-dimensional space

$\mathbf{p}, \mathbf{q}$ ... coordinate vectors of P, Q

$\mathbf{l} = \mathbf{q} - \mathbf{p}$, $\bar{\mathbf{l}} = \mathbf{p} \times \mathbf{q}$ ... direction and momentum vector of L

$L = (\mathbf{l}, \bar{\mathbf{l}})$ ... Plücker coordinates $\Longrightarrow$ interpretation as coordinates of points in $\mathbb{P}^5(\mathbb{F})$

(arbitrary commutative field $\mathbb{F}$, $\operatorname{char} \mathbb{F} \neq 2$)

Plücker coordinates of L fulfill

$$M_2^4 : \ \tfrac{1}{2}\Omega_L(L, L) = \langle \mathbf{l}, \bar{\mathbf{l}} \rangle = 0$$

- $\Omega_L(\cdot, \cdot) : \ \mathbb{F}^6 \times \mathbb{F}^6 \to \mathbb{F}$ ... polar form of the regular ruled quadric M_2^4 (of index 2)
- $\Omega_L(L, M) = 0 \Longleftrightarrow L$ and M are coplanar (intersecting).
- M_2^4 and the ambient space $\mathbb{P}^5$ serve as a point model for the set of lines in $\mathbb{P}^3(\mathbb{F})$

(and ofcourse in $\mathbb{R}^3$), M_2^4 ... Plücker's quadric, Klein's quadric, Grassmannian $\mathcal{G}_{3,1}$

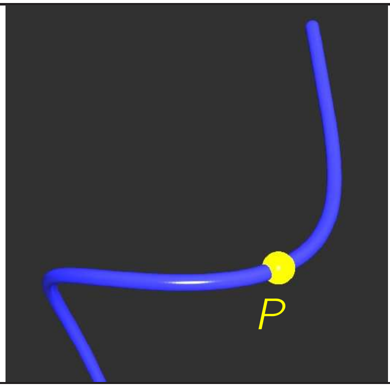
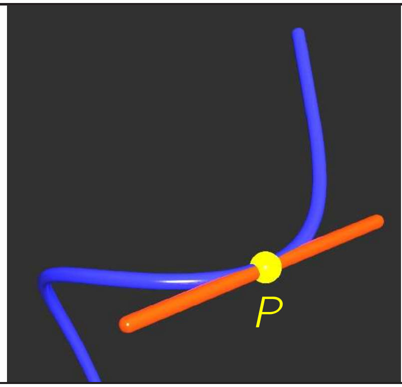
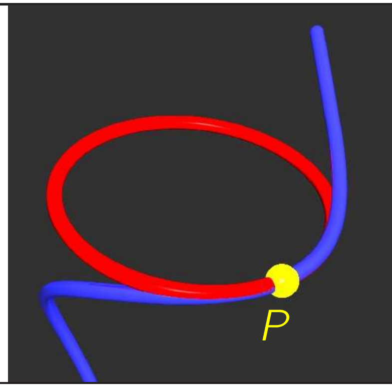
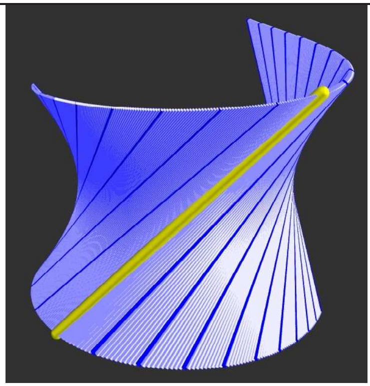
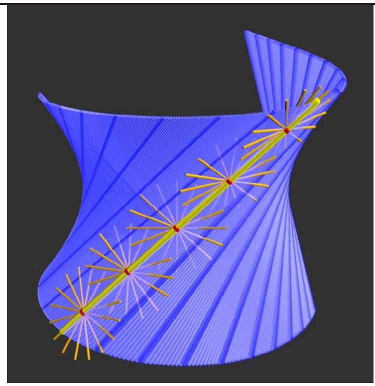
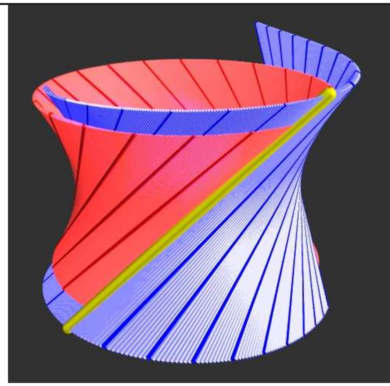
line geometry - some corresponding objects

in M_2^4	in terms of lines in $\mathbb{P}^3(\mathbb{F})$
point	(straight) line
line $\subset M_2^4$	pencil of lines
plane $\subset M_2^4$	star of lines / ruled plane
2-, 3-dim. submanifold	congruence, complex of lines
1-dim. submanifold (curve)	ruled surface
conic $\subset M_2^4$	regulus (quadratic cone, dual conic)**
algebraic curve of degree n	algebraic ruled surface of degree n
point $\in \mathbb{P}^5 \setminus M_2^4$	regular linear line complex*
line $\subset \mathbb{P}^5 \setminus M_2^4$	linear congruence of lines*
plane $\subset \mathbb{P}^5 \setminus M_2^4$	regulus*

* ...extended Klein images, actually carriers of 3-, 2-, and 1-dim. manifolds of lines

** ...only if the conic's plane is contained in M_2^4

line geometry - differential geometric properties of ruled surfaces

<table>
<tr><th>G^0</th><th>G^1</th><th>G^2</th></tr>
<tr><td>point P</td><td>point P + tangent t_P</td><td>point P (+ tangent t_P)
+ osculating plane O_P</td></tr>
<tr><td></td><td></td><td></td></tr>
<tr><td>ruling P</td><td>surface tangents
along P</td><td>one regulus of the
osculating quadric</td></tr>
<tr><td></td><td></td><td></td></tr>
</table>

G^k Hermite interpolation of ruled surfaces means . . .

sphere geometry

[1,9,21]

A sphere S in Euclidean three-space $\mathbb{R}^3$ can always be given by its equation

$$(s_6 - s_4)(x^2 + y^2 + z^2) - 2(s_1 x + s_2 y + s_3 z) + (s_6 + s_4) = 0.$$

center $C = \frac{1}{s_6 - s_4}(s_1, s_2, s_3)$, radius $R = \frac{s_5}{s_6 - s_4}$ with s_5 chosen such that

$$L_2^4 : \ \Omega_S(S, S) = s_1^2 + s_2^2 + s_3^2 + s_4^2 - s_5^2 - s_6^2 = 0$$

s_i are homogeneous $\Longrightarrow$ interpretation in as coordinates of points in $\mathbb{P}^5(\mathbb{R})$

- $\Omega_S(\cdot, \cdot) : \ \mathbb{R}^6 \times \mathbb{R}^6 \to \mathbb{R}$ ... polar form of the regular ruled quadric L_2^4 (of index 1)
- $\Omega_S(S, T) = 0 \Longleftrightarrow S$ and T are in oriented contact.
- L_2^4 and the ambient space $\mathbb{P}^5$ serve as a point model for the set of spheres in $\mathbb{R}^3$
- L_2^4 ... Lie's quadric, projectively equivalent to M_2^4 (but not over the reals)

sphere geometry

in L_2^4	in terms of spheres in $\mathbb{R}^3$
point	sphere (inluding planes, points, ...)
line $\subset L_2^4$	pencil of spheres (planes)
There are no planes in L_2^4.	
2-, 3-dim. submanifold	congruence, complex of spheres
1-dim. submanifold (curve)	envelope a channel surface
	(incl. pipe surfaces: $r = \text{const.}$)
conic $\subset L_2^4$	one family of spheres
	tangent to a Dupin cyclide
algebraic curve of degree n	envelope an algebraic channel surface
	of degree $< n(2n-1)$
point $\in \mathbb{P}^5 \setminus L_2^4$	regular linear complex of spheres *
line $\subset \mathbb{P}^5 \setminus L_2^4$	linear congruence of spheres*

* ... extended *Lie* images, actually carriers of 3-, 2-, and 1-dim. manifolds of spheres

sphere geometry - differential geometric properties of channel surfaces

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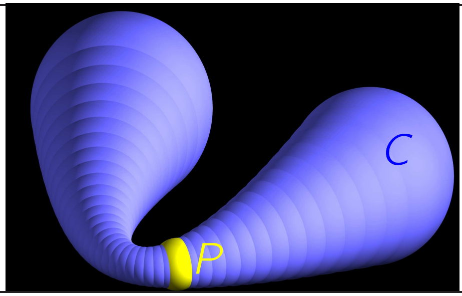
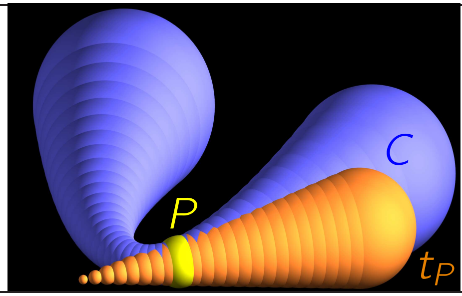
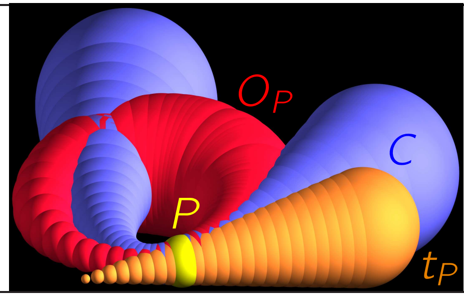
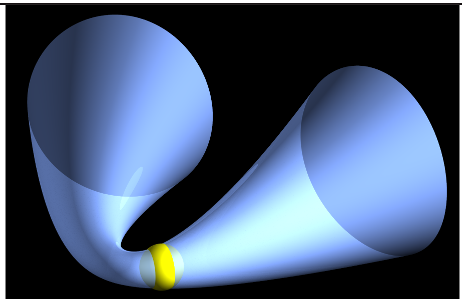
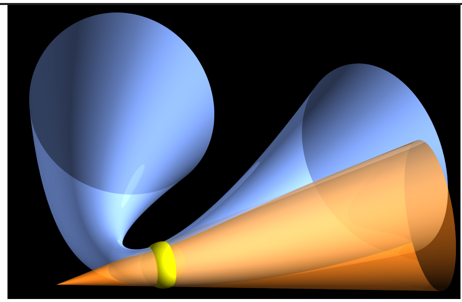
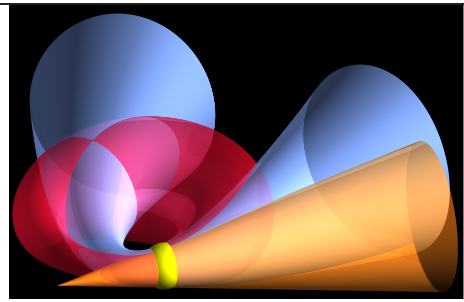
<tr><td colspan="3">one-parameter families of spheres</td></tr>
<tr><td>
</td><td>
</td><td>
</td></tr>
<tr><td>sphere P</td><td>tangent cone at P</td><td>one family of spheres enveloping the osculating Dupin cyclide</td></tr>
<tr><td colspan="3">the respective envelopes</td></tr>
<tr><td></td><td></td><td></td></tr>
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G^k Hermite interpolation with channel surfaces means ...

Bézier curves in quadrics

control points - additional constraints - polynomial condition

Bézier curves in quadrics

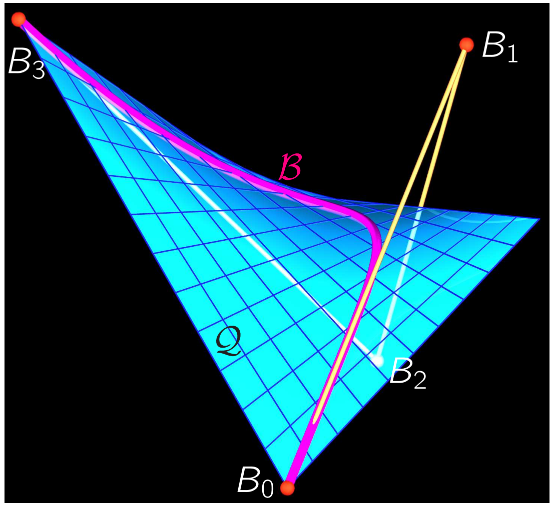
[3,4,6]

$\mathbf{b}_k$ ... (coordinate vectors of) control points B_k, $k = 0, \ldots, n$

$\varphi_k = \binom{n}{k}(1-t)^{n-k}t^k$ ... Bernstein polynomials

$\mathcal{B}(t) = \sum_{k=0}^{n} \varphi_k \, \mathbf{b}_k$ ... Bézier curve of degree n

The control points B_k of a Bézier curve $\mathcal{B}$ in a quadric $\mathcal{Q}$ are (in general) not in $\mathcal{Q}$, except the endpoints.



The quadric type and the degree of the Bézier curve have to match:

There are no cubics on oval quadrics!

$\Omega : \ \mathbb{F}^{n+1} \times \mathbb{F}^{n+1} \to \mathbb{F}$ ... symm. bilinear form, $\mathbb{F}$ commutative, char $\mathbb{F} \neq 2$

$\mathcal{Q} : \ \Omega(X, X) = 0$ ... quadric in $\mathbb{P}^n(\mathbb{F})$ (regular if its coefficient matrix is)

$\mathcal{B} \subset \mathcal{Q} \iff \mathcal{B}$ has $2n+1$ common points with $\mathcal{Q}$ or

$$P(t) = \Omega(\mathcal{B}, \mathcal{B}) \equiv 0 \quad \forall t \in \mathbb{F}$$

$\iff$ All $2n+1$ coefficients of $P(t)$ vanish (simultaneously). $\iff$ system of $2n+1$ equations (conditions)

Bézier curves in quadrics - conditions on derivatives

[4,5,19,22]

Independent of Ω, the parametrization $\mathcal{B}(t)$ and its derivatives satisfy

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t}P = \tfrac{\mathrm{d}}{\mathrm{d}t}\Omega(\mathcal{B},\mathcal{B}) \equiv 0 \quad &\Longrightarrow \quad \Omega(\mathcal{B},\dot{\mathcal{B}}) \equiv 0,\\
\frac{\mathrm{d}^2}{\mathrm{d}t^2}P = \tfrac{\mathrm{d}}{\mathrm{d}t}\Omega(\mathcal{B},\dot{\mathcal{B}}) \equiv 0 \quad &\Longrightarrow \quad \Omega(\dot{\mathcal{B}},\dot{\mathcal{B}}) + \Omega(\mathcal{B},\ddot{\mathcal{B}}) \equiv 0,\\
\frac{\mathrm{d}^3}{\mathrm{d}t^3}P = \dots \quad &\Longrightarrow \quad 3\Omega(\dot{\mathcal{B}},\ddot{\mathcal{B}}) + \Omega(\mathcal{B},\dddot{\mathcal{B}}) \equiv 0,\\
&\vdots
\end{aligned}$$

Boundary data $D = [P, \dot{P}, \ddot{P}, \dddot{P}, \dots]$ is always subject to these equations.

Since boundary data fulfills these equations, some conditions resulting from the coefficients of $P \equiv 0$ are automatically fulfilled!

Bézier curves in quadrics - why?

Any polynomial / rational parametrization can be rewritten in various bases.

It is just a linear mapping! In the space of polynomials of degree n:

$$\text{monomial} \rightarrow \text{Bernstein}: \ \text{upper triangular matrix with non}-\text{zero entries}$$
$$\left[(-1)^i \cdot \binom{n}{k} \cdot \binom{n-k}{i}\right]_{\begin{pmatrix} k = 0, \ldots, n \\ i = 0, \ldots, n-k \end{pmatrix}}$$

The coefficients corresponding to the Bézier representation give the coordinates of the control points.

Control points are helpful in any design process.

This approach keeps the degrees of the interpolants as low as possible.
The huge variety of solutions makes it possible to add constraints (side conditions).
No further construction that raises the degree and no insertion of lines is needed.

algorithms

recipe - examples - problems - solutions

algorithms - recipe

1. G^k data $D_0 = [\mathbf{p}, \dot{\mathbf{p}}, \ddot{\mathbf{p}}, \ldots]$, $D_1 = [\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}, \ldots]$
that satisfies the conditions on the derivatives (from ruled/channel surfaces)

2. D_0 ... determines the interpolant $\mathcal{B}$ at $t = 0$,
D_1 ... determines the interpolant $\mathcal{B}$ at $t = 1$

3. the ansatz of the interpolant as a Bézier curve:
$\mathcal{B}(t) = \sum_{k=0}^{n} \varphi_k \, \mathbf{b}_k$ with proper n † $\searrow$

4. control points chosen such that the G^k properties hold:
$\mathbf{b}_0 = \mathbf{p}$, $\mathbf{b}_1 = [\mathbf{p}, \dot{\mathbf{p}}]$, $\mathbf{b}_2 = [\mathbf{p}, \dot{\mathbf{p}}, \ddot{\mathbf{p}}]$, ..., $\mathbf{b}_{n-2} = [\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}]$, $\mathbf{b}_{n-1} = [\mathbf{q}, \dot{\mathbf{q}}]$, $\mathbf{b}_n = \mathbf{p}$.

5. condition for $\mathcal{B} \subset \mathcal{Q}$: $\qquad P(t) = \Omega(\mathcal{B}, \mathcal{B}) \equiv 0 \quad \forall t \in [0, 1]$
since $\deg \mathcal{B} = n \iff \deg P = 2n \implies 2n + 1$ coefficients $c_i = 0$
$\implies$ system of $2n + 1$ algebraic equations + side conditions to be solved

6. for torsal ruled surfaces: additionally $\Omega(\dot{\mathcal{B}}, \dot{\mathcal{B}}) \equiv 0 \quad \forall t \in [0, 1]$
$\implies$ system of $4n$ algebraic equations + side conditions to be solved † $\nwarrow$

algorithms - G^1

Prescribe G^1 data at the ends

$$D_0 = [\mathbf{p}, \dot{\mathbf{p}}], \; D_1 = [\mathbf{q}, \dot{\mathbf{q}}]$$

that satisfies the conditions on the derivatives*

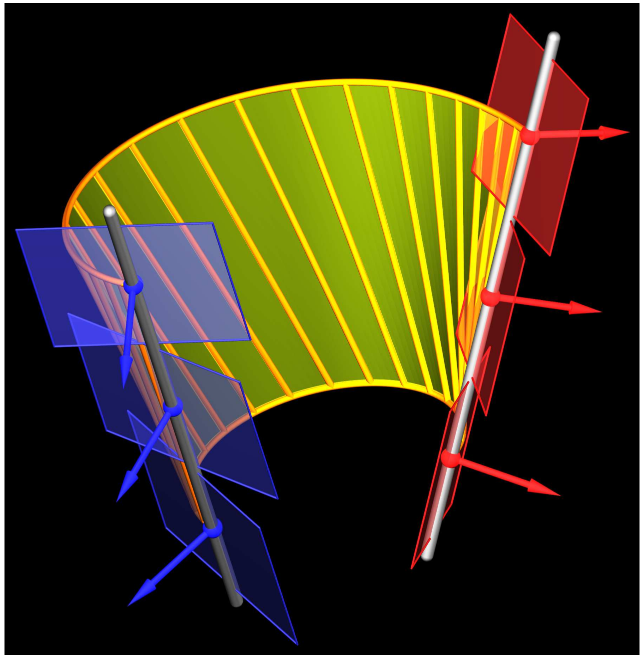
and make the ansatz

$$\mathcal{B}(t) = \sum_{k=0}^{3} \varphi_k \, \mathbf{b}_k$$

with $\mathbf{b}_0 = \mathbf{p}$, $\mathbf{b}_3 = \mathbf{q}$ and

$$\mathbf{b}_1 = \lambda_1 \, \mathbf{p} + \mu_1 \, \dot{\mathbf{p}}, \; \mathbf{b}_2 = \lambda_2 \, \mathbf{q} + \mu_2 \, \dot{\mathbf{q}}$$

(end point interpolation).



$\mathcal{B} \subset \mathcal{Q} \iff P = \Omega(\mathcal{B}, \mathcal{B})$ vanishes totally for the cubic Bézier curve $\mathcal{B}$ if, and only if,

$$2\Omega_{0,2} + 3\Omega_{1,1} = 0, \; 2\Omega_{1,3} + 3\Omega_{2,2} = 0, \; \Omega_{0,3} + 9\Omega_{1,2} = 0 \qquad \textbf{(S1)}.$$

* causes $\Omega_{0,0} = \Omega_{0,1} = \Omega_{2,3} = \Omega_{3,3} = 0$, $\Omega_{i,j} := \Omega(\mathbf{b}_i, \mathbf{b}_j)$ and $\Omega_{\mathbf{x},\mathbf{y}} := \Omega(\mathbf{x}, \mathbf{y})$

$\Longrightarrow$ three equations in four unknowns λ_i, μ_i $\Longrightarrow$ one-parameter family of solutions

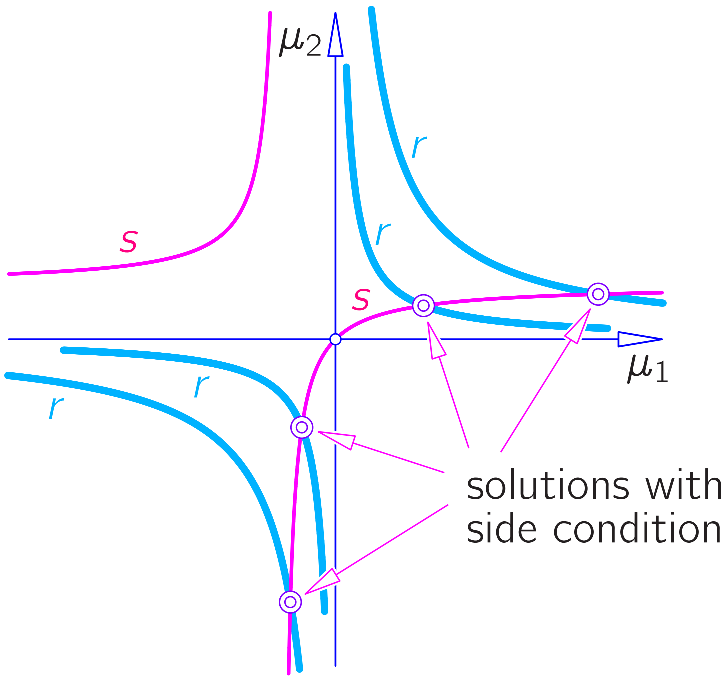
algorithms - G^1

Solving (S1) yields

$\lambda_1 = -(2\Omega_{\dot{\mathbf{p}},\mathbf{q}}\mu_1 + 3\Omega_{\dot{\mathbf{q}},\dot{\mathbf{q}}}\mu_2^2)/(2\Omega_{\mathbf{p},\mathbf{q}})$,

$\lambda_2 = -(2\Omega_{\mathbf{p},\dot{\mathbf{q}}}\mu_2 + 3\Omega_{\dot{\mathbf{p}},\dot{\mathbf{p}}}\mu_1^2)/(2\Omega_{\mathbf{p},\mathbf{q}})$,

and finally the union of two hyperbolae in the $[\mu_1, \mu_2]$-plane. $\Longrightarrow$



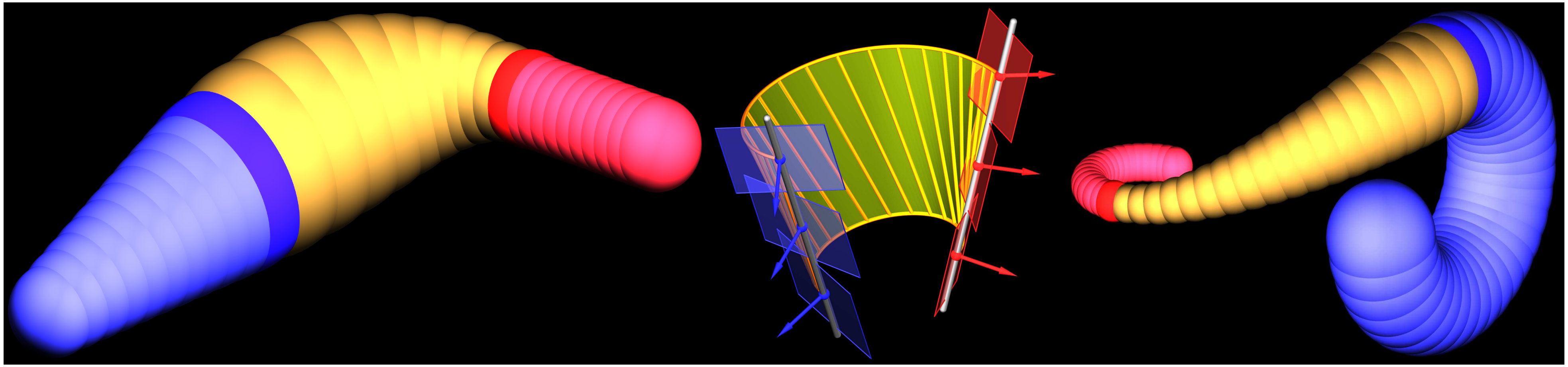
The totality of solutions to the G^1 interpolation problem in quadrics with cubics is the union of two quadratic one-parameter familes solutions.

$$r:\ 81\Omega_{\dot{\mathbf{p}},\dot{\mathbf{p}}}\Omega_{\dot{\mathbf{q}},\dot{\mathbf{q}}}\mu_1^2\mu_2^2 + 36(\Omega_{\mathbf{p},\mathbf{q}}\Omega_{\dot{\mathbf{p}},\dot{\mathbf{q}}} - \Omega_{\mathbf{p},\dot{\mathbf{q}}}\Omega_{\dot{\mathbf{p}},\mathbf{q}})\mu_1\mu_2 + 4\Omega^2_{\dot{\mathbf{p}},\mathbf{q}} = 0$$

Adding a proper side condition s simplifies the search for useful solutions or helps to avoid unwanted behaviour of solutions.

algorithms - G^1

The unifying treatment - as geometries in quadrics - allows us to apply the technique to both, ruled and channel surfaces.



The uniform parametrization of the interpolants may get lost on the way back to three-space.

algorithms - G^2

Prescribe G^2 data at the ends

$$D_0 = [\mathbf{p}, \dot{\mathbf{p}}, \ddot{\mathbf{p}}], \; D_1 = [\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}]$$

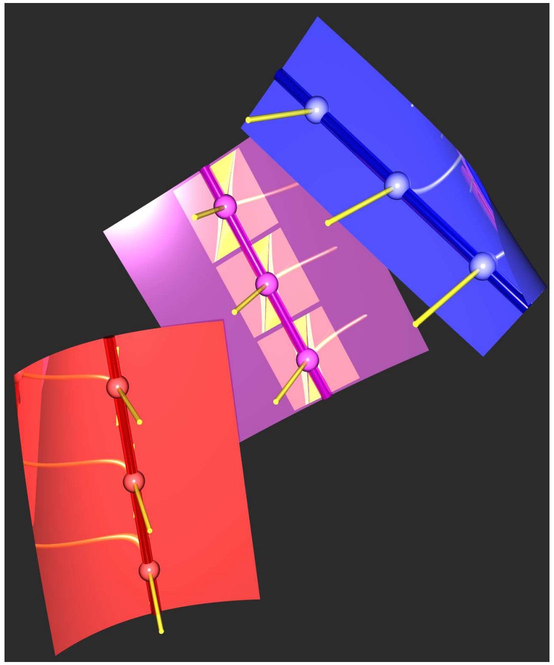
that satisfies the conditions on the derivatives*

and make the ansatz

$$\mathcal{B}(t) = \sum_{k=0}^{5} \varphi_k \, \mathbf{b}_k$$

with the afore mentioned end point conditions plus

$$\mathbf{b}_2 = \alpha_1 \, \mathbf{p} + \beta_1 \, \dot{\mathbf{p}} + \gamma_1 \, \ddot{\mathbf{p}}, \;\; \mathbf{b}_3 = \alpha_2 \, \mathbf{p} + \beta_2 \, \dot{\mathbf{p}} + \gamma_2 \, \ddot{\mathbf{p}}.$$



$\mathcal{B} \subset \mathcal{Q} \iff P = \Omega(\mathcal{B}, \mathcal{B}) \equiv 0$ for the quintic Bézier curve $\mathcal{B}$ if, and only if,

$$\begin{gathered} 4\Omega_{3i-3,3i-1} + 5\Omega_{3i-2,3i-2} = 0, \quad \Omega_{2i-2,2i+1} + 5\Omega_{2i-1,2i} = 0, \\ \Omega_{i-1,i+3} + 10\Omega_{i,i+2} + 10\Omega_{i+1,i+1} = 0, \quad \Omega_{0,5} + 25\Omega_{1,4} + 100\Omega_{2,3} = 0 \end{gathered} \qquad \textbf{(S2)}.$$

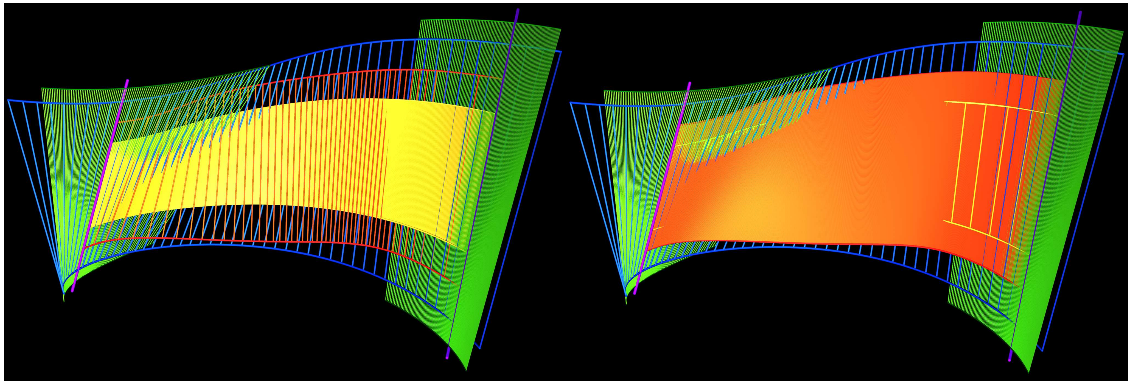
* causes $\Omega_{0,0} = \Omega_{0,1} = \Omega_{4,5} = \Omega_{5,5} = 0$, seven equations in twelve unknowns

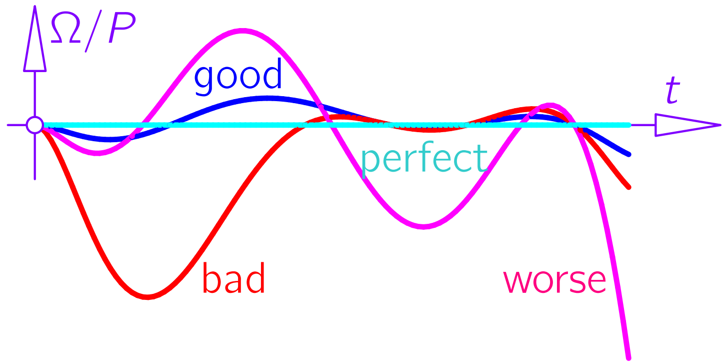
$\Longrightarrow$ 3-dim. variety of solutions of degree ≤ 150, with prescribed weights μ_i

$\Longrightarrow$ 1-dim. variety of solutions of degree eight.

algorithms - G^2

The system (of algebraic equations) **(S2)** is solved only numerically in most cases.



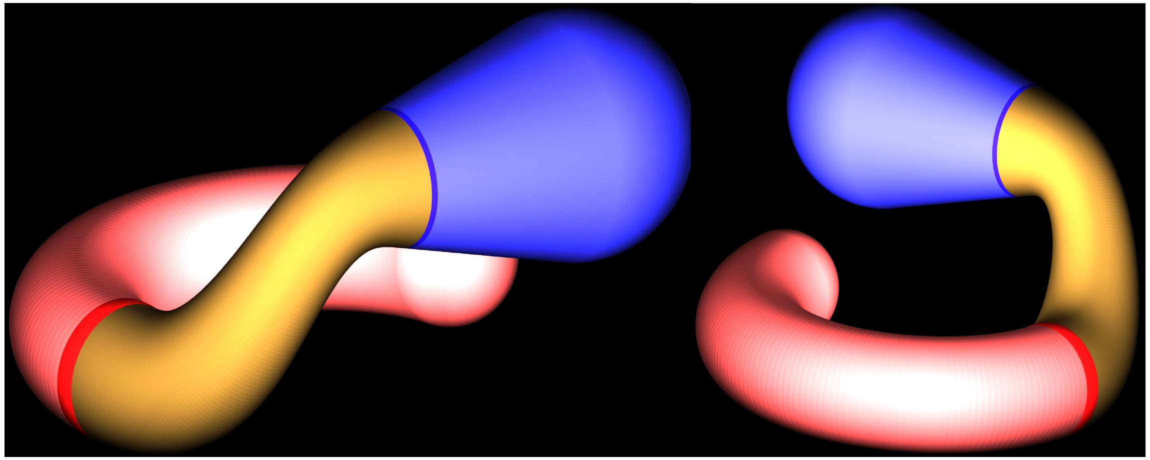


The quality of the numerical solution can be checked by looking at

$$\Omega(\mathcal{B}, \mathcal{B}) = P(t) : \ [0, 1] \to \mathbb{R}$$

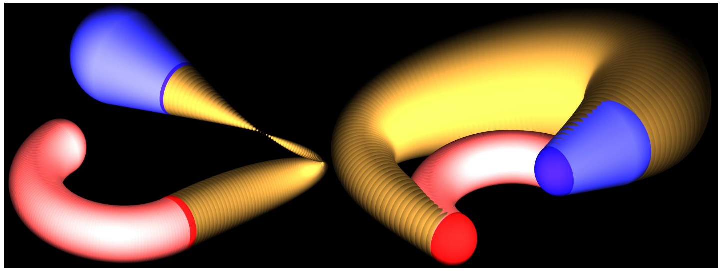
which is only almost zero over $[0, 1]$ due to numerical inaccuracies.

algorithms - G^2



Good solutions:
show only little variation in the radius function and have no self intersections.

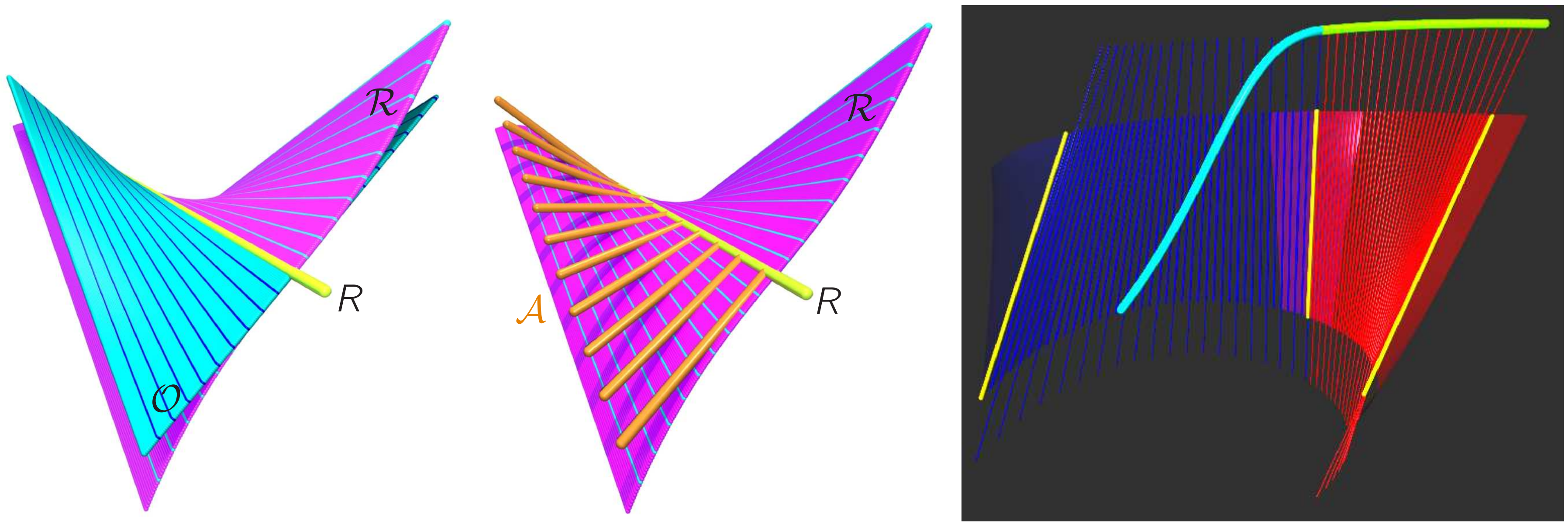
Bad solutions:
radius function has zeros in $[0,1]$ or/and channel surface looks nice at the ends, but grows into the wrong direction.



Self intersections and loops can also occur at ruled surfaces. The proper choice of μ_i can prevent the surfaces from running into the wrong direction.

algorithms - G^3

Along a (regular, non-torsal, non-inflection, . . .) ruling R, the asymptotic tangents of the ruled surface $\mathcal{R}$ form a regulus $\mathcal{A}$, *i.e.*, the regulus complementary to the primary regulus $\mathcal{O}$ of the osculating quadric. There are two asymptotic tangents (in general) which hyper-osculate the ruled surface: They intersect the ruled surface with multiplicity four (at least). These lines are called flecnodal tangents and touch the ruled surface at the flecnodes. There exists an analogon for channel surfaces.[4,5,13,19,22]



Two ruled surfaces with a G^3 join along a (regular, non-torsal, non-inflection, . . .) ruling share the flecnodes.

algorithms - G^3

We prescribe G^3 data at the ends

$$D_0 = [\mathbf{p}, \dot{\mathbf{p}}, \ddot{\mathbf{p}}, \dddot{\mathbf{p}}],\ D_0 = [\mathbf{p}, \dot{\mathbf{p}}, \ddot{\mathbf{p}}, \dddot{\mathbf{p}}]$$

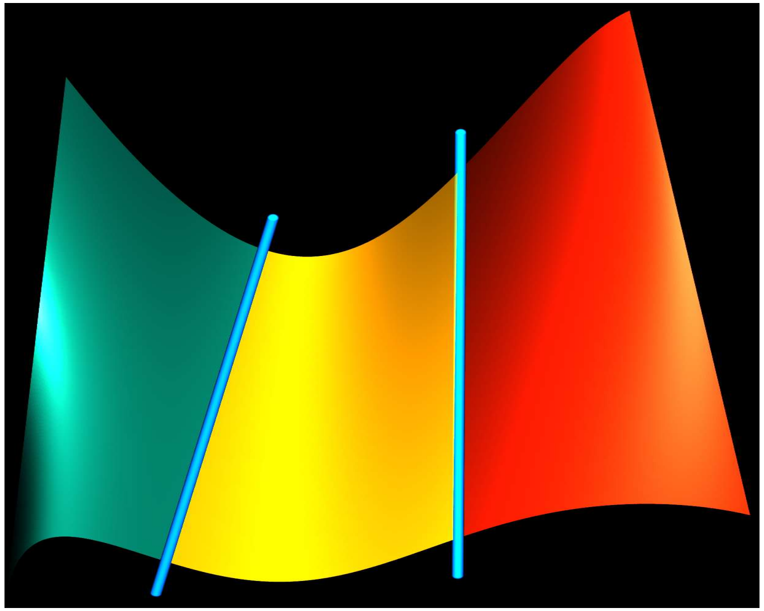
that satisies the conditions on the derivatives* and use the ansatz

$$\mathcal{B} = \sum_{k=0}^{7} \varphi_k \, \mathbf{b}_k \text{ with}$$

$\mathbf{b}_0 = \mathbf{p},\ \mathbf{b}_1 = [\mathbf{p}, \dot{\mathbf{p}}], \ldots,$

$\mathbf{b}_7 = \mathbf{q},\ \mathbf{b}_6 = [\mathbf{q}, \dot{\mathbf{q}}], \ldots.$

$\mathcal{B} \subset \mathcal{Q} \Longleftrightarrow P = \Omega(\mathcal{B}, \mathcal{B}) \equiv 0$



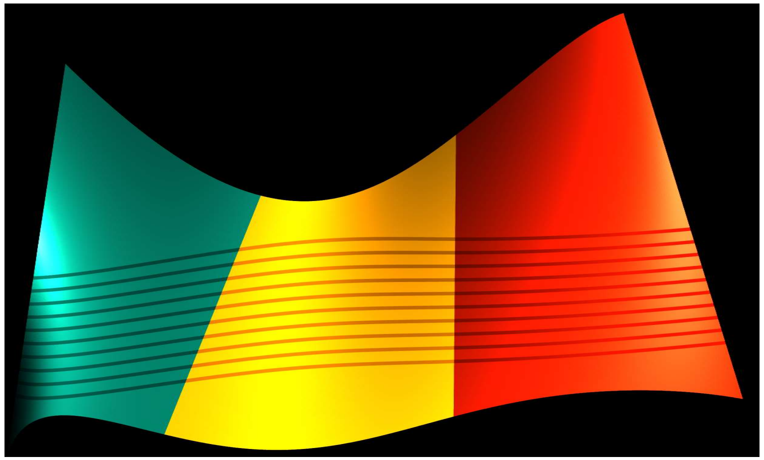
Four conditions vanish automatically due to *: $\Omega_{00} = \Omega_{01} = \Omega_{67} = \Omega_{77} = 0$.

Eleven conditions on 18 variables remain.

$\Longrightarrow$ 6-dimensional manifold of solutions.

Intelligent elimination strategy required!

Drop the degree and the dimension of the solution manifold by prescribing the right parameters!



conclusion

problems - solutions - discussion - suggestions

conclusion - problems - discussion

Interpolation problems for channel surfaces should preferably be treated in the linear cyclographic model of the geometry of spheres.

$\Longrightarrow$ advantage: simple spline interpolation

disadvantage: yields G^k joins of families of spheres,

but only G^{k-1} joins between enveloped channel surfaces

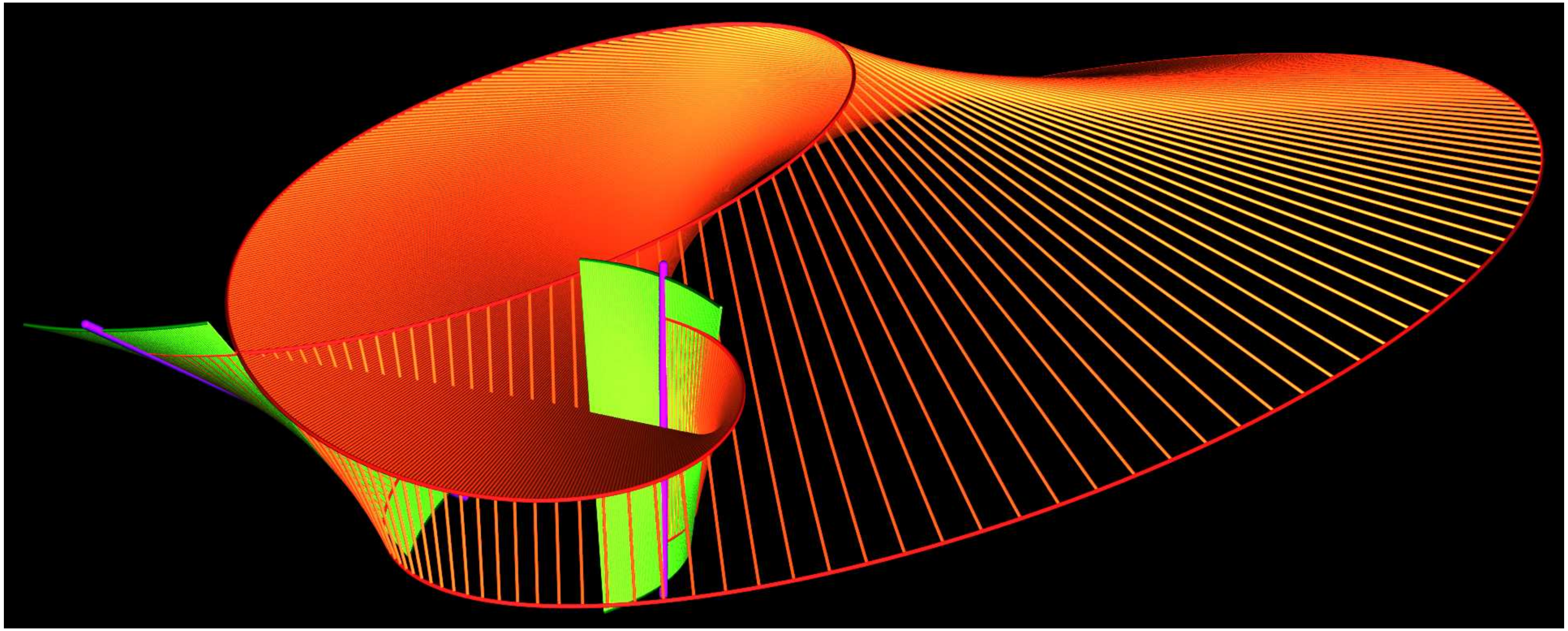
The uniform approach – Bézier curves in arbitrary (regular) quadrics – is applicable to other non-linear geometries.

$\Longrightarrow$ exact parametrization of interpolating motions

The shape parameters μ_i or the side conditions should be chosen carefully: Algebraic curves of sufficiently high degree tend to oscillate or may even have loops. So do ruled surfaces!

conclusion - problems - discussion

a bad G^2 interpolation with a nasty loop



Thank You For Your Attention!

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