The Parallelians of a Tetrahedron

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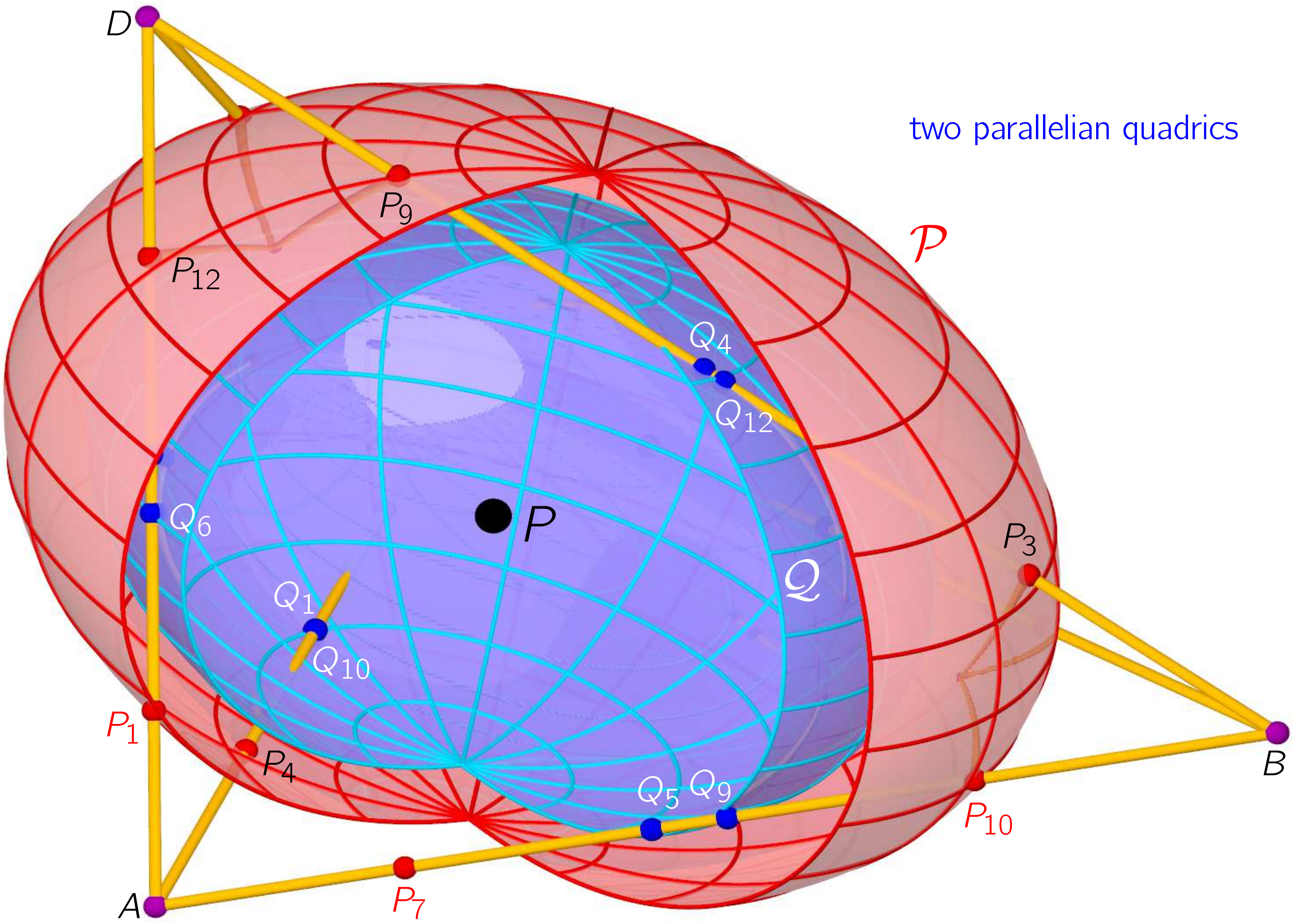
overview

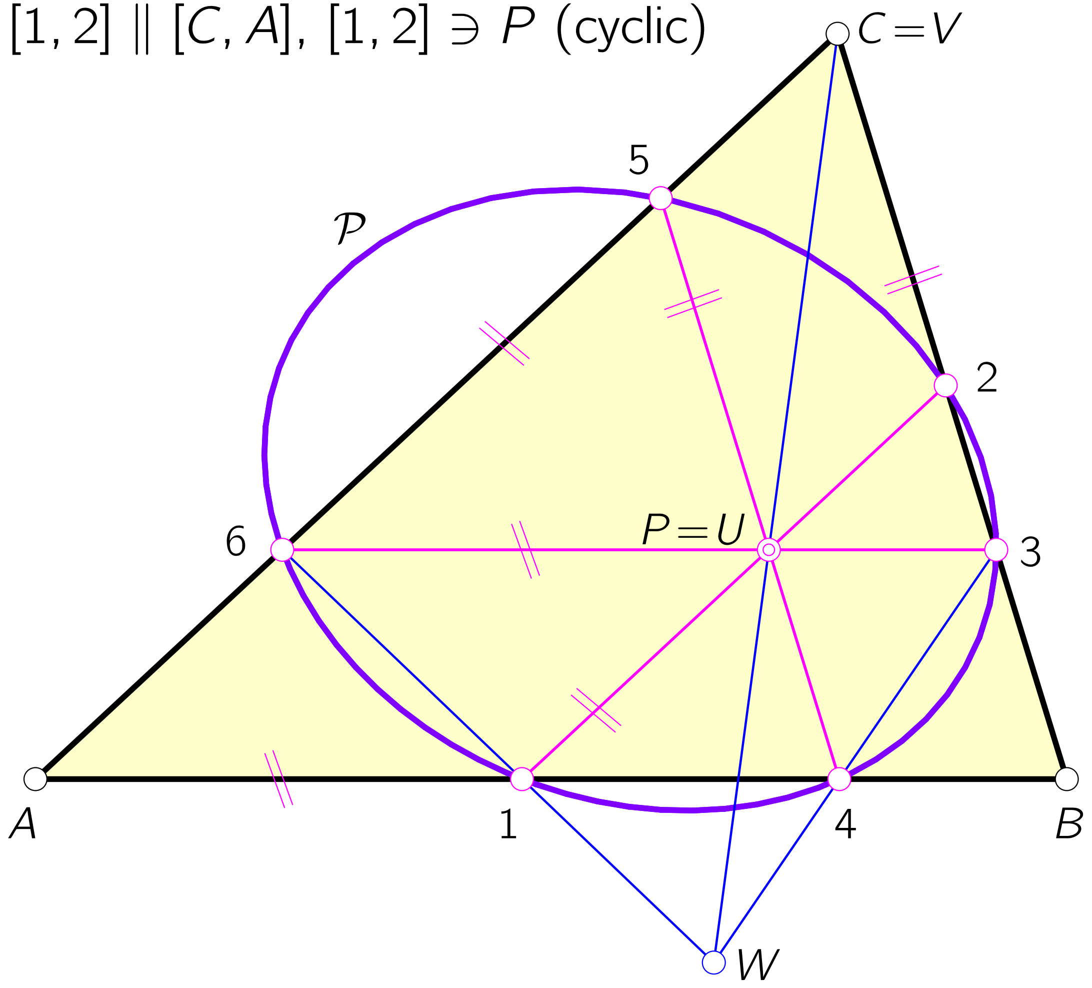
parallelians	in the common sense and more
2 parallelian quadrics	on more than 9 points
	vertices of singular quadrics:
	a degree 8 surface and Cayley's nodal cubic
projective version	very natural
dimension $n > 3$	$\lfloor \frac{n}{2} \rfloor$ different parallelian quadrics
parallelian quadrics	are objects in affine geometry
special pivots	circumcenter, centroid, Monge point, ...

two parallelian quadrics



in the plane: parallelians and parallelian conic $\mathcal{P}$

triangle $\Delta = ABC$, pivot point $P \notin \Delta^\star$
$[1,2] \parallel [C,A]$, $[1,2] \ni P$ (cyclic)



The parallelians $1,\ldots,6$
lie on a single conic $\mathcal{P}$, parallelian conic.

proof:

$1,\ldots,6$ fulfill the Pappos criterion
$[1,2] \cap [4,5] = P = U$,
$[2,3] \cap [5,6] = C = V$.
It remains to show that
$[3,4] \cap [6,1] = W \in [U,V]$
$63V \sim 14U$ and $[6,3] \parallel [1,4]$, $\ldots$

$\Longrightarrow$ $\exists$ central similarity with center
$[6,1] \cap [3,4] = W$
and $6 \mapsto 1$, $3 \mapsto 4$, $V \mapsto U$.

$\Longrightarrow$ U, V, W are collinear.

again in the plane: blowing up P

3 lines parallel to the sides of Δ intersect Δ's sides in 6 conconic points (generalized parallelians).

proof:

$1, \ldots, 6$ fulfill the Pappos criterion

$[1, 2] \cap [4, 5] = U$, $[3, 4] \cap [6, 1] = W = A$,

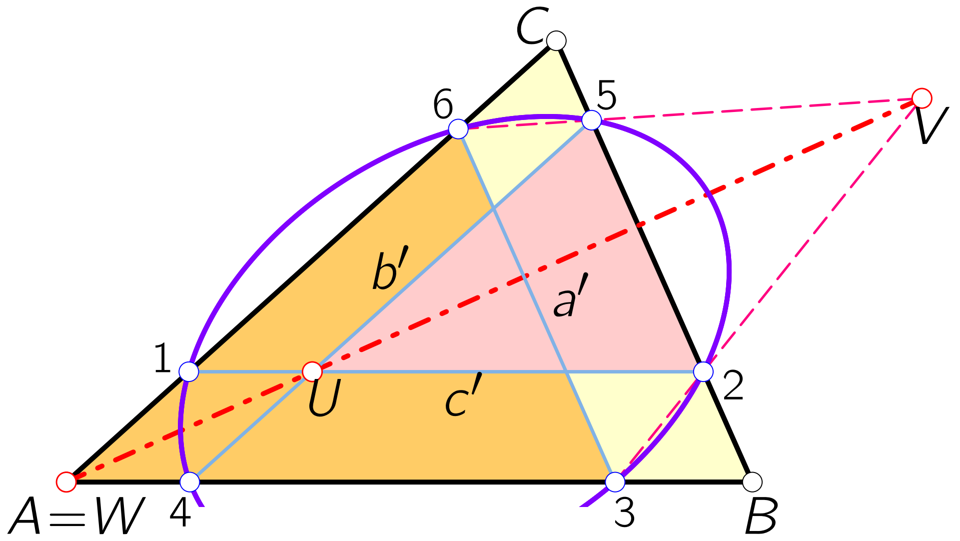
It remains to show: $[2, 3] \cap [5, 6] = V \in [U, W]$.

Note $ABC \sim A36 \sim U25$

and the parallelities.

$\Longrightarrow V =$ center of similarity

$A36 \to U25 \qquad \Longrightarrow V \in [A, U]$



3-space, simplex, and 12 parallelians

given: tetrahedron $\mathcal{T} = ABCD$ and a point P
(not in any face of $\mathcal{T}$)

4 planes through P parallel to $\mathcal{T}$'s faces intersect edges in 12 points P_i on a single quadric $\mathcal{P}$.

P_i ... parallelians, $\mathcal{P}$ ... parallelian quadric

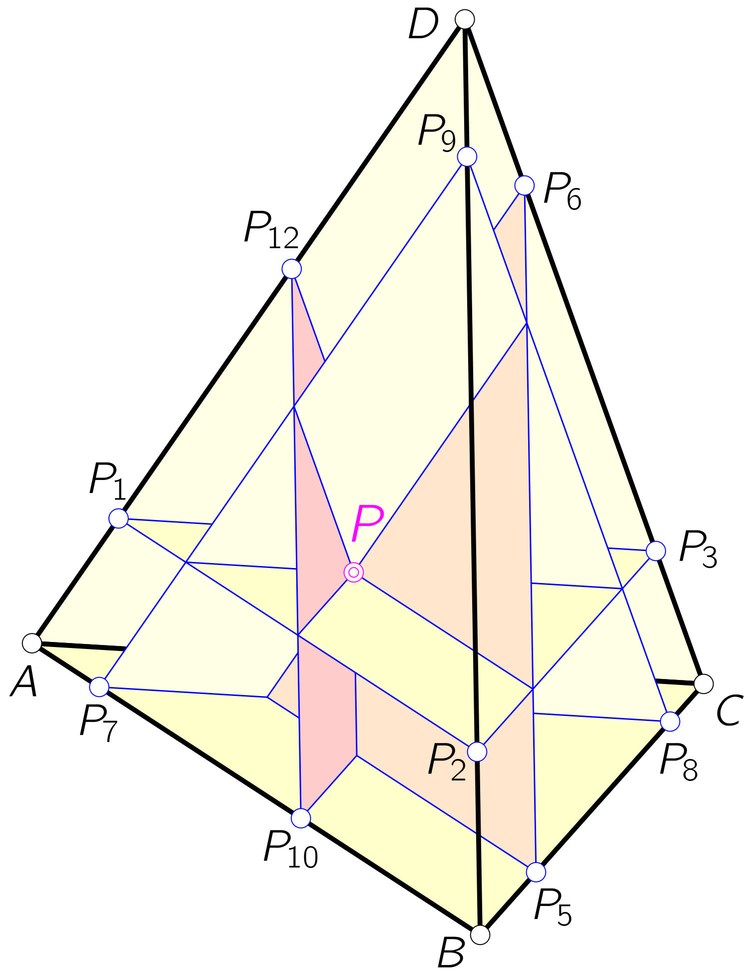
proof: note the six generalized parallelians in faces

$\Longrightarrow$ 4 conics in faces, any 2 share 2 points

6 gen. parallelians + 5 neighbours (-2 common points) define a unique quadric $\mathcal{P}'$

because of conconicity of neighbours

$\Longrightarrow$ a 10^{th} point lies on $\mathcal{P}'$ ...



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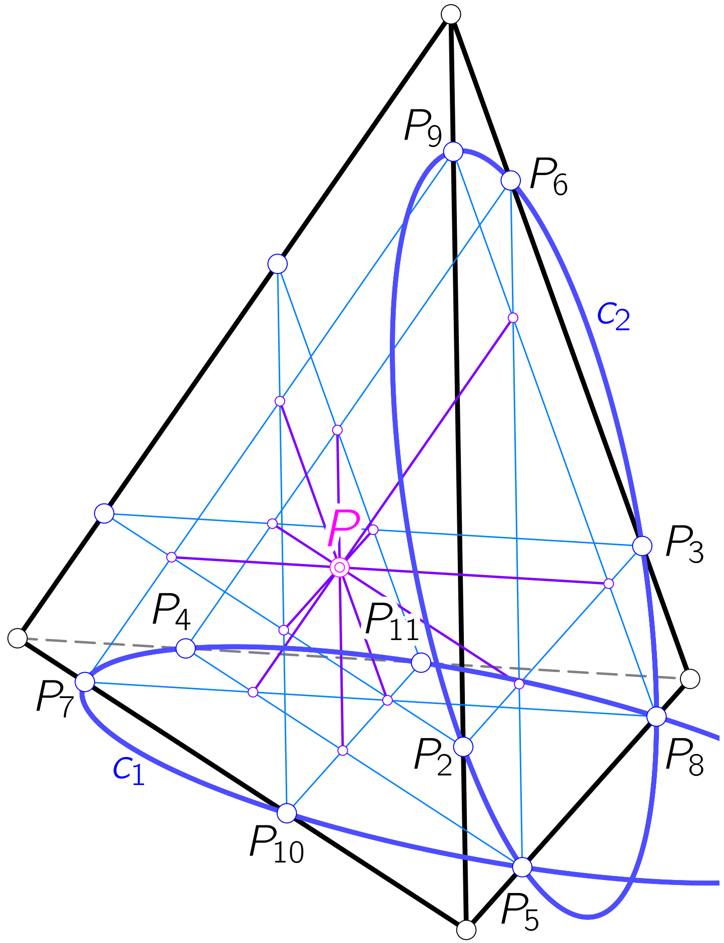


some calculations

$$\mathcal{P} : \xi\eta\zeta(\xi+\eta+\zeta) + \sum_{\text{cyclic}} \eta\zeta x^2 + \xi(2\eta+\xi+2\zeta-1)yz = 0$$

affine frame attached to $\mathcal{T}$, centered at A, unit points B, C, D, $P = (\xi, \eta, \zeta)$

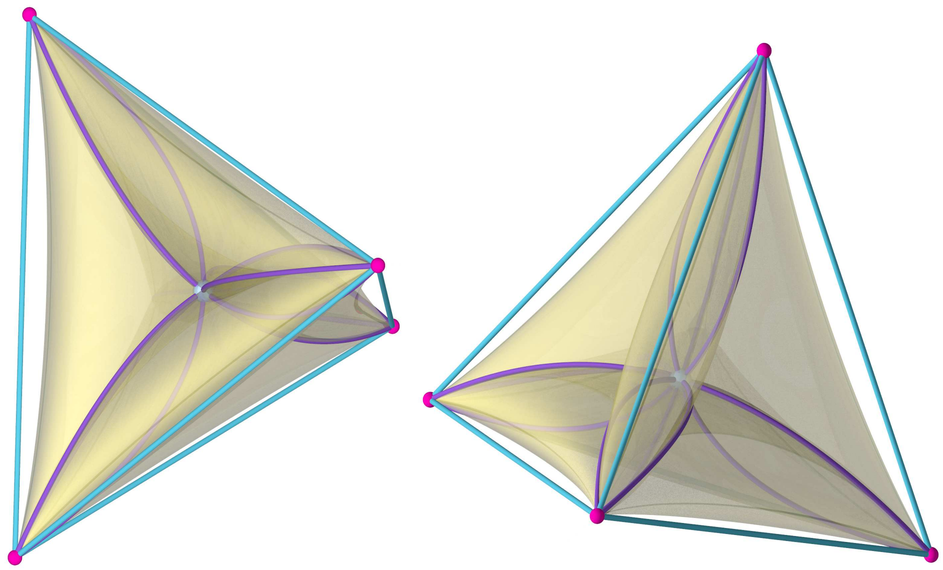
$$\mathcal{P}_{\text{sing}} : \sum_{\text{cyclic}} x^2 + yz - x = 0$$

pivot P on some Steiner circumellipsoid causes singular $\mathcal{P}$

$$\mathcal{P}_{\text{par}} : 1 + 3\sum_{\text{cyclic}} x^2 + yz - x = 0$$

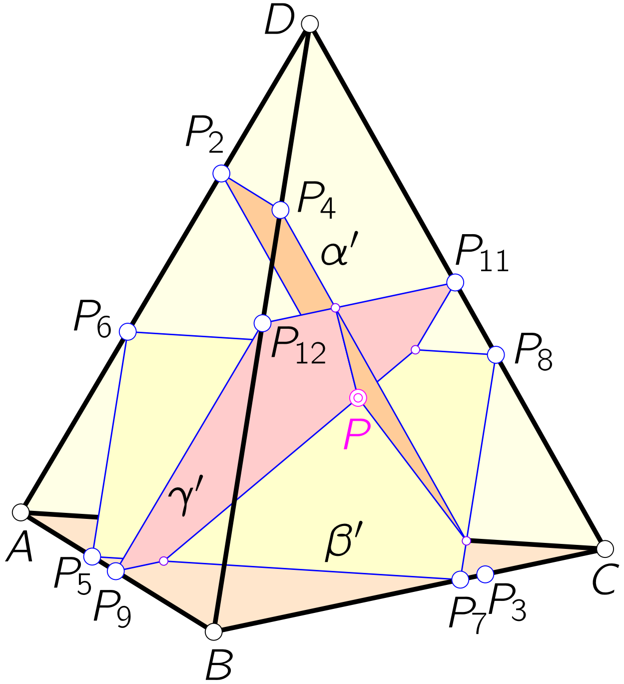
pivot P for parabolic $\mathcal{P}$

both centered at $\mathcal{T}$'s centroid $G = \frac{1}{4}(1,1,1)$



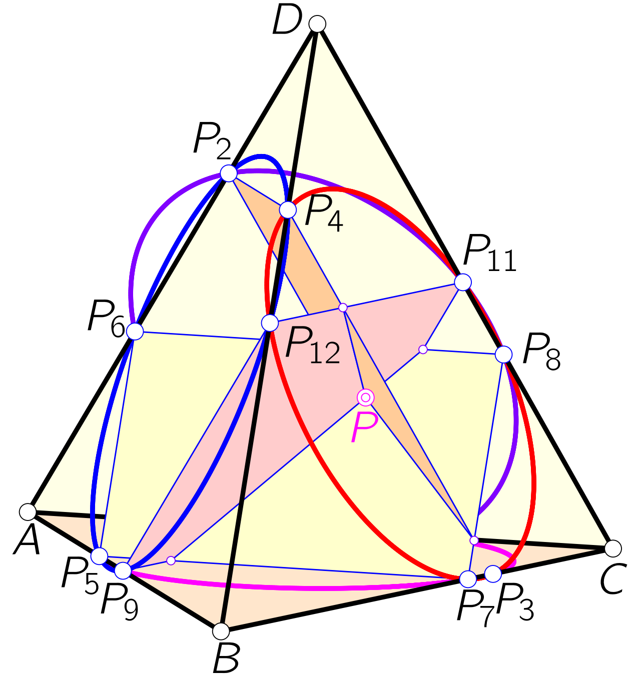
The vertices of the singular quadrics lie on a degree 8 surface with 6 parabolic double curves and a triple point at G.

the other type of parallelian quadric



3 planes through P parallel to opposite edges intersect $\mathcal{T}$'s edges in 12 points on a single quadric.

proof:
uses the same arguments



some calculations

$$\mathcal{Q}: \sum_{\text{cyclic}} (\eta+\zeta)x^2 - (1-2(\xi+\eta+\zeta))yz$$
$$- (\eta+\zeta)(2\xi+\eta+\zeta)x = -\prod_{\text{cyclic}} (\xi+\eta)$$

pivots for parabolic $\mathcal{Q}$

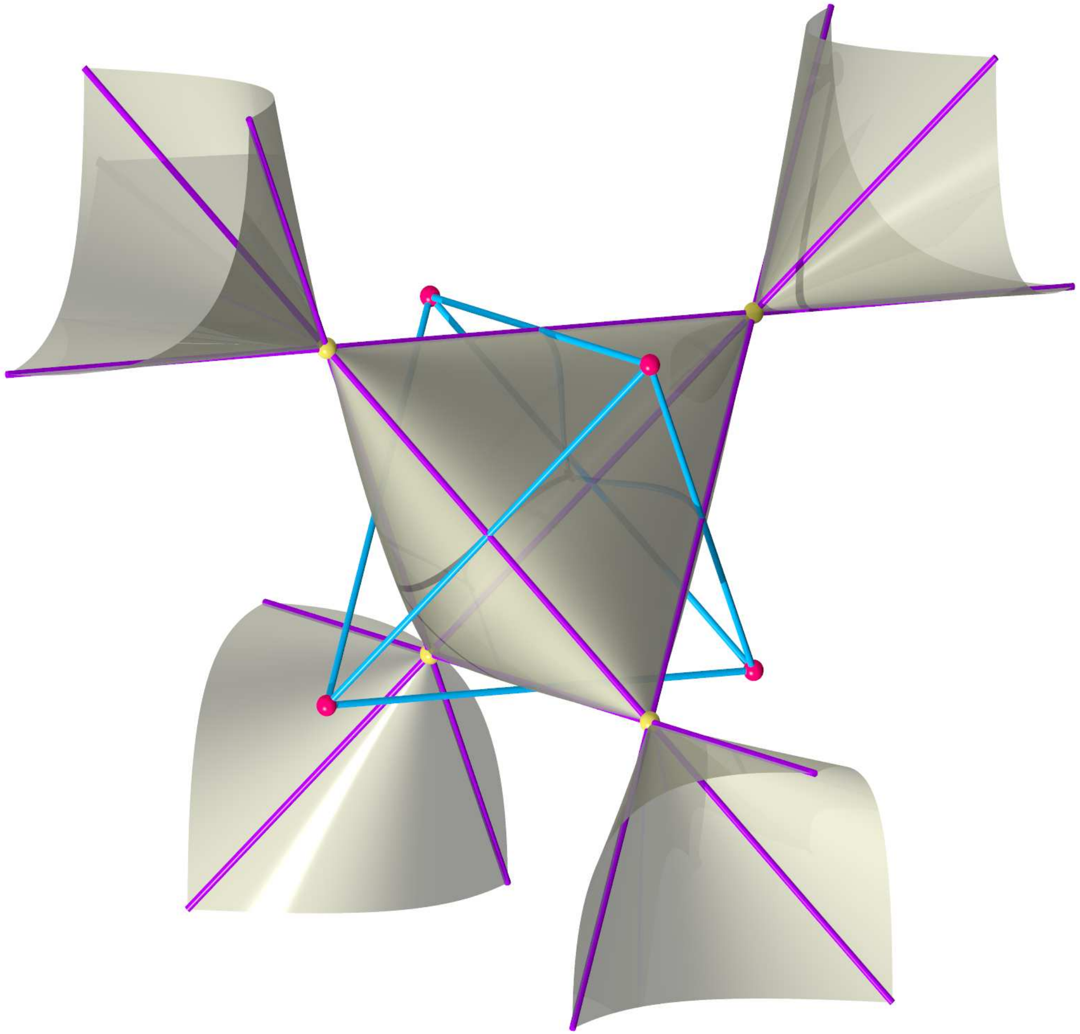
$$\mathcal{Q}_{\text{par}} : 8xyz + 4\sum_{\text{cyclic}} x^2(y+z-1) - 2yz + x = 1$$

that's Cayley's nodal cubic $\longrightarrow$

regularity of $\mathcal{Q}$:

$$\mathcal{Q}_{\text{reg}} : \prod_{\text{cyclic}} (x+y)(x+y-1) \neq 0$$

planes through edges & parallel to opposite edges



special pivots

Euclidean specialization:

$\mathcal{P}$ and $\mathcal{Q}$ are spheres only for tetrahedra with special metric properties, not for generic tetrahedra.

Euclidean specialities that deserve mention:

Pivot point	center of $\mathcal{P}$	center of $\mathcal{Q}$
centroid	centroid	centroid
circumcenter	Monge point	?

centroid, circumcenter, Monge point $\longrightarrow$ always that of $\mathcal{T}$

projective version ... is near

replace ideal plane ω by some plane λ

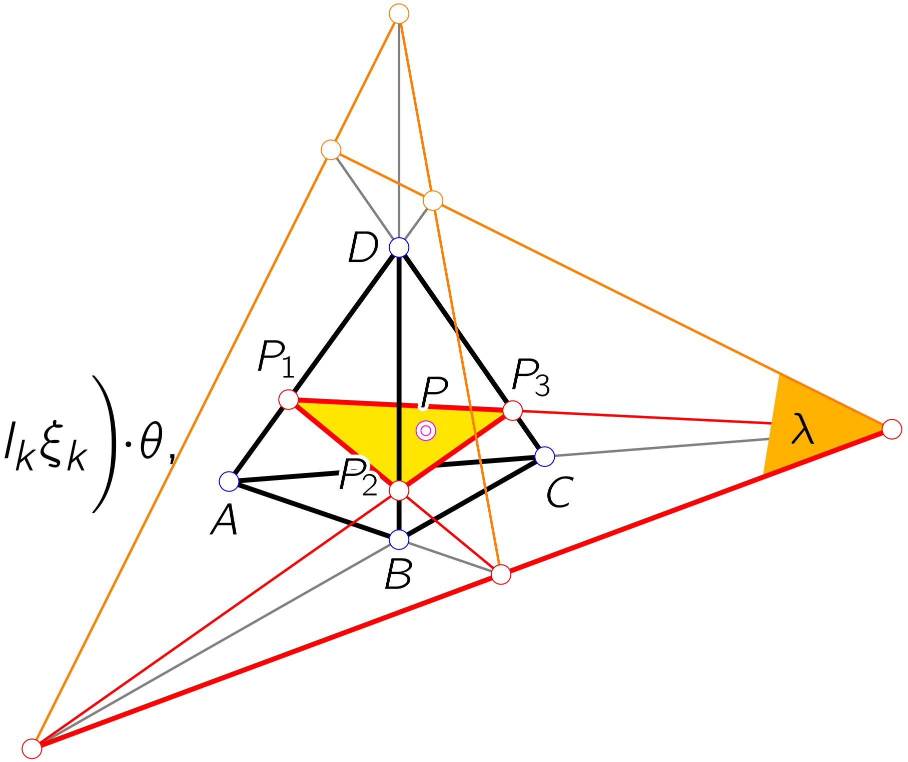
12 projectivized parallelians of a tetrahedron $\mathcal{T}$ w.r.t. a pivot point P of the first and second kind lie on two quadrics $\mathcal{P}$ and $\mathcal{Q}$:

$$\mathcal{P}: \sum_{i=0}^{3} \frac{l_i}{\xi_i}(\sigma - l_i\xi_i)x_i^2 - \sum_{i<j} \frac{x_i x_j}{\xi_i \xi_j}\theta = 0,$$

$$\mathcal{Q}: \sum_{i=0}^{3} l_i^2 x_i^2 \prod_{\substack{j<k \\ j,k \neq i}} (l_j\xi_k + l_k\xi_j)x_i^2 - \sum_{i<j} l_i l_j x_i x_j \left(\sum_{k \neq i,j} l_k \xi_k\right) \cdot \theta,$$

$$\sigma := \sum_{i=0}^{3} l_i \xi_i,\ \tau := \sum_{i<j} l_i l_j \xi_i \xi_j,$$

$$\theta := \tau + l_i l_j \xi_i \xi_j + \sum_{k,l \neq i,j} l_k l_l \xi_k \xi_l.$$



higher dimensional version . . .

. . . here, even affine geometry gets meaty!

$$\mathcal{P}_1 : \sum \frac{x_i^2}{\xi_i} + \sum_{i<j} \frac{x_i x_j}{\xi_i \xi_j}(\xi_i + \xi_j - 1 + S) - \sum x_i \xi_i (\xi_i + S) + S \cdot \prod \xi_k = 0$$

n-dimensional affine space:

$t = \lceil \frac{n}{2} \rceil$ different types of parallelian quadrics w.r.t. a simplex and a pivot point P
$N(n,k) = \binom{n}{k}(n+1)(n-k)$ with $k = 0, 1, \ldots, t$ points lying on the edges of $\mathcal{S}^n$ and on the parallelian quadric equals (to be halved in case $n - 1 = 2k$)

proof:

It takes $k+1$ vertices of $\mathcal{S}^n$ in order to span a k-face of $\mathcal{S}^k$. Then, $n-k-1$ vertices span the complementary face. There are $C_k^n = \binom{n+1}{k+1}$ combinations of vertices serving as base for k-face, and thus, $(n-k-1)(k+1)$ (non-parallel, non-incident) edges are to be intersected with C_k^n hyperplanes through P. If $n-1 = 2k$, then each k-face (and its complementary) shows up twice. It takes nested induction on k and n in order to show that $N(n,k)$ points always lie on a single quadric.

Thank You for Your Attention!

some references

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