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Contributions to the Geometry of Cam Mechanisms with Oscillating Followers

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Abstract
The geometric fundamentals for two-disk cam mechanisms operating two rigidly connected oscillating followers with straight-line or circular-arc profiles are developed. Special attention is paid to the existence of single-disk cam mechanisms of this kind. For this purpose the problem of plane curves with an isoptic circle is discussed. Finally single-disk cam mechanisms with two oscillating roller followers joined by a coupler link are considered.

Zusammenfassung—Beiträge zur Geometrie von Nockentrieben mit Schwinghebeln: W. Wunderlich.
Die geometrischen Grundlagen für Doppelnockentriebe, die zwei starr verbundene Schwinghebel mit geraden oder kreisförmigen Flanken betätigen, werden entwickelt. Besonderes Augenmerk wird der Existenz von Einscheiben-Nockentrieben dieser Art zugewandt. Zu diesem Zweck wird das Problem von ebenen Kurven mit einem isoptischen Kreis erörtert. Schliesslich werden Einscheiben-Nockentriebe mit zwei Schwinghebeln betrachtet, die durch ein Koppelglied gelenkig verbunden sind.

Резюме — Вклад в геометрию кулачковых механизмов с коромыслами: В. Вундерлих.
В работе изложены геометрические основы двухдисковых кулачковых механизмов, очерченных прямыми линиями или дугами окружностей и приводящих в движение два жестко соединенных коромысла. Особенное внимание уделено существованию однодискового кулачкового механизма этого рода. С этой целью обсуждается проблема плоских кривых с изоптическим кругом. В конце рассматривается однодисковый кулачковый механизм с двумя коромыслами, соединенными шарниром.

1. Cam Mechanisms with Oscillating Flat-Faced Followers

A PLANE cam mechanism operating an oscillating follower consists of three systems: The fixed system or frame Σ_0; the cam disk Σ_1 with a characteristic profile c, usually rotating uniformly about a fixed pivot $O \in \Sigma_0$; the follower Σ_2 turning about another fixed pivot $P \in \Sigma_0$ and being in steady contact with the cam. The corresponding profile of the follower will at first be supposed as a straight line p through P. The rotation of the cam system Σ_1 induces an oscillating motion of the follower system Σ_2. In most cases cam mechanisms serve as dwell mechanisms; then a certain part of the cam profile c is an arc of a circle with center O (Fig. 1).

The shape of the cam profile c may be described by its "support function" $u(\tau)$, relating the central distance u of the profile tangent p with its angle of direction τ. Thus

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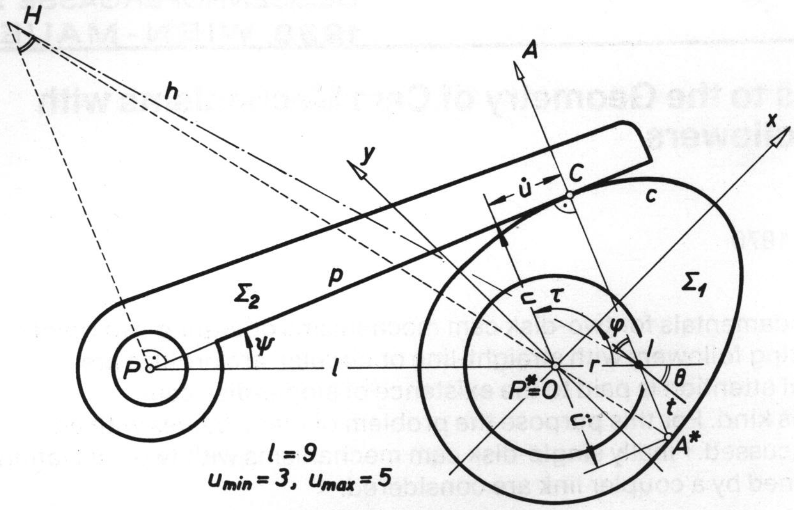

Figure 1. Cam mechanism with oscillating flat-faced follower.

the cartesian equation of the tangent line p in a rectangular coordinate system x, y with origin O and belonging to Σ_1 is given by

$$x \cos\tau + y \sin\tau = u(\tau). \tag{1.1}$$

By differentiation we get the normal of the envelope c of the straight-line set (1.1) with

$$-x \sin\tau + y \cos\tau = \dot{u}(\tau) = \frac{\mathrm{d}u}{\mathrm{d}\tau}. \tag{1.2}$$

Solving the pair of equations (1.1) and (1.2) we arrive at a parametric representation of the cam profile c:

$$x = u \cos\tau - \dot{u} \sin\tau, \qquad y = \dot{u} \sin\tau + \dot{u} \cos\tau. \tag{1.3}$$

Differentiating once more we get the normal of the evolute of c

$$-x \cos\tau - y \sin\tau = \ddot{u}(\tau) = \frac{\mathrm{d}^2u}{\mathrm{d}\tau^2} \tag{1.4}$$

and thus the radius of curvature of c:

$$\rho = u + \ddot{u}. \tag{1.5}$$

To ensure the convexity of the cam profile c, necessary in the case of flat-faced followers, $u(\tau)$ must be a periodic function with period 2π and $u + \ddot{u} \geqq 0$.

The motion of the cam mechanism is characterized by the relation between the angles of rotation φ and ψ of cam and follower, respectively. If these angles are measured from the line connecting P with O, we see from Fig. 1:

$$\varphi + \tau = \psi + \frac{\pi}{2}, \qquad u(\tau) = l \cdot \sin\psi, \tag{1.6}$$

where l means the known distance between the pivots O and P. Equations (1.6) define the characteristic function $\psi(\varphi)$ by means of the parametric function $\varphi(\tau)$ and $\psi(\tau)$. Figure 2 shows the diagram of the motion law $\psi(\varphi)$ corresponding to the cam mechanism of Fig. 1, where the non-circular part of the disk profile c has been assumed to be an ellipse.

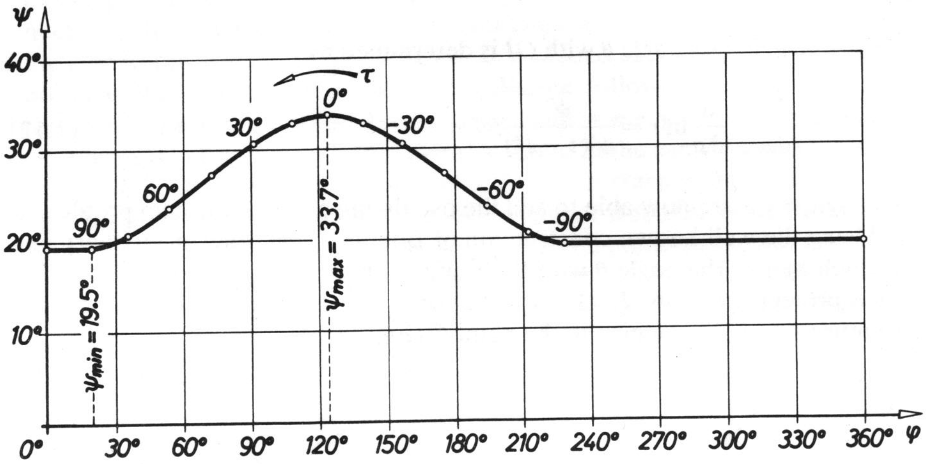

Figure 2. Motion law for the cam mechanism of Fig. 1.

In practice the motion law $\psi(\varphi)$ will be prescribed, and the problem consists in finding the cam profile c, i.e. its support function $u(\tau)$. The analytical way is evident: Starting with arbitrary pairs of corresponding values φ, ψ we get from (1.6):

$$\tau = \psi - \varphi + \frac{\pi}{2}, \qquad u = l \cdot \sin \psi. \tag{1.7}$$

The coordinates x, y of cam points $C \in c$ may then be calculated by means of (1.3) with

$$\dot{u} = l\dot{\psi} \cdot \cos \psi = \frac{l\psi' \cos \psi}{\psi' - 1}, \tag{1.8}$$

where the prime indicates the derivative with respect to φ. The value of $\dot{\psi}$ was determined from the relations

$$\dot{\psi} = \psi' \dot{\varphi} = \dot{\varphi} + 1. \tag{1.9}$$

The construction procedure is also simple. We consider the relative motions of Σ_0 and Σ_2 with respect to the cam system Σ_1. The point $P \in \Sigma_0$ describes a circle o with center O and radius l. Any position $P \in o$ determines the angle φ by $\measuredangle xOP = \pi - \varphi$. The corresponding angle ψ determines the follower line p, as $\measuredangle OPp = \psi$. Proceeding in this way, we get the desired cam profile c as the envelope of its tangent lines p. To add the points of contact we mark the instant center I of Σ_2/Σ_1. According to the theorem of Aronhold–Kennedy it is situated on the line PO at a distance $\overline{OI} = r$, defined by the proportion

$$r : (l + r) = \mathrm{d}\psi : \mathrm{d}\varphi = \psi' : 1. \tag{1.10}$$

The perpendicular to p, issued from I, cuts the line p in its contact point C with c (Fig. 1).

The polar equation

$$r = r(\varphi) = \frac{l\psi'}{1-\psi'} \tag{1.11}$$

defines the centrode in Σ_1. Its angle θ with OI is determined by

$$\cot\theta = \frac{\mathrm{d}r}{r\mathrm{d}\varphi} = \frac{\mathrm{d}}{\mathrm{d}\varphi}\ln r = \frac{\psi''}{\psi'(1-\psi')}. \tag{1.12}$$

By aid of this angle θ we are now able to add the osculating circle of the cam profile c at any point C. Using the well-known construction of Bobillier* we draw an auxiliary line h through I which makes the angle θ with the profile normal IC, we cut h with the line $PH \| IC$ and we project the point $H \in h$ from O onto the normal IC; thus we get the center of curvature A^* of c, belonging to the point C (Fig. 1).

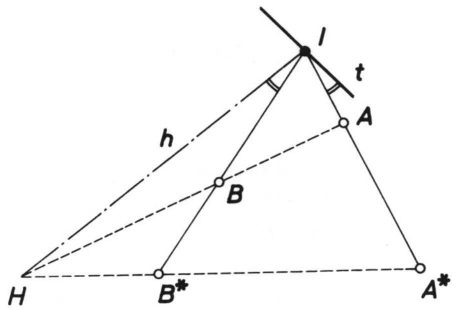

Figure 3. Bobillier's construction.

If we prefer to calculate the radius of curvature $\rho = \overline{A^*C}$ we can start from formula (1.5). Differentiating equation (1.8) and making use of (1.9) we get

$$\rho = \frac{l}{(1-\psi')^3}(1-\psi')(1-2\psi')\sin\psi + \psi''\cos\psi. \tag{1.13}$$

A slight modification enables us to apply the developed theory to oscillating followers with a straight-line profile not passing through the pivot P: adding to $u(\tau)$ a constant a (positive or negative) we get the support function of a cam profile c which is

*Figure 3 recalls the general construction of E. Bobillier. At any moment of a constrained plane motion there exists a (quadratic) correspondence T: $X \to X^*$, relating each point X of the moving plane with the center of curvature, X^*, of the path of X traced in the fixed plane. This transformation T relates also the corresponding centers of curvature of a moving curve and its envelope. The transformation T has the following essential property: If two pairs of corresponding points A, A^* and B, B^* are on different pole rays, the lines AB and A^*B^* meet on an axis h which passes through the instant center I and forms there the same angle with one pole ray as the other pole ray with the centrode tangent t (Fig. 3). Thus it is possible to construct the centrode tangent t, if two point pairs A, A^* and B, B^* on different pole rays are known. Conversely, if the instant center I, the centrode tangent t and one pair A, A^* are known, it is possible to find any other pair B, B^*. In Fig. 1 we know indeed the pole I, the centrode tangent t and the pair of corresponding points $P, P^* = O$; hence we can construct the center of curvature, A^*, of the profile c as the point corresponding to the point at infinity, $A \perp p$, which is the center of curvature of the moving line p generating the profile c.

parallel to c at the distance a and appropriate to work with a follower line $\bar{p} \| p$ that is offset from P by the distance a (see Fig. 11).

2. Double-Disk Cams with Flat-Faced Followers

The cam disk c operating the straight-line follower p is well able to achieve the forward motion. To avoid springs or other elastic links maintaining the contact of p and c during the return motion, we may arrange a second straight-line follower $\bar{p}$ rigidly connected with p and forming in P a constant angle $\pi-\gamma$ with p (Fig. 4). This line $\bar{p}$ belonging also to the system Σ_2, generates in Σ_1 a second cam profile $\bar{c}$ which is appropriate to produce the return motion of the oscillating follower, provided the cam center O remains in the interior of the moving angle $\measuredangle \bar{p}p = \pi - \gamma$.

The point of contact, $\bar{C}$, of $\bar{p}$ and $\bar{c}$ is found in the same way as before by issuing the normal $I\bar{C} \perp \bar{p}$ form the instant center I. The corresponding center of curvature, $\bar{A}^*$, may be added again by aid of the construction of Bobillier using the centrode tangent t. Both of the centers A^* and $\bar{A}^*$ are situated, as all points related to points at infinity,† on the so-called "cusp circle" which touches the centrode in I.† Thus the center of curvature $\bar{A}^*$ can simply be found by intersecting the normal $I\bar{C}$ with the cusp circle determined by I, t and A^* (Fig. 4).

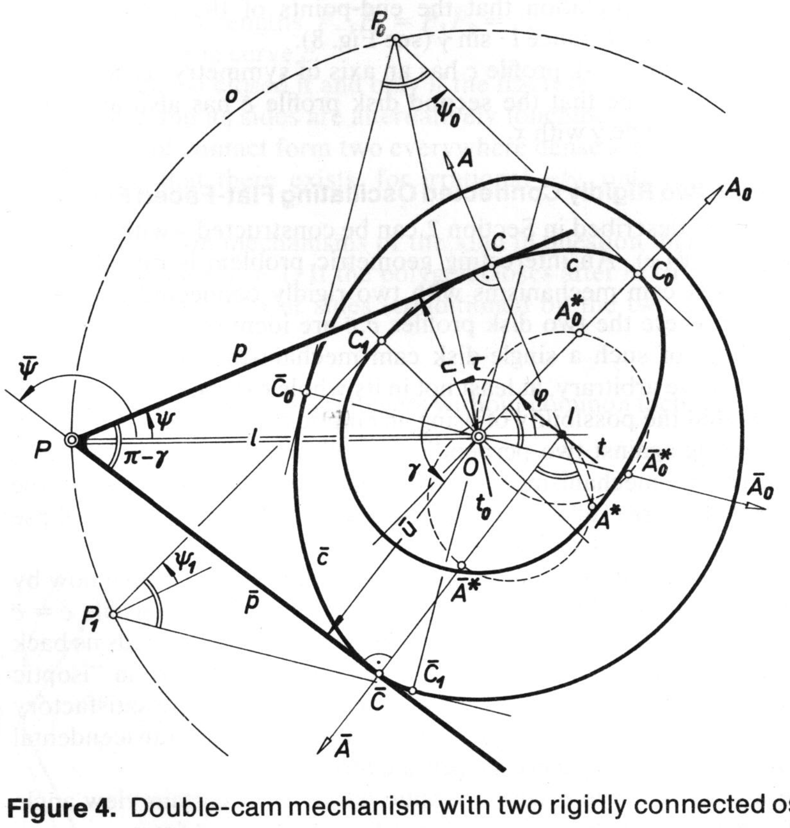

Figure 4. Double-cam mechanism with two rigidly connected oscillating flat-faced followers.

†The cusp circle, to be considered as a locus in the fixed plane, is analogous to the well-known "inflection circle" of the moving plane; it is identical with the inflection circle of the inverted motion. The cusp circle comprises all cusps which at the moment may appear at envelopes generated by moving straight lines; these lines form a pencil.

The characteristic function for the motion of $\bar{p}$ is described by

$$\bar{\psi}(\varphi) = \psi(\varphi) + \gamma. \tag{2.1}$$

From this relation the support function $\bar{u}(\bar{\tau})$ of the second cam profile $\bar{c}$ can be derived by analogy to (1.7); thus we arrive at

$$\bar{\tau} = \bar{\psi} - \varphi + \frac{\pi}{2} = \tau + \gamma, \qquad \bar{u} = l \cdot \sin \bar{\psi} = l \cdot \sin(\psi + \gamma). \tag{2.2}$$

Developing the last expression and comparing it with (1.7) we find

$$\bar{u} = u \cos \gamma + \sqrt{l^2 - u^2} \sin \gamma, \tag{2.3}$$

which leads to the algebraic identity

$$u^2 + \bar{u}^2 - 2u\bar{u} \cos \gamma = l^2 \sin^2 \gamma = \text{const.} \tag{2.4}$$

This means in geometrical interpretation that the end-points of the perpendiculars $\overline{Op} = u$ and $\overline{O\bar{p}} = \bar{u}$ have constant distance $l \cdot \sin \gamma$ (see Fig. 8).

If, as in Figs. 1 and 4, the first disk profile c has an axis of symmetry, x, containing the cam center O, it is easy to see that the second disk profile $\bar{c}$ has also an axis of symmetry, $\bar{x}$, forming in O the angle γ with x.

3. Single-Disk Cams with two Rigidly Connected Oscillating Flat-Faced Followers

Double-disk cams as just described in Section 2 can be constructed—within certain limits—for any motion law $\psi(\varphi)$. An interesting geometric problem is raised by the question whether there exist cam mechanisms with two rigidly connected oscillating straight-line followers $p, \bar{p}$ where the two disk profiles $c, \bar{c}$ are identical. Of course the characteristic function $\psi(\varphi)$ of such a single-disk cam mechanism will then be of a rather special kind and no more arbitrary, at least not in its whole extent.

M. Goldberg[1] indicated the possibility of using an elliptic cam profile $c = \bar{c}$ having its center in O and operating against two perpendicular follower lines $p \perp \bar{p}$ ($\gamma = \pi/2$, Fig. 5). The existence of this mechanism, recently studied in [6], is based upon the well-known fact, due to de la Hire (1685), that all rectangles circumscribed to an ellipse c are inscribed in a concentric circle o, called the "orthoptic circle" of c.

Considering for a mechanism of this kind the relative motion Σ_2/Σ_1, defined now by the sliding of the sides $p, \bar{p}$ of a rigid angle $\measuredangle \bar{p}Pp = \pi - \gamma$ along the cam profile $c = \bar{c}$ while the vertex P of the angle wanders on a circle o, the general problem leads us back to the question whether there exist convex plane curves c which possess an "isoptic circle" o. From this point of view, Green[2] gave a general, but not quite satisfactory solution: The particular examples derived by his method are of strongly transcendental character and rather different from Goldberg's algebraic prototype.

Let us investigate a convex curve c whose isoptic curve for a certain view angle $\pi - \gamma$ is a circle o; this means that the pair of tangent lines $p, \bar{p}$ issued from any point $P \in o$ to c forms always the constant angle $\measuredangle \bar{p}Pp = \pi - \gamma$. Starting from an arbitrary tangent p to c with intersection points P_0, P_1 on o we may construct an equiangular polygon $\ldots P_{-1}P_0P_1P_2 \ldots$ inscribed in the circle o and having at each corner the interior angle $\pi - \gamma$. Such a chord polygon of o has periodic structure and in general sides of

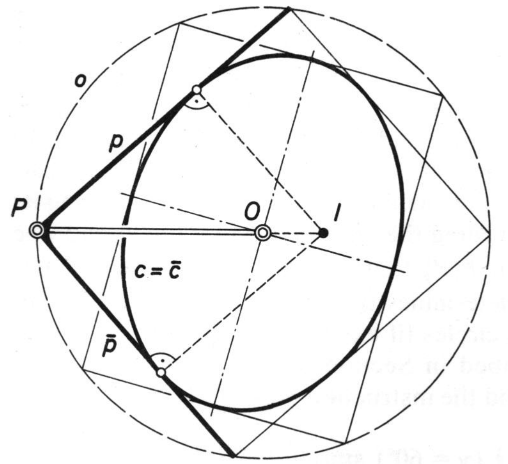

Figure 5. Goldberg's elliptical single-cam mechanism with two perpendicular oscillating flat-faced followers.

only two different lengths $\overline{P_{-1}P_0} = \overline{P_1P_2} = \ldots$ and $\overline{P_0P_1} = \overline{P_2P_3} = \ldots$ (Fig. 6). All sides must touch the curve c.

This polygon is closed if and only if the fraction π/γ is rational. Otherwise the polygon is infinite and its sides are alternatively touching two circles $k,\bar{k}$ concentric with o. As the points of contact form two everywhere dense sets on k and $\bar{k}$, we see, by reasons of continuity, that there exists, for irrational π/γ, only the trivial solution $c = k = \bar{k}$ ($\psi = \text{const}$).

Non-trivial cam mechanisms of the kind in question therefore are possible only for rational values of $\pi/\gamma > 1$. If the polygon closes after m circuits, we have, with respect to the even number $2n$ of sides (conditioned by the two sorts of sides), the relation $m \cdot 2\pi = 2n \cdot \gamma$ or

$$\gamma = \frac{m}{n}\pi \qquad (m,n \text{ integers without common factor}). \tag{3.1}$$

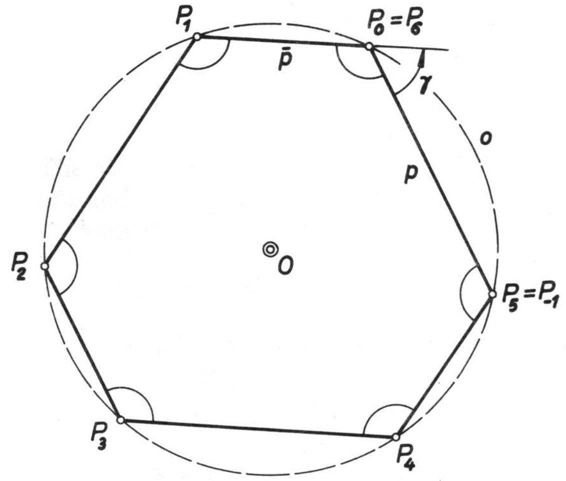

Figure 6. Closed equiangular chord polygon of a circle.

For $m = 1$ the chord polygons are convex and the sought-for cam profile $c = \bar{c}$ may be determined as the envelope of an appropriate continuous family of such polygons. A family of this kind can be constructed as follows: We start with an arbitrary equiangular chord polygon $P_1P_2 \ldots P_{2n}$ of the given circle o and mark an instant center I_1 on the radius OP_1; let C_1 and C_{2n} be the end-points of the perpendiculars dropped from I_1 to the sides P_1P_2 and P_1P_{2n}, respectively. Now draw at random a convex arc C_1C_{2n} as part of c, having no other points in common with the sides P_1P_2 and P_1P_{2n} and opening its concave side to O. Any chord $P_1'P_2'$ of o touching the chosen arc C_1C_{2n} is then to be completed to an equiangular chord polygon $P_1'P_2' \ldots P_{2n}'$ of o. The one-parametric family of polygons constructed in this way determines the rest of c as the envelope of the sides. The contact points and osculating circles (if existing for the initial arc) may be added by kinematical methods as described in Section 2.—For $m > 1$ (odd, see Section 4) we have star polygons (Fig. 10) and the instruction prescribing the initial arc of c has to be slightly modified.

The example shown in Fig. 7 for $m/n = \frac{1}{3}$ ($\gamma = 60°$) started from the regular polygon existing, because of reasons of continuity, in each family of chord polygons of the kind in question. Then a parabola was used as initial arc C_1C_6, touching the sides P_1P_2 and P_1P_6 to avoid corners of the profile c. The additional arc C_1C_2, derived from the parabola C_6C_1, is part of an algebraic curve of class 4.

Since each of the generating chord polygons $P_1'P_2' \ldots P_{2n}'$ is mapped onto itself by the rotation S with center O and angle 2γ, the enveloped cam profile $c = \bar{c}$ possesses the same rotational symmetry, i.e. is invariant with respect to the cyclic transformation group S^i $(i = 1, 2, \ldots n)$ of order n.

4. Analytic Treatment of Curves with an Isoptic Circle

For the analytic treatment of the disk problem considered in Section 3 let us put, for brevity, $l = 1$. Describing the disk profile $c = \bar{c}$ by means of its support function $u(\tau)$

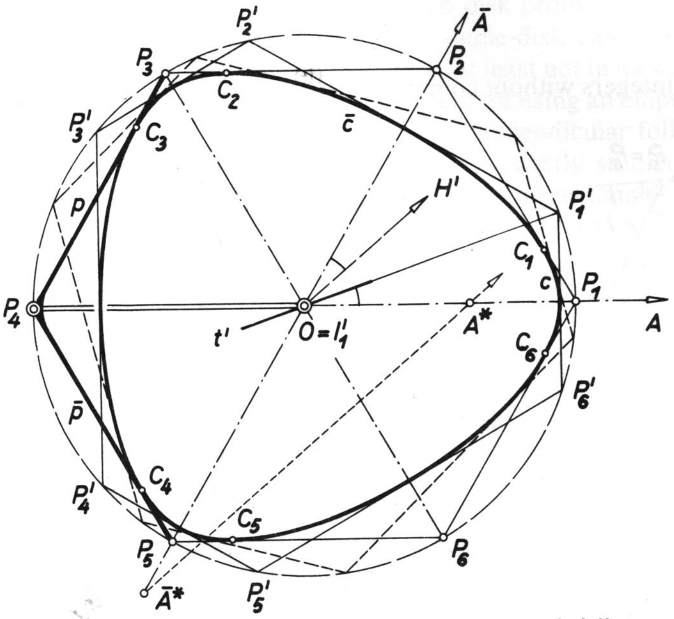

Figure 7. Single-cam mechanism with two rigidly connected oscillating flat-faced followers.

which is to be periodic with period 2γ, we have, in accordance with (2.4), the relation

$$u^2+\bar{u}^2-2u\bar{u}\cdot\cos\gamma=\sin^2\gamma \tag{4.1}$$

for the central distances $u=u(\tau)$ and $\bar{u}=u(\tau+\gamma)$ of two adjacent profile tangents p and $\bar{p}$ forming the prescribed view angle $\measuredangle \bar{p}Pp=\pi-\gamma$ in the point P of the isoptic circle o of c. This relation expresses the elementary fact that the segment joining the end-points of the distances u and $\bar{u}$ has constant length $\sin\gamma$; it represents a chord subtending the angle γ at the circumference of an auxiliary circle with unit diameter OP (Fig. 8).

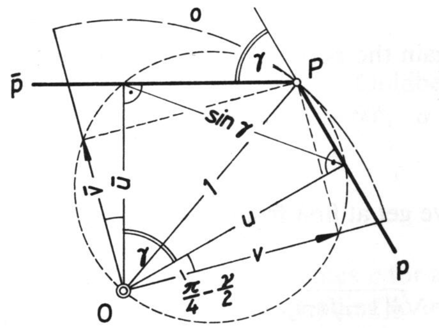

Figure 8. Transformation of coordinates.

As the relation (4.1) is valid for all values of the independent variable τ it is an identity in τ and a functional equation for the unknown function $u(\tau)$.

If we consider the quantities u and $\bar{u}$ as special coordinates of the point O, localizing it by its distances from the axes p and $\bar{p}$ for an observer in the system Σ_2, the relation (4.1) may be considered as the equation of that circle $\bar{o}$ with center P and radius $l=1$ which is the path of O in the course of the relative motion Σ_1/Σ_2 (Fig. 9). For simplification it seems advisable to introduce normal cartesian coordinates $v,\bar{v}$ by means of the linear transformations

$$\begin{aligned} u&=\alpha v+\beta\bar{v} \\ \bar{u}&=\beta v+\alpha\bar{v} \end{aligned} \quad \text{with} \quad \begin{aligned} \alpha&=\cos\left(\frac{\pi}{4}-\frac{\gamma}{2}\right), \\ \beta&=\sin\left(\frac{\pi}{4}-\frac{\gamma}{2}\right). \end{aligned} \tag{4.2}$$

The new axes have common bisectors with $p,\bar{p}$ and Fig. 8 shows the geometric meaning of v and $\bar{v}$.

Now the equation of the unit circle $\bar{o}$ reads simply

$$v^2+\bar{v}^2=1. \tag{4.3}$$

This condition is equivalent to (4.1) and might be verified by direct substitution. The function

$$v(\tau)=\frac{\alpha u-\beta\bar{u}}{\alpha^2-\beta^2}=\frac{\alpha u(\tau)-\beta u(\tau+\gamma)}{\sin\gamma} \tag{4.4}$$

has the same period 2γ as $u(\tau)$. Calculating its value for $\tau+\gamma$ we find, with attention to the periodicity of $u(\tau)$:

$$v(\tau+\gamma)=\frac{\alpha u(\tau+\gamma)-\beta u(\tau+2\gamma)}{\sin\gamma}=\frac{\alpha\bar{u}-\beta u}{\alpha^2-\beta^2}=\bar{v}. \tag{4.5}$$

Hence the structure of the original functional equation (4.1) has been essentially simplified: The new equation (4.3) is free of the mixed term.

Knowing with (4.3) the square sum of v and $\bar{v}$, we may yet prescribe the square difference

$$v^2-\bar{v}^2=f(\tau). \tag{4.6}$$

This function, arbitrary in principle, must have again the period 2γ and further satisfy the condition

$$f(\tau+\gamma)=\bar{v}^2-v^2=-f(\tau). \tag{4.7}$$

Having now chosen an appropriate function $f(\tau)$, we get at first from (4.3) and (4.6) the auxiliary functions

$$v(\tau)=\sqrt{\tfrac{1}{2}[1+f(\tau)]},\qquad \bar{v}(\tau)=v(\tau+\gamma)=\sqrt{\tfrac{1}{2}[1-f(\tau)]} \tag{4.8}$$

and then by means of (4.2) the support function

$$u(\tau)=\alpha v(\tau)+\beta\bar{v}(\tau) \tag{4.9}$$

of the sought-for cam profile c, possessing the unit circle o as isoptic curve. For a convex cam profile c the functions $u(\tau)$, $v(\tau)$ and $f(\tau)$ must have not only the period 2γ, but also 2π. We see again, in accordance with (3.1), that γ/π has to be rational; otherwise we should have two incommensurable periods and the continuous function $u(\tau)$ would reduce to a constant, characterizing the trivial circle profile.

In the case of a rational fraction $\gamma/\pi=m/n$ (3.1) any linear combination of 2γ and 2π with integral coefficients λ, μ is also a period:

$$\lambda\cdot 2\gamma+\mu\cdot 2\pi=(\lambda m+\mu n)\,\frac{2\pi}{n}. \tag{4.10}$$

Since m and n are relatively prime, there exist integers λ and μ such that $\lambda m+\mu n=1$; hence

$$\frac{2\pi}{n}=\frac{2\gamma}{m} \tag{4.11}$$

is a period of $u(\tau)$. If m were even, γ would be a period, i.e. $u=\bar{u}$ and therefore $u=$ const, which means again only the trivial circle profile. Consequently m is to be supposed as odd[2]. With respect to (4.11) and (4.7) appropriate functions $f(\tau)$ may be given by a trigonometric series (finite or convergent)

$$f(\tau)=\sum_i\,(a_i\cos in\tau+b_i\sin in\tau),\quad i=1,3,5,\ldots \tag{4.12}$$

The simplest examples are furnished by

$$f(\tau) = a \cdot \cos n\tau. \tag{4.13}$$

The corresponding profile curves c are algebraic, but not necessarily convex. To get convex curves the constant a (which can be supposed to be positive) must be sufficiently small. With $m/n = \frac{1}{2}$ ($\gamma = 90°$, $\alpha = 1$, $\beta = 0$) we are led from

$$u(\tau) = v(\tau) = \sqrt{\tfrac{1}{2}(1 + a\cos 2\tau)} \tag{4.14}$$

by use of (1.3) to

$$\frac{x^2}{1+a} + \frac{y^2}{1-a} = \frac{1}{2}. \tag{4.15}$$

This is, provided $0 < a < 1$, Goldberg's elliptic cam with eccentricity $\sqrt{a}$ (Fig. 5). In the case $m/n = \frac{1}{3}$ ($\gamma = 60°$, $\alpha = (\sqrt{3}+1)/2\sqrt{2} = 0.9659$, $\beta = (\sqrt{3}-1)/2\sqrt{2} = 0.2588$) the function

$$f(\tau) = a \cdot \cos 3\tau \tag{4.16}$$

leads to algebraic cam profiles c for a flat-faced follower pair with the angle of 120°. A convex specimen corresponding to $a = \frac{1}{2}$ is to be seen in Fig. 9. Following (1.1), (4.8) and (4.9) it was constructed as the envelope of the straight-line set

$$x\cos\tau + y\sin\tau = \alpha v + \beta\bar{v} \text{ with } v = \sqrt{\tfrac{1}{2}(1 + a\cos 3\tau)}, \qquad \bar{v} = \sqrt{\tfrac{1}{2}(1 - a\cos 3\tau)}. \tag{4.17}$$

To determine the class of c, i.e. the invariant number of (real and imaginary) tangents

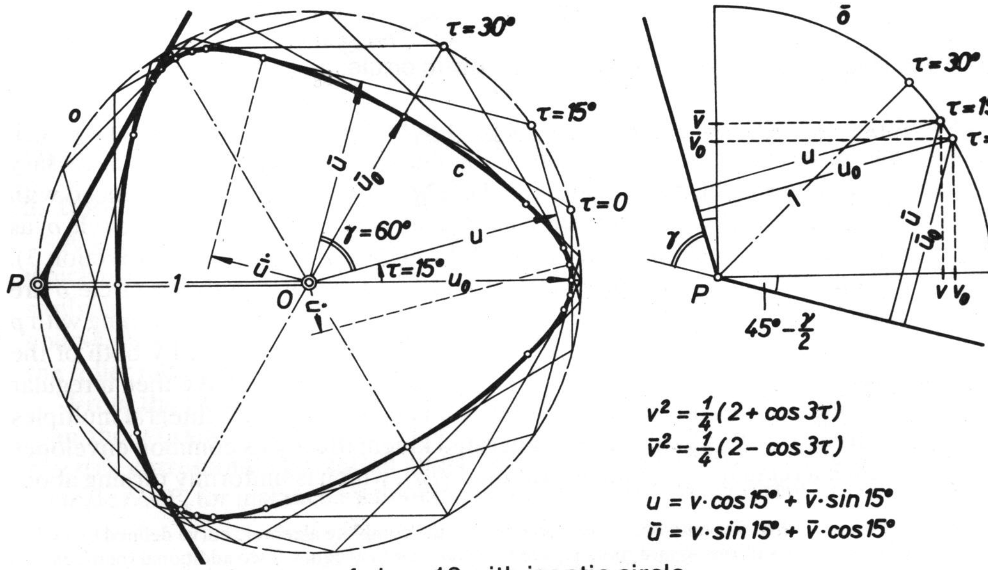

Figure 9. Algebraic curve of class 12 with isoptic circle.

through any point of the plane, we have to fix x and y in (4.17) and then to solve the equation for τ. By means of the complex substitution $\exp i\tau = t$ we arrive after elimination of the square roots at an algebraic equation of the 12th degree in t. Hence the class of c is 12. In general the class is $4n$ for n odd, and $2n$ for n even, with exception of $n = 2$ (ellipse: class 2).* The value $m/n = \frac{3}{4}$ ($\gamma = 135°$, $\alpha = 0.9239$, $\beta = -0.3827$) leads by way of

$$f(\tau) = a \cdot \cos 4\tau \tag{4.18}$$

to algebraic cam profiles of class 8 for a flat-faced follower pair with the angle of 45°. A convex specimen corresponding to $a = 1/20$ is shown in Fig. 10.

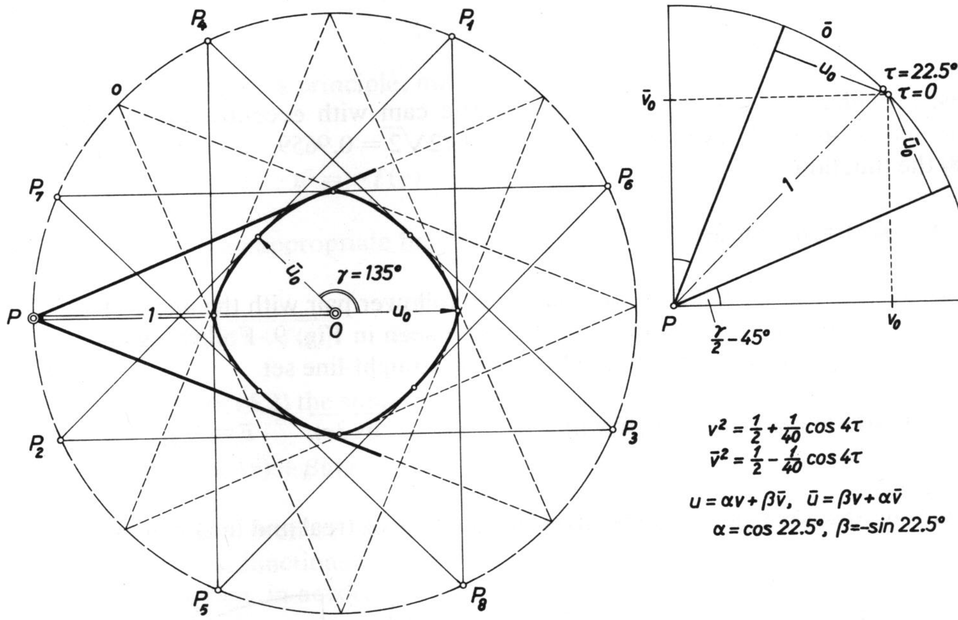

Figure 10. Algebraic curve of class 8 with isoptic circle.

It should be noticed that the method pursued here does not furnish all curves with an isoptic circle, because the assumption of the period 2γ for $u(\tau)$ is not a necessary consequence of the fundamental equation (4.1). This assumption was introduced with respect to the existence of circumscribed equiangular chord polygons of the circle o, as they are required by the desired convexity of the sought-for cam profiles c (Section 3). In this case both of the intersection points of a profile tangent p with the circle o are equivalent, and in each of them p is met by one of the adjacent tangents forming with p the angle $\pi - \gamma$. The other possibility to be considered is that p is met by both of the adjacent tangents in the same point on o. In every point $P \in o$ we have then a regular star of tangent lines of c forming with each other all angles which are integral multiples of γ. Hence curves c of this kind may be generated kinematically as common envelopes of all rays of such a regular star (finite for rational γ/π), which is uniformly turning about

*The curve drawn in Fig. 9 is only one of four branches of the complete algebraic curve defined by (4.17), where all sign combinations of the square roots $v, \bar{v}$ are to be taken into account. Two additional (non-convex) branches are to be seen in [7]. For $a = 1$ the curve splits into two six-cusped hypocycloids, each of class 6.

its vertex P, while P is moving periodically on the circle o. Since these curves cannot be convex, they will not be further discussed here (see [7]). Well-known examples are the cusped cycloids, generated by a uniform motion of P on o.

A cam mechanism derived from the three-cusped hypocycloid c whose vertex circle o is orthoptic ($\gamma = 90°$) is shown in Fig. 11: A convex parallel curve $\tilde{c}$ of c could (theoretically) operate a right-angled follower yoke with straight-line profiles. As $\tilde{c}$ is of constant breadth, the yoke might be completed to form a square. The cam and the square then work like a pair of gears according to the law $\psi = 3\varphi/4$.

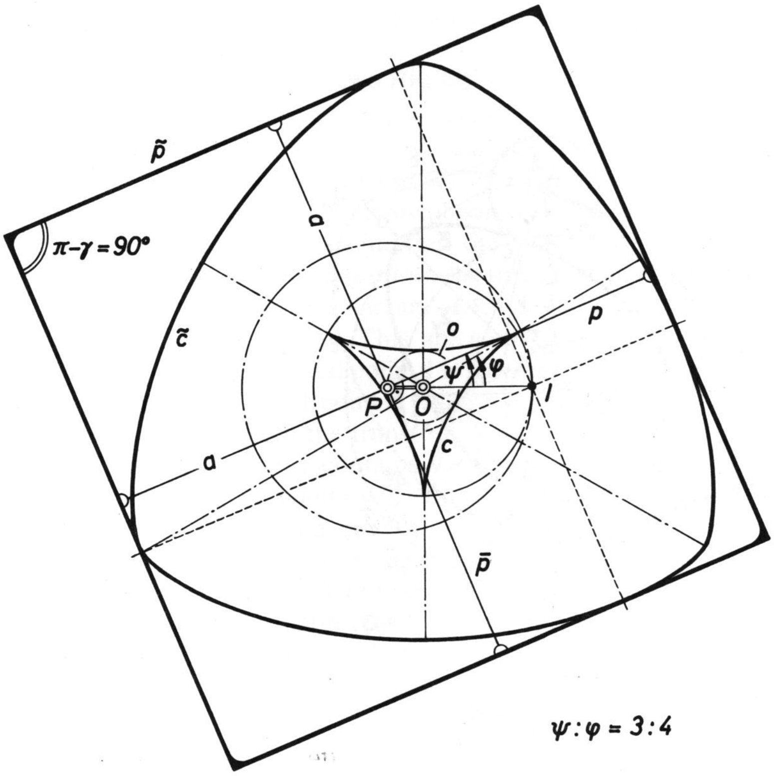

Figure 11. Cam disk of constant breadth with square yoke.

5. Double-Disk Cams with two Rigidly Connected Roller Followers

A commonly used cam mechanism has at the end of its follower a circular roller which is in steady contact with the rotating cam disk. The profile of the disk – not necessarily convex – is a parallel curve $\tilde{c}$ of the path c described by the roller center A in the course of the relative motion Σ_2/Σ_1. The constant distance between c and $\tilde{c}$ is equal to the roller radius a (Fig. 12).

From the geometric point of view the curve c (the "pitch profile") is more important than the working cam profile $\tilde{c}$ itself. When c is given by means of its polar equation $r = r(\tau)$, corresponding rotation angles φ, ψ of cam and follower are obtainable from the triangle OAP, for instance by double application of the cosine theorem:

$$\cos(\varphi + \tau) = \frac{L^2 - l^2 - r^2(\tau)}{2lr(\tau)}, \quad \cos\psi = \frac{L^2 + l^2 - r^2(\tau)}{2lL}. \qquad (5.1)$$

The constants $L = \overline{PA}$ and $l = \overline{OP}$ denote the length of the follower lever and the central distance of the pivot P, respectively. Formulae (5.1) determine the motion law $\psi(\varphi)$ in parametric form by means of the auxiliary functions $\varphi(\tau)$ and $\psi(\tau)$.

If the motion law $\psi(\varphi)$ is prescribed, the pitch curve c can be obtained as follows: Starting with arbitrary pairs of corresponding values φ, ψ we get from the triangle OAP (Fig. 12) the polar coordinates r, τ by

$$r^2 = l^2 + L^2 - 2lL\cos\psi, \qquad \sin(\tau+\varphi) = \frac{L}{r}\sin\psi. \tag{5.2}$$

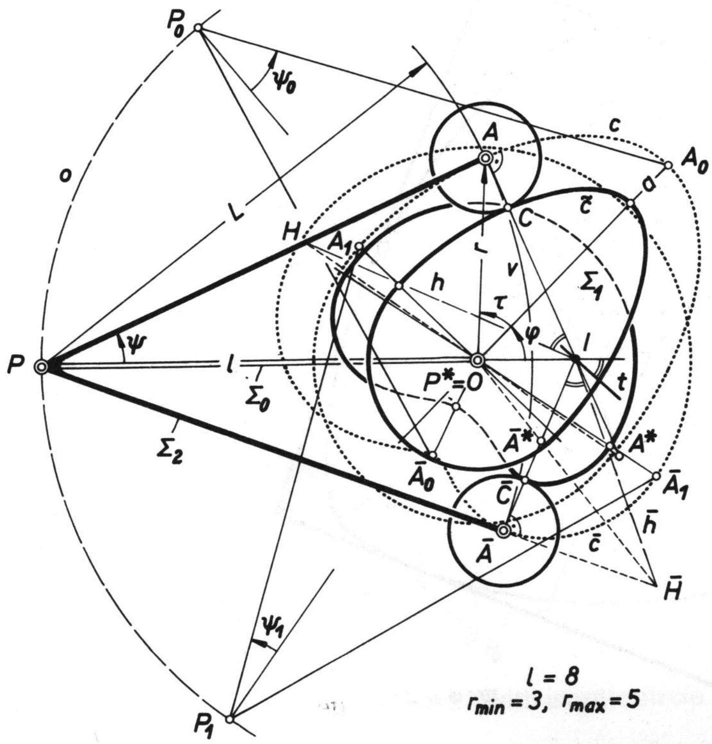

Figure 12. Double-cam mechanism with two rigidly connected oscillating roller followers.

To provide the return motion a second follower $P\bar{A}$ may be added which is rigidly connected with the first follower PA and hence turning about the same pivot P. Its contact roller—with the same radius a for simplicity—is operated by a second cam disk, whose profile is the parallel curve in a distance a inside the path $\bar{c}$ described by the roller center $\bar{A}$ during the relative motion Σ_2/Σ_1. It is sufficient to know the pitch profiles c and $\bar{c}$, the working cam profiles being parallel curves of c and $\bar{c}$ offset by the distance a; c and $\bar{c}$ may be considered as ideal cam profiles for vanishing roller radius.

If the first pitch profile c is given directly or determined by the motion law $\psi(\varphi)$, it is a mere routine task to derive the second pitch profile $\bar{c}$. We have only to consider the relative motion Σ_2/Σ_1 of the triangle $AP\bar{A}$: This motion is defined by the path c of A and the circle path o of P (center O, radius l). Using a sheet of transparent paper (vellum) for the moving system Σ_2, the path $\bar{c}$ of $\bar{A}$ is quickly constructed. An axial symmetry of

c, as assumed in Fig. 12, does not induce a symmetry of $\bar{c}$. Tangents of $\bar{c}$ are simply added by means of the instant center I which is the point of intersection of the normal of c in A with the line OP; the tangent of $\bar{c}$ in $\bar{A}$ then is perpendicular to $I\bar{A}$. If the center of curvature A^* of c belonging to A (and valid also for $\tilde{c}$ in C) is known, the center of curvature $\bar{A}^*$ of $\bar{c}$ belonging to $\bar{A}$ (and valid also for the corresponding point $\bar{C}$ of the cam profile) is to be found again by Bobillier's construction (Fig. 12).

Devices for automatic grinding of the second cam profile have been indicated by Hagedorn[3]. The idea consists in fixing the system Σ_2 and replacing the second roller by a grindwheel which is steered by the first roller led along the first profile.

6. Single-Disk Cams with two Rigidly Connected Roller Followers

A problem analogous to that which was treated in Section 3 concerns the question whether it is possible that the two cam disks operating a pair of rigidly connected oscillating followers as considered in Fig. 12, may coincide. The answer depends on the question whether there exist identical non-trivial pitch profiles c and $\bar{c}$.

For simplicity let us assume that the two follower levers PA and $P\bar{A}$ be of equal length L. To get a better insight into the transformation $V: A \rightarrow \bar{A}$ which maps c onto $\bar{c}$, we may introduce the circular arc $A\bar{A} = v$ with center P and radius L as a material element into the system Σ_2. This arc v envelopes a circle w in Σ_1 with center O and radius $L-l$. When the initial point A is led along the curve c and the arc v is sliding along the fixed circle w, the end-point $\bar{A}$ will trace the curve $\bar{c}$.

Applying repeatedly the transformation V to an arbitrary point A_0 of a curve $c = \bar{c}$, we see that such a curve contains with each point A_0 also the points $\bar{A}_0 = A_1$, $\bar{A}_1 = A_2$ etc. In general these points will constitute two infinite sequences $A_0, A_2, A_4, \ldots$ and $A_1, A_3, A_5, \ldots$ distributed everywhere densely along two circumferences with the common center O (Fig. 13). In particular the named sequences will be finite and closed, if and only if the ratio $\measuredangle A_0OA_1 : \pi$ is rational. Since this ratio depends on the varying central distance $\overline{OA}_0$, a non-trivial curve $c = \bar{c}$—different from a circle with center O—would cut each circle of an infinite concentric set in infinitely many points. It is evident that there exists no simply closed curve, as needed for the pitch profile, with such property.

Now the situation is completely different, if—as compatible with the geometric character of the transformation V—the arcs $A_0A_1, A_1A_2, \ldots$ all belong to the same circle v. Provided the sequence $A_0, A_1, \ldots, A_n = A_0$ is closed, its points form a rigid regular polygon and may well be wandering along a common path $c = \bar{c}$.

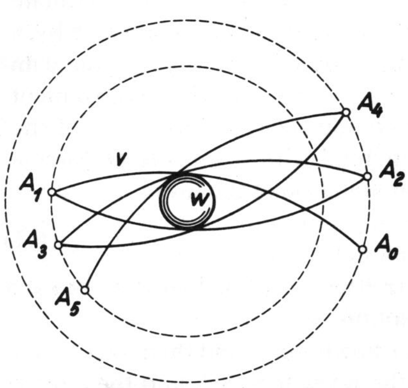

Figure 13. Iterated transformation V.

An example of this kind is shown in Fig. 14 for $n = 3$. An equilateral triangle $A_0A_1A_2$ is turning about its center P with constant angular velocity 1, while P is running on a circle o (center O) with angular velocity 3. In the course of this "planetary motion" the corners A_0, A_1, A_2 describe simultaneously a two-lobed trochoid $c = \bar{c}$. Three equal circles with the centers A_0, A_1, A_2 and common radius a generate as a common envelope the parallel curve $\tilde{c}$ of $\bar{c}$ offset by the distance a. Fixing now the system OP $(=\Sigma_0)$, the cam $\tilde{c}$ $(=\Sigma_1)$ would operate the three rigidly connected roller followers PA_0, PA_1 and PA_2 $(=\Sigma_2)$. The systems Σ_1 and Σ_2 work like gears with velocity ratio 3:2; hence the motion law of the mechanism is $\psi = 2\varphi/3$.

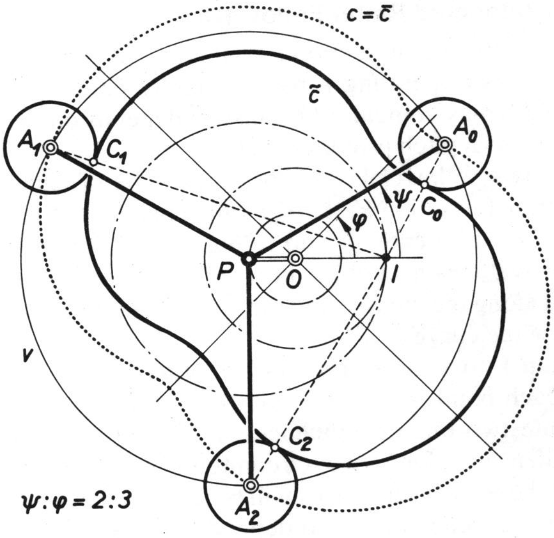

Figure 14. Two-lobed cam operating three rigidly connected roller followers.

This example is scarcely of high practical value, though there exists a certain connection with the famous Wankel motor[6]. Anyway it is of theoretical importance as it demonstrates in principle the possibility of single-cam mechanisms with rigidly connected roller followers, a fact which sometimes is denied.

It will be objected that the follower movement in Fig. 14 is not an oscillating one, and it seems to be impossible indeed to get single-disk cam mechanisms of this kind for oscillating output. Nevertheless a certain success is obtainable with approximate solutions. As indicated by Hinkle[4] and Hagedorn[3] it is possible to operate by a single-disk cam a pair of equal rollers oscillating along a straight line through the cam center O and having an invariant central distance $2b$ from each other. This arrangement corresponds to an infinite lever length $L = \infty$. The common relative path $c = \bar{c}$ of the roller centers A and $\bar{A}$ is a curve which is its own conchoid with respect to O. Such a "curve of constant diameter" may be defined by a polar equation of the form

$$r = f(\tau) + b, \tag{6.1}$$

where $f(\tau)$ is a periodic function with the property $f(\tau + \pi) = -f(\tau)$. Thus the forward motion of the follower determines already its return motion.

The idea is now to construct a pitch profile c according to (6.1) and then to combine it with a finite yoke $AP\bar{A}$. It is advisable to choose the lever length L and the central distance $\overline{OP} = l$ of the pivot P so that

$$L^2 - l^2 = b^2, \tag{6.2}$$

to have exact coincidence in the symmetric middle position $A_0P\bar{A}_0$ of the yoke. In other positions there is only approximate coincidence: A and $\bar{A}$ cannot belong to c at the same time, but the error is rather small. This is to be seen in Fig. 15 which illustrates an example constructed by means of the function $f(\tau) = e \cdot \cos\tau$. In this case the working cam profile $\tilde{c}$ is a curve parallel to a Pascal snail c:

$$r = e \cdot \cos\tau + b. \tag{6.3}$$

The pitch profile c (6.3) is convex if $b \geqq 2e > 0$. A numerical investigation of the errors occurring would be desirable.

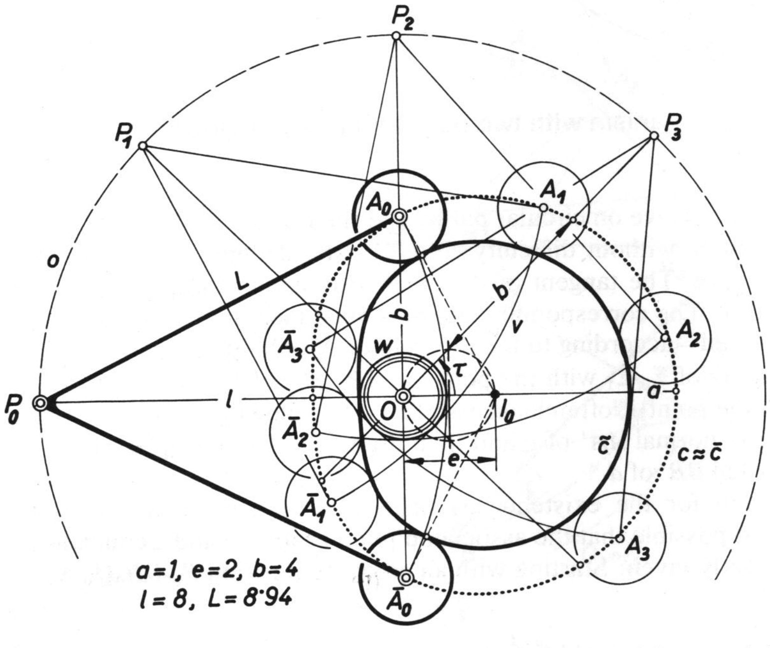

Figure 15. Approximate single-cam mechanism with two rigidly connected oscillating roller followers.

7. Cam Mechanisms with two Oscillating Roller Followers Joined by a Coupler Link

A new kind of single-cam mechanism was recently proposed by Jackowski and Dubil[5]: Two oscillating followers PA and QB of equal length L and with equal end rollers (centers A and B, common radius a) are operated by a cam disk c whose center O is the mid-point of the segment $\overline{PQ} = 2l$ joining the follower pivots P and Q. A supplementary coupler link with joints in A and B connects the two followers (Fig. 16). This mechanism—whose possibility is not sufficiently clarified—consists of five links: Frame Σ_0 (PQ), cam $\Sigma_1(\tilde{c})$, followers Σ_2 (PA) and Σ_3 (QB), and coupler Σ_4 (AB).

The existence of double cam mechanisms of this kind is evident, even in the case of different follower lengths, unequal roller radii and arbitrarily arranged pivots. In the course of the relative motion Σ_4/Σ_1 which has to be considered, the point $A \in \Sigma_4$ is led around a curve c in Σ_1 parallel to the (given) cam profile $\tilde{c}$. The points P and Q are

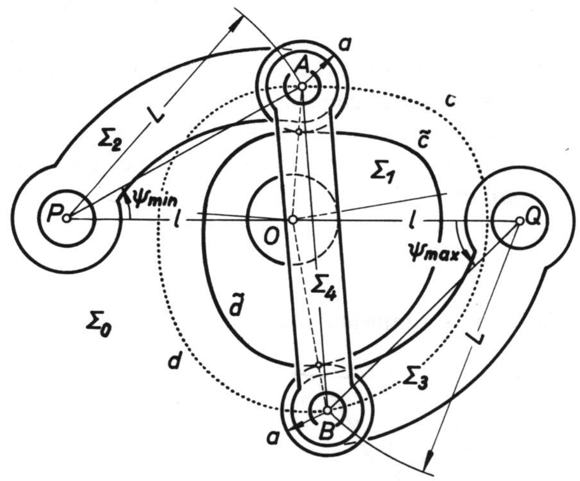

Figure 16. Single-cam mechanism with two oscillating roller followers joined by a coupler link.

constrained by Σ_2 and Σ_0 to move on circular paths, and the point $B \in \Sigma_4$ traces in Σ_1 a certain path d, constructible without difficulty (Fig. 17) and defining the second disk profile $\tilde{d}$ as a parallel curve. The tangent of d in B is also simple to derive from the (known) tangent of c in A: The corresponding normals must meet in the instant center I of Σ_4/Σ_1 which is situated – according to the theorem of Aronhold–Kennedy – on the line connecting the pole O of Σ_0/Σ_1 with the pole $J = AP \cdot BQ$ of Σ_4/Σ_0. Another construction which avoids the point J, often inaccessible, is the following (Fig. 17): We cut the line $OA' \| PA$ with the normal AA' of c and then draw parallels $OB' \| QB$ and $A'B' \| AB$; thus we get the normal BB' of d.*

The decisive question for the existence of single-disk cam mechanisms of the present kind is now: Is it possible that the associated pitch profiles c and d coincide? A positive answer is promptly given: Starting with an arbitrary position $P_0A_0B_0Q_0$ of the

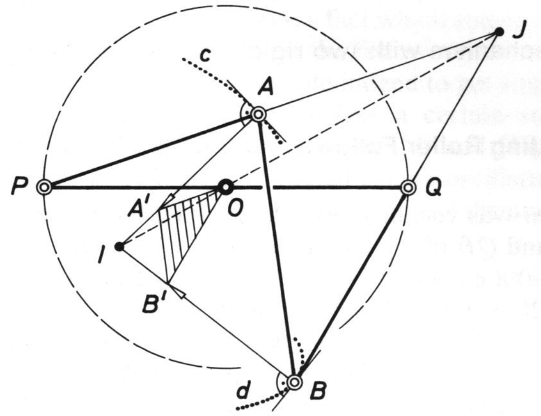

Figure 17. Construction of double-disk cams for follower linkage.

*This construction is based upon the homothetic triangles $OA'B' \sim JAB$ and may be interpreted as a velocity diagram.

linkage, any smooth arc from A_0 to B_0 may serve as part of $c = d$; leading the joint A on this arc c, the pivots P and Q are varying on the circle o (center O, diameter P_0Q_0) and the joint B traces the complementary arc d from B_0 to A_0 (Fig. 18). To avoid corners in A_0 and B_0 the arc c is to be chosen in such a way that the corresponding normals meet in a point I_0 on the line through O and the pole $J_0 = P_0A_0 \cdot Q_0B_0$.

It is advisable to begin with the extremal position $P_0A_0B_0Q_0$, i.e. to make $\measuredangle OP_0A_0 = \psi_{min}$ and consequently $\measuredangle OQ_0B_0 = \psi_{max}$. The end tangents of the arc $c = A_0B_0$ then are perpendicular to OA_0 and OB_0, respectively, as $I_0 = O$. The shape of c may be deter-

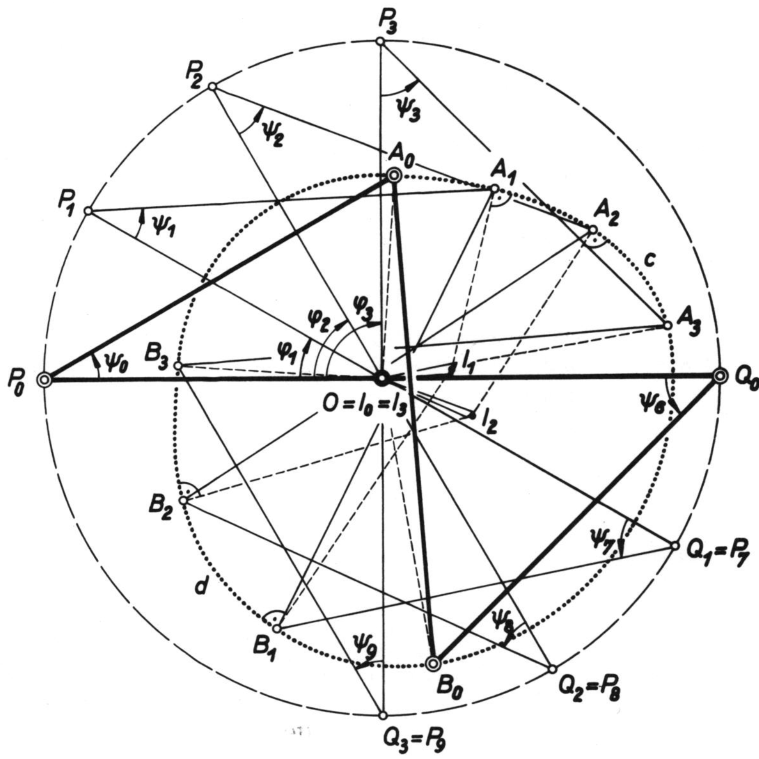

Figure 18. Construction of the pitch profile for the single-cam mechanism of Fig. 16.

mined by a prescribed law $\psi(\varphi)$ defining the advance motion of the follower PA from ψ_{min} to ψ_{max}. Thus the return motion is already determined; it is equivalent to the movement of the follower QB during the advance motion of PA. In Fig. 18 the part A_3B_0 of the arc A_0B_0 was chosen as a circle quadrant with center O to get a dwell mechanism; consequently the corresponding part B_3A_0 of the complementary arc B_0A_0 is also a circle quadrant with center O.

The working profile of the definite cam disk consists of curves $\tilde{c}$, $\tilde{d}$ parallel to the pitch arcs c and d at a distance a equal to the roller radius (Fig. 16). When the cam is driven in the opposite sense the characters of the forward and return motions will be interchanged.

It is improbable that there exist closed algebraic or analytic cam profiles for this single-disk mechanism.

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