Jnl. Mechanisms Volume 3, pp.391–396/Pergamon Press 1968/Printed in Great Britain

Concerning the Trajectory of the Center of Mass of the Four-Bar Linkage and the Slider-Crank Mechanism*

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Abstract
The path of the combined center of mass of a four-bar linkage is shown to be a coupler curve which is similar to a coupler curve of this mechanism. Also, the trajectory of the total center of mass of a slider-crank linkage is shown to be a connecting-rod curve which is similar to a connecting-rod curve of this mechanism. These properties are extended in order to determine the necessary conditions for redistributing the link masses such that the total center of mass of a given linkage remains stationary.

1. LET $A_0A_1A_2A_3$ be a plane four-bar linkage with the fixed pivots A_0 and A_3 (Fig. 1). The change of position of the moving links is best described by considering the linkage in the complex plane, and by expressing the variable vectors A_0A_1, A_1A_2, A_2A_3 as the complex numbers u_1, u_2, u_3 which have constant absolute magnitude and a constant sum.

$$|u_k| = a_k = \text{const.}, \quad (k=1, 2, 3), \qquad u_1+u_2+u_3=a=\text{const}. \tag{1}$$

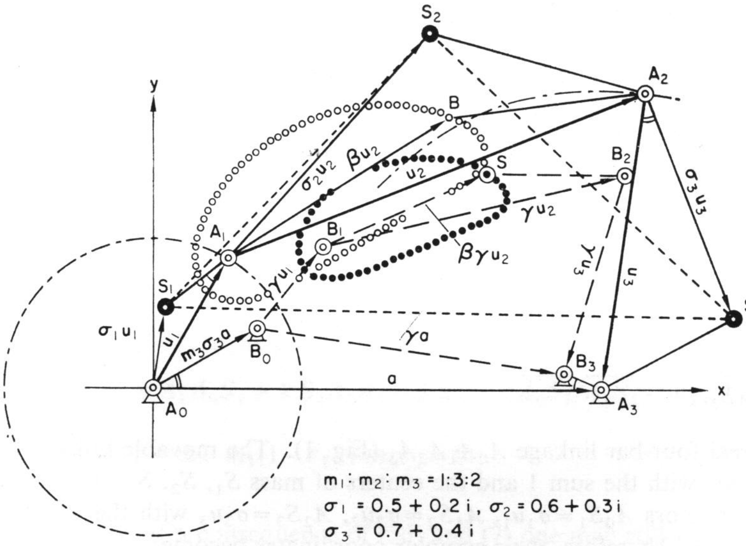

Figure 1. Four-bar linkage with center of mass trajectory and coupler curve which is similar to it.

*_Buletinul Institutului Politehnic din Iasi_ Serie Nouă, Tomul X (XIV), Fasc. 1–2, 285–291 (1964).

A point B which is rigidly connected to coupler A_1A_2 is defined by the vector $A_1B=\beta u_2$ with $\beta=$const., and this point describes a coupler curve during the motion. This well known algebraic curve, which is dependent on six mechanism parameters and is in general of sixth degree and tri-circular,* is a coupler curve. It finds a great deal of application in machine design because of its simple generation as well as of its great variety of forms. If one lets point A_0 be the origin of the complex plane, one may very easily describe the coupler curve with the complex equation

$$z=u_1+\beta u_2\,; \tag{2}$$

where the variables u_k are based on equation (1).†

For the subsequent material, we need the following:

First Auxiliary Theorem

If u_1, u_2, u_3 are three complex variables with invariant absolute values as well as a constant sum $a\neq 0$, then each linear combination $Z=\alpha_0+\alpha_1u_1+\alpha_2u_2+\alpha_3u_3$ with constant coefficients α_k represents, in general, a coupler curve which is similar to a coupler curve which has been created by a four-bar linkage where $u_1+u_2+u_3=a$.

Elimination of u_3 leads to the linear polynomial

$$Z=(\alpha_0+\alpha_3a)+(\alpha_1-\alpha_3)u_1+(\alpha_2-\alpha_3)u_2 \tag{3}$$

in u_1 and u_2 only, which may be presented as a linear function of z (2):

$$Z=c+\gamma z \quad \text{with} \quad c=\alpha_0+\alpha_3a \quad \text{and} \quad \gamma=\alpha_1-\alpha_3\,, \tag{4}$$

where $\beta=(\alpha_2-\alpha_3)/(\alpha_1-\alpha_3)$.

The position vector Z comes from the coupler curve z through the similarity transformation $Z=c=\gamma z$. The coupler curve Z which is created by the four-bar linkage $B_0B_1B_2B_3$ has the fixed bearing points $B_0(Z_0=c)$ and $B_3(Z_3=c+\gamma a=\alpha_0+\alpha_1a)$ as well as the moving links $v_k=\gamma u_k$, $(k=1, 2, 3)$.

The position vector Z is reduced accordingly to a circle whenever:

$$\alpha_2=\alpha_3\,, \quad (\beta=0,\ \text{centerpoint } B_0) \tag{5a}$$

$$\alpha_1=\alpha_2\,, \quad (\beta=1,\ \text{centerpoint } B_3) \tag{5b}$$

$$\alpha_1=\alpha_3\,, \quad (\beta=\infty,\ \text{centerpoint } B_0=B_3)\,. \tag{5c}$$

The position vector shrinks finally to a point whenever:

$$\alpha_1=\alpha_2=\alpha_3\,, \tag{5d}$$

namely to the point $B_0=B_3$.

2. We now consider a real four-bar linkage $A_0A_1A_2A_3$ (Fig. 1). The movable links may have the masses m_1, m_2, m_3 with the sum 1 and the centers of mass S_1, S_2, S_3. Let their positions be given by the vectors $A_0S_1=\sigma_1u_1$, $A_1S_2=\sigma_2u_2$, $A_2S_3=\sigma_3u_3$ with the general complex constants σ_k defined. Therefore, their complex coordinates become:

$$S_1\colon\ z_1=\sigma_1u_1\,, \qquad S_2\colon\ z_2=u_1+\sigma_2u_2\,, \qquad S_3\colon\ z_3=u_1+u_2+\sigma_3u_3 \tag{6}$$

* Compare references [1] and [2]. A well developed theory of the coupler curve by complex numbers was given by A. Haarbleicher [3].

† The author gave in [4] a short analytic proof, based on this complex notation, of the classical theorem of S. Roberts which concerns itself with the three-fold generation of coupler curves of four-bar linkages.

The total center of mass S of the movable parts can therefore be given by:

$$\left.\begin{aligned} &S\colon\ Z = m_1 z_1 + m_2 z_2 + m_3 z_3 = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 \\ \text{with}\quad &\alpha_1 = m_1\sigma_1 + m_2 + m_3 = 1 + m_1(\sigma_1 - 1), \\ &\alpha_2 = m_2\sigma_2 + m_3, \qquad \alpha_3 = m_3\sigma_3 . \end{aligned}\right\} \tag{7}$$

On the basis of Auxiliary Theorem No. 1, one may note:

Theorem No. 1. The path of the combined center of mass of a four-bar linkage is, in general, a coupler curve which is similar to a coupler curve of the particular mechanism.

With this one proves in general a fact which, under limiting assumptions (i.e. for real σ_k) and from a purely geometric point of view, was given by R. Kreutzinger [5].

For the fixed bearing points B_0 and B_3 of the four-bar linkage $B_0B_1B_2B_3$, which creates the path of the center of mass, and which at each moment is similar to four-bar linkage $A_0A_1A_2A_3$, one finds, with consideration of $\alpha_0 = 0$ and (7), the coordinates

$$B_0\colon\ Z_0 = m_3\sigma_3 a, \qquad B_3\colon\ Z_3 = [1 + m_1(\sigma_1 - 1)]a. \tag{8}$$

Through this, the position of this four-bar linkage is fully defined. The characteristic quantity β with which one defines the coupler point S, which describes the coupler curve has, according to (4) and (7), the value:

$$\beta = \frac{(\alpha_2 - \alpha_3)}{(\alpha_1 - \alpha_3)} = \frac{m_2\sigma_2 + m_3(1 - \sigma_3)}{m_1\sigma_1 + m_2 + m_3(1 - \sigma_3)}. \tag{9}$$

With the help of these formulae one may find the trajectory of the center of mass of the four-bar linkage of Fig. 1. Positions where the tangent is horizontal belong to such equilibrium positions of the mechanism whose stability or lack of stability can be recognized easily.

With the criteria given in (5a)–(5d) one obtains information concerning the exceptions to Theorem No. 1. The trajectory of the center of mass becomes a circle when $\beta = 0$, 1, or ∞. This has the following consequences in the light of the real and positive values of m_k:

$$\begin{aligned} &\beta = 0\colon\ \sigma_2/(\sigma_3 - 1) = m_3/m_2\,; \\ &\sphericalangle A_2A_1S_2 + \sphericalangle S_3A_3A_2 = \pm\pi, \qquad \overline{A_1S_2}/\overline{A_3S_3} = m_3 \,.\, \overline{A_1A_2}/(m_2 \,.\, \overline{A_2A_3}), \end{aligned} \tag{10a}$$

$$\begin{aligned} &\beta = 1\colon\ \sigma_1/(\sigma_2 - 1) = m_2/m_1\,; \\ &\sphericalangle A_1A_0S_1 + \sphericalangle S_2A_2A_1 = \pm\pi, \qquad \overline{A_0S_1}/\overline{A_2S_2} = m_2 \,.\, \overline{A_0A_1}/(m_1 \,.\, \overline{A_1A_2}), \end{aligned} \tag{10b}$$

$$\beta = \infty\colon\ m_1(1 - \sigma_1) + m_3\sigma_3 = 1. \tag{10c}$$

Finally, as a consequence of (5d) and (7) one may still state the remarkable:

Theorem No. 2. The total center of mass of a four-bar linkage is stationary when the following conditions are satisfied:

$$\sigma_2 = 1 + \frac{m_1}{m_2}\sigma_1 \quad\text{and}\quad \sigma_3 = 1 + \frac{m_2}{m_3} + \frac{m_1}{m_3}\sigma_1. \tag{10d}$$

Its position is given by $m_3\sigma_3 a$. The double conditions of (10d) are equivalent to the two conditions of (10a) and (10b). These also furnish the geometric significance. The condition may certainly be fulfilled since the individual centers of mass of each link can be brought to any desired point on the link through the appropriate link design and distribution of mass. Such a distribution of centers of mass for the mechanism of Fig. 1 with real σ_k is shown by Fig. 2. A four-bar linkage which is constructed according to Theorem No. 2, which has an invariant total center of mass, is in neutral equilibrium in all positions. It maintains this property also after changing the bearing points A_0 and A_3 since the condition (10d) concerns itself only with masses and centers of mass of the three links.

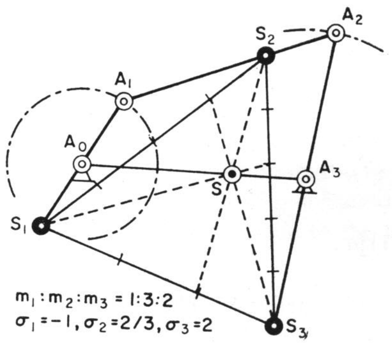

Figure 2. Four-bar linkage with stationary center of mass.

3. The slider-crank mechanism $A_0A_1A_2A_2'$ (Fig. 3), with fixed bearing point A_0 and point A_2 translating in a straight line, may be interpreted as the limiting form of a four-bar linkage which has the second bearing point A_3 at infinity. The analytic treatment now needs a certain modification. Let A_0 be the origin of a complex plane and assign the vectors A_0A_1, A_1A_2 to be expressed by the complex variables u_1, u_2 with fixed absolute values. Then let u_3 be a complex constant which represents the vector A_2A_2' whose terminal lies on a straight line through A_0 which is parallel to the path of A_2. One may describe the mechanism by:

$$|u_k| = a_k = \text{const.}, \qquad (k=1,2), \qquad u_3 = \text{const.},$$
$$\arg.\,(u_1+u_2+u_3) = \alpha = \text{const.} \tag{11}$$

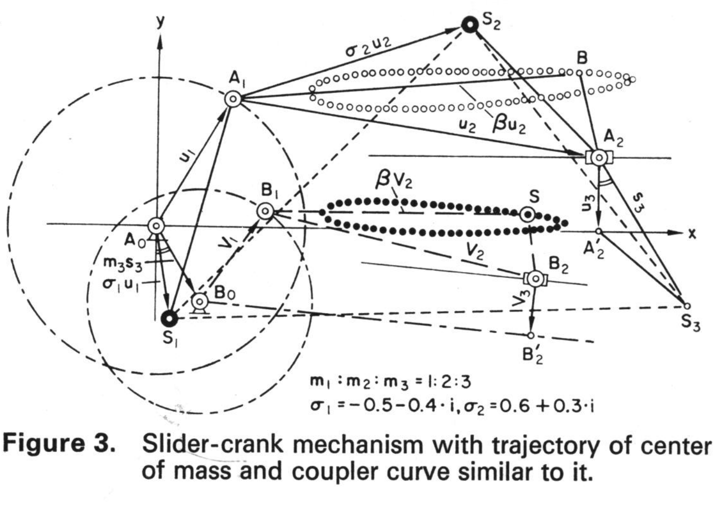

Figure 3. Slider-crank mechanism with trajectory of center of mass and coupler curve similar to it.

If one places, as shown in Fig. 3, the real axis parallel to the path of A_2, one obtains $\alpha=0$. For the usual inline slider-crank, one may assume $u_3=0$.

Point B, which is rigidly connected to the coupler A_1A_2, which may be defined by the vector $A_1B=\beta u_2$ with $\beta=$const., describes during the motion a so-called-connecting rod curve. This well known algebraic curve, which is dependent on five parameters, is in general of fourth degree and monocircular [1]–[3]. It again may be described by the complex equation:

$$z=u_1+\beta u_2, \tag{12}$$

where the variables u_k must now satisfy (11).

One notes the corresponding:

Second Auxiliary Theorem

Whenever u_1, u_2 are two complex variables with fixed absolute values, and whenever their sum is increased by a constant u_3 and that sum has the fixed argument α, then every linear combination $Z=\alpha_0+\alpha_1u_1+\alpha_2u_2$ with constant coefficients α_k is in general a connecting rod curve of the slider-crank which is similar to a connecting-rod curve created by the slider-crank mechanism of $\arg(u_1+u_2+u_3)=\alpha$.

The transformation

$$Z=\alpha_0+\alpha_1(u_1+\beta u_2) \text{ with } \beta=\alpha_2/\alpha_1, \quad (\alpha_1\neq 0), \tag{13}$$

shows that the position vector Z of the connecting-rod curve z (12) originates from the similarity transformation $Z=\alpha_0+\alpha_1 z$. The slider-crank mechanism $B_0B_1B_2B_2'$ which generates the connecting-rod curve Z has the fixed bearing point $B_0(Z_0=\alpha_0)$, the movable links $B_0B_1=v_1=\alpha_1u_1$, $B_1B_2=v_2=\alpha_1u_2$, and the guided straight line for B_2 which goes through the point $\alpha_0-\alpha_1u_3$ and has the direction $\alpha+\arg\alpha_1$.

The curve given by the position vector Z is reduced to a circle (centerpoint B_0) in the case:

$$\alpha_1=0, (\beta=\infty) \text{ or } \alpha_2=0, (\beta=0). \tag{14a, b}$$

It shrinks to the point B_0 in the case:

$$\alpha_1=\alpha_2=0. \tag{14c}$$

4. Consider now the material slider-crank mechanism $A_0A_1A_2A_2'$ (Fig. 3). The movable links A_0A_1 (crank), A_1A_2 (connecting rod), and A_2A_2' (piston) may have the relative masses m_1, m_2, m_3 with the sum 1 and the centers of mass S_1, S_2, S_3. Their positions are given by the vectors $A_0S_1=\sigma_1u_1$, $A_1S_2=\sigma_2u_2$ with, in general, the complex constants σ_k, and also by $A_2S_3=s_3=$const. The complex coordinates of the individual centers of mass are therefore:

$$S_1\colon\ z_1=\sigma_1u_1, \qquad S_2\colon\ z_2=u_1+\sigma_2u_2, \qquad S_3\colon\ z_3=u_1+u_2+s_3 \tag{15}$$

The total center of mass S of the movable parts is then given by:

$$\begin{aligned}&S\colon\ Z=m_1z_1+m_2z_2+m_3z_3=\alpha_0+\alpha_1u_1+\alpha_2u_2\\&\text{with } \alpha_0=m_3s_3, \qquad \alpha_1=m_1\sigma_1+m_2+m_3=1+m_1(\sigma_1-1),\\&\alpha_2=m_2\sigma_2+m_3.\end{aligned} \tag{16}$$

Based on the Auxiliary Theorem No. 2 one may therefore state:

Theorem No. 3. The trajectory of the total center of mass of the slider-crank mechanism is, in general, a connecting-rod curve which is similar to a connecting-rod curve of the mechanism.

For the fixed bearing point B_0 of the mechanism $B_0B_1B_2B_2'$ which generates the trajectory of the center of mass, and which at all times is similar to the mechanism $A_0A_1A_2A_2'$, one finds according to [3]:

$$B_0: \; Z_0 = \alpha_0 = m_3 s_3 . \tag{17}$$

For the moving links the vectors are:

$$B_0B_1 = v_1 = \alpha_1 u_1 , \qquad B_1B_2 = v_2 = \alpha_1 u_2 , \qquad B_2B_2' = v_3 = \alpha_1 u_3 . \tag{18}$$

Since the straight line of the path of B_2' passes through B_0, one may define this new mechanism completely by the above. The characteristic quantity β, which describes that point S on the connecting rod which generates the connecting-rod curve, has according to (13) and (16) the following value:

$$\beta = \frac{\alpha_2}{\alpha_1} = \frac{m_2\sigma_2 + m_3}{m_1\sigma_1 + m_2 + m_3} . \tag{19}$$

The trajectory of the center of mass of the slider-crank mechanism shown in Fig. 3 was determined with the help of these formulae.

To determine the exceptions of Theorem No. 3, one uses the criteria (14a)–(14c). The trajectory of the center of mass becomes a circle whenever $\beta = \infty$ or 0. This takes place when:

$$\sigma_1 = -\frac{(m_2 + m_3)}{m_1} \tag{20a}$$

$$\sigma_2 = -\frac{m_3}{m_2} . \tag{20b}$$

In the first case one finds the center of mass of the crank S_1 on the extension of the crank radius A_0A_1, while in the second case the center of mass of the connecting rod S_2 lies on the extension of the connecting rod A_1A_2. The position of the center of mass S_3 of the piston is of no importance in these considerations.

If the conditions (20a) and (20b) are fulfilled at the same time, one obtains according to (14c) a slider-crank mechanism which has a stationary total center of mass at B_0 (17).

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