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On Burmester's Focal Mechanism and Hart's Straight-line Motion

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Abstract
The article develops in a simple and modern manner the essential geometric properties of Burmester's focal mechanism with applications to the driving of double-rocker linkages and the synthesis of Hart's straight-line mechanism.

Zusammenfassung—Über den Burmesterschen Brennpunktsmechanismus und der Hartschen Geradführung.
Der Artikel entwickelt in einfacher und moderner Weise die wesentlichen geometrischen Eigenschaften des Burmesterschen Brennpunktsmechanismus mit Anwendungen auf den Antrieb von Doppelschwingen und die Synthese der Hartschen Geradführung.

Резюме—О фо кальных механизмах Вурместера и прямиле Харта—В. Вундерлих.
В настоящей работе развиваются простым образом существенные геометрические свойства фокальных механизмов Бурместера с применением к движению двухкоромысловаго механизма и синтезу прямила Харта.

Section 1

LET $MPFQN$ be a plane five-bar linkage with sizes $\overline{MP}=m$, $\overline{PF}=p$, $\overline{FQ}=q$, $\overline{QN}=n$ and $\overline{MN}=d$, M and N being fixed. To reduce the freedom $f=2$ of the mechanism, an additional condition must be imposed. Such a condition, as introduced by Hart [1] and utilized also by Darboux [2], Burmester [3], Dixon [4] and others, consists in establishing a certain relation between the angles at P and Q, for instance

$$\sphericalangle MPF=\omega, \qquad \sphericalangle FQN=\pi-\omega. \tag{1.1}$$

To determine the path of F we shall make use of isotropic coordinates $z=x+iy$, $\bar{z}=x-iy$ derived from cartesian coordinates x, y with origin in M, the x-axis going through N (Fig. 1). By means of the abbreviations

$$e^{i\phi}=u, \qquad e^{i\psi}=v, \qquad e^{i\omega}=w \quad (u\bar{u}=v\bar{v}=w\bar{w}=1), \tag{1.2}$$

ϕ and ψ denoting the angles of the bars MP and NQ with the base line MN, respectively, the position of F is characterized by

$$z=mu-puw, \qquad \bar{z}=m\bar{u}-p\bar{u}\bar{w} \tag{1.3}$$

and

$$z-d=nv+qvw, \qquad \bar{z}-d=n\bar{v}+q\bar{v}\bar{w}. \tag{1.4}$$

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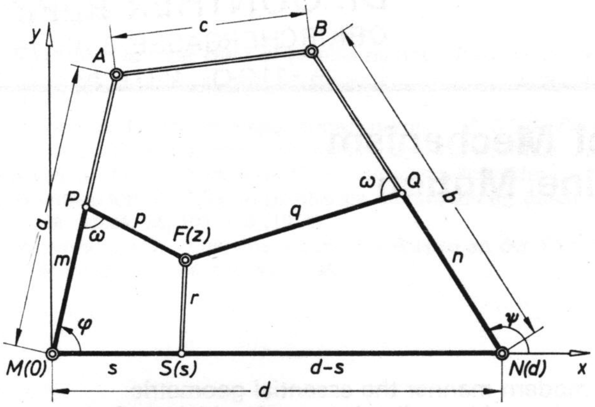

Figure 1. Five-bar linkage of Hart.

Eliminating u and v we arrive at the equations

$$z\bar{z}=m^2+p^2-mp(w+\bar{w}),$$
$$(z-d)(\bar{z}-d)=n^2+q^2+nq(w+\bar{w})\,; \tag{1.5}$$

these express in complex form the cosine theorems for the triangles MFP and NFQ. Elimination of $w+\bar{w}$ finally leads to

$$(mp+nq)z\bar{z}-mpd(z+\bar{z})=mp(n^2+q^2-d^2)+nq(m^2+p^2) \tag{1.6}$$

or

$$(z-s)(\bar{z}-s)=r^2 \tag{1.7}$$

with

$$s=\frac{mpd}{mp+nq},$$
$$r^2=\frac{m^2n^2(p^2+q^2)+(m^2+n^2)p^2q^2+mnpq(m^2+n^2+p^2+q^2-d^2)}{(mp+nq)^2} \tag{1.8}$$

This shows that the locus of F is a circle $|z-s|=r$ with center $S(x=s,\ y=0)$ and radius r (Fig. 1). Thus the condition (1.1) may be realized by adding a link SF fixed in that point S which divides the segment MN according to

$$\overline{MS}:\overline{NS}=s:(s-d)=mp:-nq\,. \tag{1.9}$$

Section 2

The relation between the angles ϕ and ψ is described by the equation

$$(mu-nv-d)(m\bar{u}-n\bar{v}-d)=(pu+qv)(p\bar{u}+q\bar{v}), \tag{2.1}$$

derived by elimination of z, $\bar{z}$ and w from (1.3) and (1.4). Reducing it to

$$(mn+pq)(u\bar{v}+\bar{u}v)+md(u+\bar{u})-nd(v+\bar{v})=m^2+n^2-p^2-q^2+d^2 \tag{2.2}$$

we see that it is of the same form as the relation between adjacent angles of a four-bar linkage.

Thus it may be tried to connect the systems MP and NQ by a link AB with joints A on MP and B on NQ. Denoting the lengths $\overline{MA}$, $\overline{NB}$, $\overline{AB}$ with a, b, c respectively, we get from Fig. 1, starting with $\overline{AB}=|au-bv-d|=c$:

$$(au-bv-d)(a\bar{u}-b\bar{v}-d)=c^2 \tag{2.3}$$

or

$$ab(u\bar{v}+\bar{u}v)+ad(u+\bar{u})-bd(v+\bar{v})=a^2+b^2-c^2+d^2. \tag{2.4}$$

Comparing the equations (2.2) and (2.4) we find that they define the same relation when the coefficients differ only by a common factor λ:

$$\frac{ab}{mn+pq}=\frac{a}{m}=\frac{b}{n}=\frac{a^2+b^2-c^2+d^2}{m^2+n^2-p^2-q^2+d^2}=\lambda. \tag{2.5}$$

This leads to

$$a=\lambda m, \qquad b=\lambda n, \qquad c^2=a^2+b^2+d^2-\lambda(m^2+n^2-p^2-q^2+d^2)$$
$$\text{with } \lambda=1+(pq/mn). \tag{2.6}$$

These formulas confirm the possibility, firstly proved in full generality by Darboux [2], that the condition (1.1) can be realized by means of a coupler AB connecting the links MP and NQ as shown in Fig. 1.

Section 3

If now, in the mechanism of Fig. 1, the system AB is considered as fixed (instead of MN), the five-bar linkage $BQFPA$ with sizes $\overline{BQ}=(\lambda-1)n=pq/m$, $\overline{QF}=q$, $\overline{FP}=p$, $\overline{PA}=(\lambda-1)m=pq/n$ and $\overline{AB}=c$ is again subject to an angular condition analogous to (1.1):

$$\measuredangle BQF=\omega, \qquad \measuredangle FPA=\pi-\omega. \tag{3.1}$$

Utilizing the result of §1, it follows that the joint F can be connected with the bar AB by a link FT, where T divides the segment AB in a ratio calculated by analogy to (1.9):

$$\overline{AT}:\overline{BT}=\frac{p^2q}{n}:-\frac{pq^2}{m}=mp:-nq=\overline{MS}:\overline{NS}. \tag{3.2}$$

With this step we have arrived at the eight-bar linkage of Fig. 2, known as the "focal mechanism" of Burmester [3]. It consists of a four-bar linkage $MABN$, completed by four additional bars PF, QF, SF, TF. Opposite sides of the quadrangle $MABN$ are divided by the new joints P, Q and S, T in equal proportions.

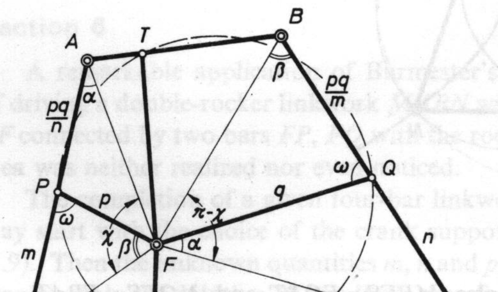

Figure 2. Focal mechanism of Burmester.

Section 4

Conversely it is always possible to complete an arbitrary four-bar linkage $MABN$ to a focal mechanism, and that in infinitely many ways. The four side-lengths a, b, c, d being given, any value may be chosen for the factor λ, and then the quantities m, n, p, q can be calculated from the four equations (2.5).

If we consider the quadrangle $MABN$ in one of its possible positions, the question arises, which points of the plane can be taken for the fourfold joint F. Noticing the similarity of the triangles $MPF \sim FQB$ ($\sphericalangle MPF = \sphericalangle BQF = \omega$, $\overline{PM} : \overline{PF} = \overline{QF} : \overline{QB} = m : p$), we state that

$$\sphericalangle PFM = \sphericalangle FBQ = \beta . \tag{4.1}$$

Analogously we find triangles $NQF \sim FPA$ and

$$\sphericalangle NFQ = \sphericalangle PAF = \alpha . \tag{4.2}$$

From this it follows (Fig. 2):

$$\sphericalangle AFM = \omega - \alpha + \beta = \chi , \qquad \sphericalangle NFB = \alpha + \pi - \beta - \omega = \pi - \chi . \tag{4.3}$$

In other words the situation of F is characterized by the property that opposite sides of the quadrangle $MABN$ subtend angles with sum π, at F if F is on the inside; if F is on the outside, the sum would be 0 or 2π. Generally we can write:

$$\sphericalangle AFM + \sphericalangle NFB \equiv 0 \pmod{\pi} . \tag{4.4}$$

Points with this property can be found at the intersection of pairs of auxiliary circles passing through M, A and N, B respectively, and forming similar figures with the chords MA and NB (Fig. 3). Any point F constructed in this way determines the angles α and β at A and B (Fig. 2) and allows the finding of P and Q by means of (4.2) and (4.1). Points S and T are added analogously.

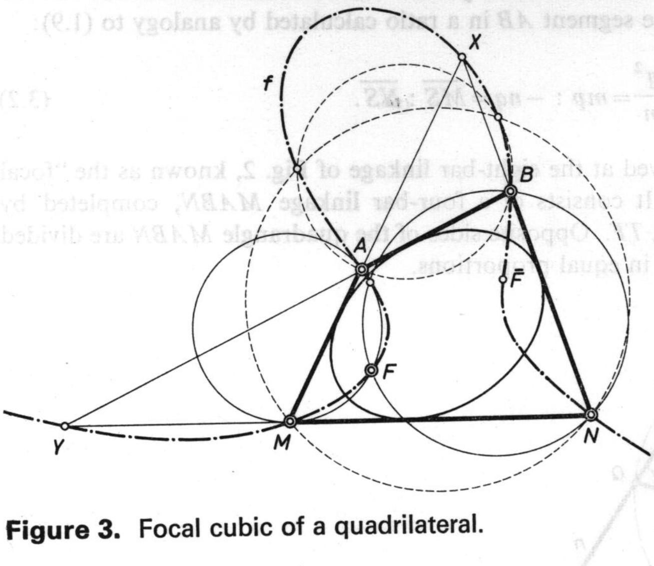

Figure 3. Focal cubic of a quadrilateral.

Noticing the similarity of the quadrangles $MPFS \sim FQBT$ and $NQFS \sim FPAT$—an immediate consequence of the similar triangles considered above—it is easy to state that the quadrangle $PSQT$ is inscribed in a circle, because opposite angles have the sum π (Fig. 2).

Making use again of the complex notation (§1), the angle condition (4.4) for the point $F(z)$ is expressed by

$$\frac{z-z_A}{z}:\frac{\bar{z}-\bar{z}_A}{\bar{z}}=\frac{z-z_B}{z-d}:\frac{\bar{z}-\bar{z}_B}{\bar{z}-d} \tag{4.5}$$

or

$$z(\bar{z}-d)(\bar{z}-\bar{z}_A)(z-z_B)=\bar{z}(z-d)(z-z_A)(\bar{z}-\bar{z}_B)\,. \tag{4.6}$$

This is the equation of the locus f of admissible points F in isotropic coordinates z, $\bar{z}$. As the highest term $z^2\bar{z}^2$ vanishes, f is an algebraic curve of order 3.

From the generation by means of similar circle pencils with base points M, A and N, B it follows that the cubic f passes through M, N, A, B and the points $X=MA\cdot NB$ and $Y=MN\cdot AB$. There is a second generation of f, based upon the condition

$$\measuredangle MFN+\measuredangle BFA\equiv 0 \pmod{\pi} \tag{4.7}$$

and utilizing similar circle pencils with base points M, N and A, B (dotted circles in Fig. 3).

An immediate consequence of $\measuredangle MPF=\measuredangle BQF=\omega$ (Fig. 2) is the fact that the four points P, F, Q and X are situated on a circle. Analogously S, F, T and Y are concyclic too.

Section 5

Consider now the linear system ("range") of all conics inscribed to the quadrilateral $MABN$.* A well known theorem of projective geometry, due to Desargues, says that the pairs of tangents issuing from a point G to all conics of the range form a (quadratic) involution [5]. This involution contains also the line pairs GM, GB and GN, GA representing the tangents from G to the degenerate conics (MB) and (NA) of the system.

Applying this theorem to the point F in Fig. 2, we state by aid of (4.3) that the line pairs FM, FB and FN, FA form angles with common symmetrals. Therefore the ray involution in F is a symmetric involution and comprises also the pair of isotropic lines through F (with directions $x:y=\pm i$). This means that F is a focus of one of the conics of the range (Fig. 3).

Thus the curve f (4.6) may be defined as the locus of the foci of all conics inscribed to $MABN$. Therefore f is called the "focal cubic" of the quadrilateral, and this was also the reason for the name of "focal mechanism".

Section 6

A remarkable application of Burmester's focal mechanism consists in the possibility of driving a double-rocker linkwork $MABN$ with revolving coupler AB by means of a crank SF connected by two bars FP, FQ with the rockers MA, NB. It seems that up till now this idea was neither realized nor even noticed.

The completion of a given four-bar linkwork of the indicated kind ($a+b>c+d$, $c<d$) may start with the choice of the crank support S on MN. This fixes the ratio $\rho=mp:nq$ (1.9). Then the unknown quantities m, n and p, q can be calculated from (2.5); this program requires only the solution of quadratic equations. The crank radius r is finally determined by the second formula (1.8).

* The term "inscribed" is to be understood in the sense of projective geometry: the range comprises all conics touching the unlimited lines MA, AB, BN and NM.

Let us consider in detail the symmetric case $a=b$, $\rho=1$ (Fig. 4). The quantities $m=n$ and $p=q$ are determined, according to (2.5), by

$$m^2-am+p^2=0, \qquad 2am^2-(2a^2-c^2+d^2)m-2ap^2+ad^2=0. \tag{6.1}$$

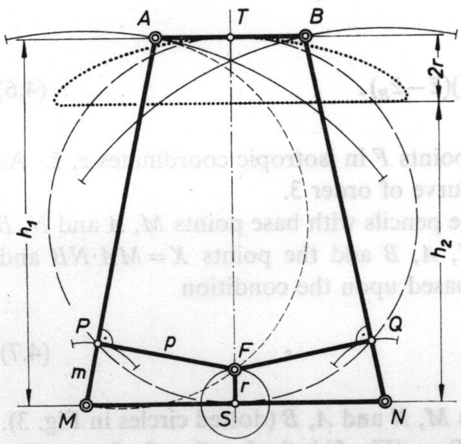

Figure 4. Crank driving of a symmetric double rocker linkwork.

Introducing the distances h_1, h_2 of base MN and coupler AB in parallel positions, given by

$$h_1^2=a^2-\tfrac{1}{4}(d-c)^2, \qquad h_2^2=a^2-\tfrac{1}{4}(d+c)^2, \tag{6.2}$$

the solutions may be written in the form

$$m=\frac{4a^2-c^2+d^2-4h_1h_2}{8a}, \qquad p=\frac{(c+d)h_1-(c-d)h_2}{4a}, \qquad r=\frac{h_1-h_2}{2}. \tag{6.3}$$

A graphic solution may proceed as follows: the location of the point S in the middle of MN induces the position of T by reason of (3.2) to be in the middle of AB. Considering now the symmetric form of the quadrilateral $MABN$ in Fig. 4, the circle $SPTQ$ mentioned in §4 is determined by its diameter ST and cuts the sides MA and NB in P and Q, respectively. The angles ω and $\pi-\omega$ at P and Q being equal because of the symmetry, we have $\omega=\pi-\omega=\pi/2$. Thus the perpendiculars $PF\perp MA$ and $QF\perp NB$ can be drawn and meet on the axis of symmetry in F. Another possibility is based upon (4.3): the angles χ and $\pi-\chi$ being equal to $\pi/2$, F may be found at the intersection of the two auxiliary circles having diameters MA and NB. The diameter $2r=2\cdot\overline{SF}$ of the crank circle is got in the simplest way as the distance between the two coupler positions parallel to the base MN; this follows immediately from the invariance of the length FT (Fig. 4).

Fig. 4 illustrates the special example $a=b=5$, $c=2$, $d=4$ of a Chebyshev linkwork for approximate straight-line motion realized by the coupler curve of the point T [6]. Formulas (6.3) give with $h_1=2\sqrt{6}$ and $h_2=4$:

$$m=n=\tfrac{2}{5}(7-2\sqrt{6})\approx 0{\cdot}8404; \qquad p=q=\tfrac{1}{5}(2+3\sqrt{6})\approx 1{\cdot}8697;$$
$$r=\sqrt{6}-2\approx 0{\cdot}4495. \tag{6.4}$$

The author's assistant Fuhs found this mechanism by pure intuition during the design of a demonstration model as shown in Fig. 5 [7].

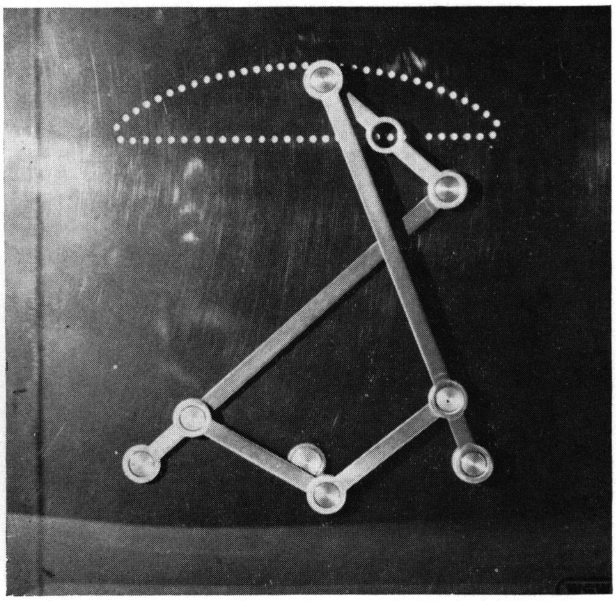

Figure 5. Demonstration model of Chebyshev's approximate straight-line motion.

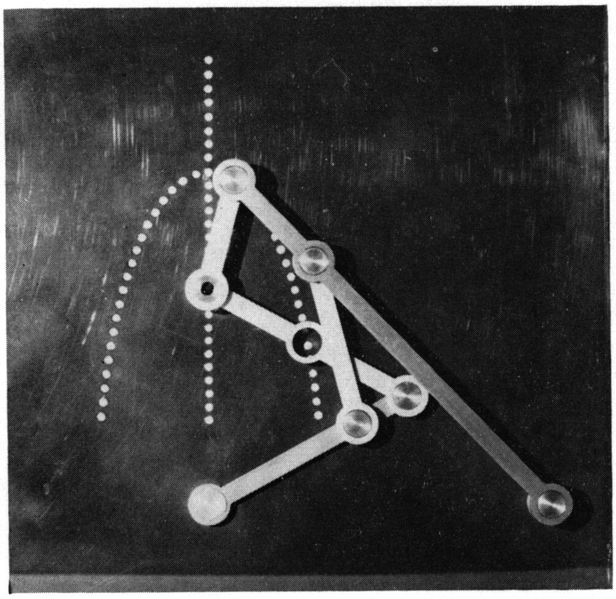

Figure 7. Demonstration model of Hart's straight-line linkwork.

Section 7

The exact straight-line linkwork of Hart [1] is nothing other than the special case of Burmester's focal mechanism inw hich $r=\infty$. The corresponding condition which follows is from (1.8) is

$$mp+nq=0, \tag{7.1}$$

and so we put

$$p=\kappa n, \qquad q=-\kappa m. \tag{7.2}$$

From (2.6) we get

$$\lambda=1-\kappa^2 \tag{7.3}$$

and with this value

$$a=(1-\kappa^2)m, \qquad b=(1-\kappa^2)n, \qquad c=\kappa d. \tag{7.4}$$

These formulas allow one in a simple way to derive straight-line mechanisms with rational (or integer) sizes.* We have only to start with rational values for m, n, d and κ.

The straight line described by the point F is given by (1.6)—with vanishing term $z\bar{z}$—and therefore has the equation

$$x=\frac{d}{2\kappa}+\frac{1-\kappa^2}{2\kappa}\cdot\frac{m^2-n^2}{d}. \tag{7.5}$$

With S the point T also becomes a point at infinity. This means that, fixing the bar AB, Hart's six-bar linkage moves the joint F along a straight line perpendicular to AB.

If $m=p$ and $x=0$, the link PF is performing an elliptic motion. The two conditions require

$$m=\kappa n, \qquad d=(1-\kappa^2)n \tag{7.6}$$

and induce $a=c$ and $b=d$. Thus $MABN$ is a kite quadrangle.

Figures 6 and 7 illustrate the special example $n=8$, $\kappa=\frac{1}{2}$. The other quantities, calculated by means of (7.6), (7.2) and (7.4), are:

$$m=p=4, \qquad b=d=6, \qquad q=-4, \qquad a=c=3. \tag{7.7}$$

The demonstration model, constructed again by Fuhs [7], shows besides the straight path of F also the ellipse described by the midpoint of PF.

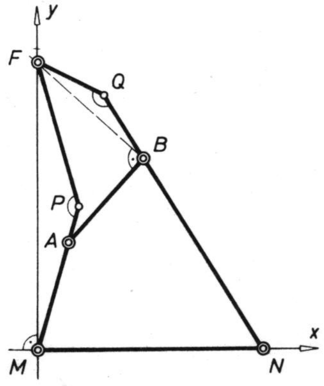

Figure 6. Hart's six-bar linkage for exact straight-line motion.

* Negative signs, as in (7.2) at q, cause certain modifications of the meaning of the angle condition (1.1).

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