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Contribution to the Geometry of Elliptic Gears

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Abstract
It is shown that the motion of elliptic wheels, usually derived from the antiparallelogram, can be constrained by a crossed transmission belt which is wound around two equal elliptic disks, confocal with the rolling ellipses. By considering the relative paths of belt points, one arrives at a simple gearing system for wheels of this kind which is the natural generalization of the usual involute system for circular wheels.

1. Antiparallelogram and elliptic wheels

LET $E_1F_2E_2F_1$ be an antiparallelogram, i.e. a special four-bar linkage with equal opposite sides: $E_1F_2 = F_1E_2 = 2a$, $E_1F_1 = E_2F_2 = 2e$ (Fig. 1). It is well-known that, provided $a > e$, the pole curves for the motion of the coupler E_2F_2 with respect to the fixed base E_1F_1 are two equal ellipses p_1, p_2 with the foci E_1, F_1 and E_2, F_2 respectively, and the major axis of common length $2a$ [2, section 8.6], [4, section 19]. During this motion p_2 is rolling on p_1 without slip. The varying contact point P (instantaneous center or pole) is the intersection point of the arms E_1F_2 and F_1E_2. The ellipses p_1 and p_2 are always mirror images with respect to the common tangent t in P which is the axis of symmetry for the quadrangle.§

For an observer on the crank E_1F_2 this link seems to be fixed, whereas the ellipses p_1, p_2 turn around E_1, F_2, respectively, and roll along each other. This modified mechanism is the basis of elliptic wheels. A primitive realization could be obtained by materializing the ellipses p_1, p_2 in the form of simple disks, although "friction wheels" of such a kind would not work during a complete

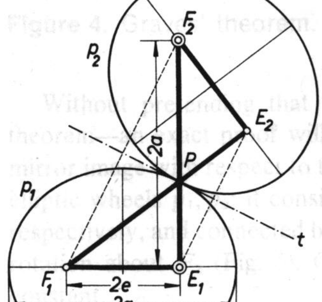

Figure 1. Elliptic wheels derived from the antiparallelogram.

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§This symmetry leads to the relation $E_1P + PF_1 = E_1P + PF_2 = E_1F_2 = 2a$ which proves that p_1, the locus of P, is the ellipse with foci E_1, F_1 and major axis $2a$.

revolution. Corresponding gears could be constructed by means of the classical principle of Reuleaux: a suitable tooth-profile c_1 is arbitrarily chosen in the plane of p_1, and the corresponding profile c_2 is then generated as the envelope of all positions which c_1 takes in the plane of p_2 during the rolling motion of p_1 on p_2; at any moment the common normal in the varying contact point of c_1 and c_2 must pass through the instantaneous pole P [4, section 46].

2. Transmission belts

Without disturbing the motion of the elliptic disks p_1, p_2 an endless belt might be wound around them, crossing itself at the contact point P. In a similar way a crossed belt of length $4(a + e)$, encompassing the quadrangle $E_1F_2E_2F_1$ along its four sides, might be introduced.

The second author raised the question whether there possibly exists an oval g_1 between the ellipse p_1 and its focal segment E_1F_1 which could be connected with a corresponding oval g_2 by a closed crossed belt. Using the three-pole theorem of Aronhold–Kennedy [4, section 25] it becomes clear that in any case the crossing point of the straight parts of the belt must coincide with the pole P.

To answer Zenow's question, let us begin with an arbitrary straight line through the pole P (Fig. 2). This line, representing an initial position of one straight part of the belt, cuts the ellipse p_1 in a second point P_1', forming there with p_1 a certain angle α'. After a certain time, P_1' coincides with the corresponding point P_2' of p_2, symmetric to P_1' with respect to the common ellipse tangent t at P. At this moment $P_1' = P_2'$ is the new instantaneous center, and the new position of the segment PP_1' determines the other straight part of the belt. Its protraction $P_2'P_2''$ forms at P_2' the same angle α' with p_2 and meets p_2 in a second point P_2''. Proceeding in this way we see from Fig. 2 that there arises a reflection polygon $PP_1'P_1'' \ldots$ of p_1 whose sides are all tangents of the required curve g_1, and an equal reflection polygon $PP_2'P_2'' \ldots$ of p_2 is circumscribed to the corresponding curve g_2.

Now a well-known theorem, due to J. V. Poncelet (1822), states that any reflection polygon of a conic is circumscribed to a confocal conic (which may degenerate to the pair of focal points). This theorem, usually derived by means of the projective theory of confocal conic systems [1, section 30], will be proved by elementary methods. To find the tangents from a point P to a given ellipse g, the following construction is commonly used (Fig. 3): Draw an auxiliary circle h with center P and passing through one of the foci E, F of g, say E, and cut it by a second circle q with center F, having the major axis $2a$ of g as radius; the two intersection points Q_1, Q_2 then determine the required tangents t_1, t_2 as the bisectors of the angles EPQ_1 and EPQ_2, respectively. If the points Q_1 and Q_2, situated symmetrically to the line FP, coincide in Q on FP, the bisector of the angle EPQ gives the tangent t of the confocal ellipse p which passes through P. As the angles EPQ and EPQ_1 have a common leg and differ by $\sphericalangle Q_1PQ = \sphericalangle QPQ_2 = \sigma$, their bisectors t and t_1 form the angle $\sigma/2$; the same is valid for t and t_2. Hence the light ray t_1, tangent to the ellipse g, is reflected in P on the confocal ellipse p into another tangent t_2 of g; obviously this is the root of Poncelet's theorem.

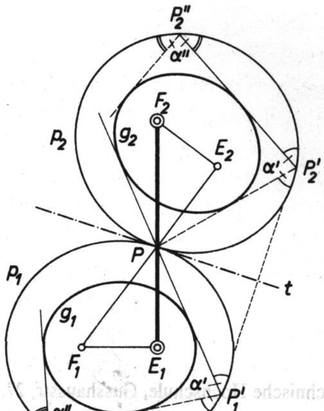

Figure 2. Reflection polygons in the elliptic pitch curves.

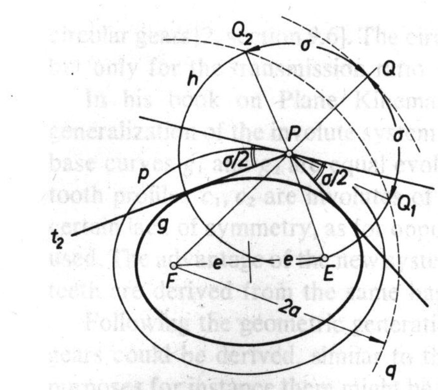

Figure 3. Poncelet's theorem.

There is another famous theorem, due to Ch. Graves (1841) and closely connected with the previous facts: If a closed rope, longer than the perimeter of an ellipse g, is slung around g and spanned by a pin P, then P can be led along an ellipse p, confocal with g. In fact the system is in equilibrium, if the spanning force T applied at P is acting in the direction of the internal bisector of the straight parts of the rope, as the tension forces T_1, T_2 in these parts have the same value (Fig. 4). Hence the pin P can move only orthogonally to this bisector if the direction of T is changed, which means that the external bisector t of T_1 and T_2 is the tangent of the path p of P.† According to Poncelet's theorem, p is an ellipse. The classical "gardener's construction" of the ellipse appears as a limit case of Graves' theorem, i.e. when the elliptical base g is reduced to the focal segment EF.

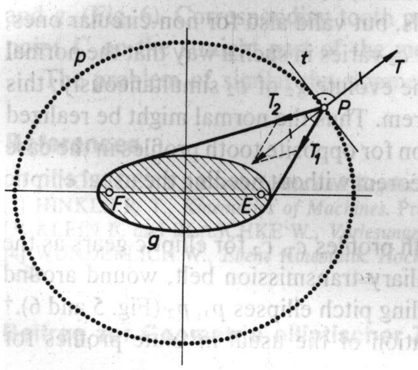

Figure 4. Graves' theorem.

Without pretending that these heuristic considerations are a rigorous proof of Graves' theorem—an exact proof will be indicated in section 3—we can complete Fig. 4 by adding its mirror image with respect to the tangent t and thus obtain a mechanism equivalent to the pair of elliptic wheels p_1, p_2: it consists of two equal elliptic disks g_1 and g_2, confocal with p_1 and p_2, respectively, and connected by a closed crossed belt which transmits the rotation about E_1 into a rotation about F_2 (Fig. 5). Of course the transmission ratio of the angular velocities is not constant.

The reasoning was essentially based upon the fact that the rolling curves p_1 and p_2 are always symmetrically situated with respect to their common tangent t. It is easy to see that this symmetry occurs only for elliptic wheels. Nevertheless the transmission problem can be posed for other noncircular wheels too, e.g. for those which were investigated in [4, section 45]. The existence of closed transmission belts has been led back by Zenow to a difficult system of functional equations, but it seems that there do not exist solutions in the general case.

†The same is also true for any other base curve g, different from an ellipse.

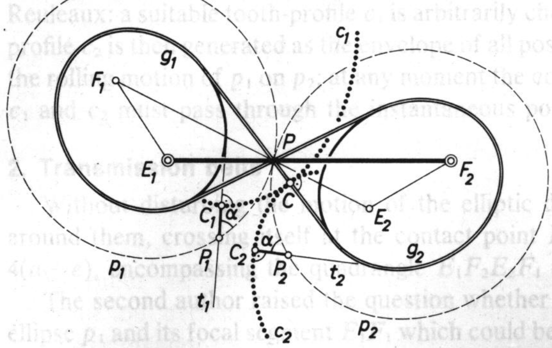

Figure 5. Rotating elliptic disks connected by a crossed belt.

3. Elliptic gears

As already mentioned in section 1, elliptic wheels can be supplied with teeth by means of Reuleaux's method, i.e. by choosing a suitable tooth profile c_1 and constructing its envelope c_2 during the motion which is determined by the rolling of the ellipse p_1 on the ellipse p_2.

We propose to take for c_1 an involute of some ellipse g_1, confocal with p_1. Thus each normal t_1 of c_1 is tangent to g_1 and cuts p_1 in a first point P_1. According to Reuleaux this normal t_1 has to be transferred to the corresponding point P_2 of p_2, forming there the same angle α with p_2 as t_1 forms with p_1. This position t_2 is then a normal of the second tooth profile c_2. Corresponding to Poncelet's theorem (section 2), t_2 is a tangent of the ellipse g_2 which is equal to g_1 and confocal with p_2. Hence the profile c_2 is an involute of g_2 (Fig. 5). Corresponding points C_1 on c_1 and C_2 on c_2 are determined by equal segments P_1C_1 of t_1 and P_2C_2 of t_2, as they become the contact point of the tooth profiles when, after a certain rotation of the wheels, P_1 coincides with P_2 (on the line E_1F_2). As pointed out in [4, section 40] for circular wheels, but valid also for non-circular ones, the contact element of corresponding tooth profiles c_1 and c_2 varies in such a way that the normal of the element is rolling on the evolute g_1 of c_1 and on the evolute g_2 of c_2 simultaneously; this fact is again a simple consequence of the three-pole theorem. Thus the normal might be realized by a belt wound around g_1 and g_2. Repeating this conclusion for opposite tooth profiles in the case of elliptic wheels, we have a rigorous proof for Graves' theorem without needing the usual elliptic integrals.

Summarizing we may consider our corresponding tooth profiles c_1, c_2 for elliptic gears as the paths of a point C fixed on the straight part of an auxiliary transmission belt, wound around elliptic base curves g_1, g_2 which are confocal with the rolling pitch ellipses p_1, p_2 (Fig. 5 and 6).† This is a natural generalization of the known interpretation of the usual involute profiles for

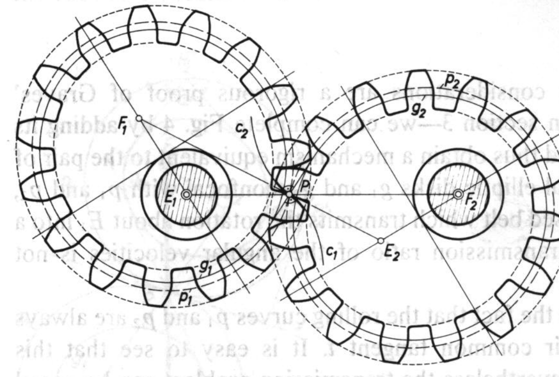

Figure 6. Involute system for elliptic gears.

†It seems attractive to use the degenerate ellipses $g_1 = E_1F_1$ and $g_2 = E_2F_2$, as then the tooth profiles c_1 and c_2 would become circular arcs, but it is obvious that teeth of this simple shape would not work in critical phases of the motion. In fact, in those positions where E_1, E_2, F_1, F_2 are on the same straight line, the common normal in the contact point of corresponding circular tooth profiles passes through the rotation centers E_1 and F_2, and thus these profiles are not able to transmit rotation.

circular gears [2, section 9.6]. The circular limit case is obtained for vanishing eccentricity ($e = 0$), but only for the transmission ratio -1.

In his book on Plane Kinematics [4, section 46] the first author has studied another generalization of the involute system to elliptic gears, imposing the condition $\alpha = \text{const}$. Then the base curves g_1 and g_2 are equal evolutoids of the pitch ellipses p_1 and p_2, respectively, and the tooth profiles c_1, c_2 are involutes of these evolutoids g_1, g_2. A disadvantage of this system is a certain lack of symmetry, as for opposite profiles different (although equal) evolutoids have to be used. The advantage of the new system is to be seen in the fact that both left and right sides of the teeth are derived from the same base ellipse g_1 or g_2.

Following the geometric generation of the teeth, appropriate ways of manufacturing elliptic gears could be derived, similar to those for circular involute gears [2, section 9]. For finishing purposes for instance there might be used a grinding wheel with a plane face normal to its axis; if it is moved in such a way that a line parallel to this axis is constrained to roll on the base ellipse g, the flat side of the tool would grind the involute face of a tooth. A similar concept would serve for the motion of cutters or hobs generating the tooth spaces in the gear blank.

Corresponding developments could be performed for elliptic bevel gears by means of operations on the sphere analogous to those used here in the plane.

4. Conclusion

Elliptic gear wheels are based upon the motion of an antiparallelogram $E_1F_2E_2F_1$ and its pole curves which consist of two equal ellipses p_1 and p_2 with foci E_1, F_1 and E_2, F_2 respectively (Fig. 1). Any two smaller equal ellipses g_1 and g_2, confocal with the rolling ellipses p_1 and p_2 and rigidly connected with them, have the property that a crossed transmission belt slung around them does not disturb the original motion (Fig. 5). This fact gaves the possibility for an apparently new gear system for elliptic wheels which represents a natural generalization of the well-known involute system for circular wheels: The tooth profiles are throughout involutes of the "base" ellipses g_1 and g_2 (Fig. 6). Corresponding tooth profiles c_1, c_2 may be considered as the relative paths of a point C on the straight part of the mentioned transmission belt (Fig. 5).

The problem of similar developments for other non-circular wheels remains open.

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Beitrag zur Geometrie elliptischer Zahnräder

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Kurzfassung—Die zur Relativbewegung der beiden kürzeren Seiten eines gelenkigen Antiparallelogramms gehörigen Polkurven sind bekanntlich zwei kongruente Ellipsen, die ihre Brennpunkte in den Gelenken haben und deren Hauptachsenlänge mit jener der längeren Antiparallelogrammseiten übereinstimmt (Bild 1, gleichläufiges Zwillingskurbelgetriebe). Diese Ellipsen p_1, p_2 bilden als Wälzkurven die Grundlage der elliptischen Zahnradpaare. Sie könnten durch einen geschlossenen, sich im Wälzpunkt P kreuzenden Treibriemen umschlungen werden, ohne dass die Bewegung behindert wird, und Gleiches gilt für einen das Gelenkviereck umschlingenden Treibriemen. Der zweite Verfasser hat die Frage nach Zwischenformen aufgeworfen, bei welchen die Bewegung durch einen gekreuzten Transmissionsriemen vermittelt wird, der zwei innerhalb der Wälzkurven angeordnete ovale Scheiben g_1, g_2 verbindet.

Während das verallgemeinerte, für beliebige Wälzkurven gestellte Problem sehr schwierig ist und vermutlich überhaupt keine Lösung besitzt, gelingt die Lösung für elliptische Wälzkurven wegen deren symmetrischer Lage bezüglich der Wälztangente elementar. Anhand von Bild 2 wird zunächst gezeigt, dass die Scheibenränder g_1, g_2 Reflexionspolygonen der Ellipsen p_1, p_2 eingeschrieben sein müssen. Gestützt auf einen

Satz von Poncelet, demzufolge ein an einer Ellipse p wiederholt reflektierter Lichtstrahl immer wieder denselben, zu p konfokalen Kegelschnitt g berührt (Bild 3), ergibt sich eine elliptische Gestalt der Scheiben g_1 und g_2. Ein damit zusammenhängender Satz von Graves besagt ferner, dass ein Stift P, der mittels eines gespannten geschlossenen Fadens um eine Ellipse herumgeführt wird, eine zu g konfokale Ellipse p beschreibt (Bild 4). Spiegelt man schliesslich diese Figur an der Bahntangente t von P, so gelangt man zu dem gesuchten Riementrieb (Bild 5).

Die von einem Punkt C eines geradlinigen Riemenstücks in den beiden rotierenden Radsystemen durchlaufenen Relativbahnen c_1, c_2 stellen geeignete Zahnprofile für die Ellipsenräder dar (Bild 5). Damit ist ein einfaches Verzahnungssystem fur Ellipsenräder gefunden, bei dem die Zahnprofile Evolventen von kongruenten, zu den Wälzkurven p_1, p_2 konfokalen Ellipsen g_1, g_2 sind, die sowohl für die rechten als auch für die linken Zahnflanken brauchbar sind (Bild 6). Dieses neue System stellt eine natürliche Verallgemeinerung der bei kreisrunden Rädern gebräuchlichen Evolventenverzahnung dar und eröffnet auch technisch realisierbare Möglichkeiten zur Herstellung elliptischer Zahnräder.

Unter Berufung auf das bekannte Verzahnungsprinzip von Reuleaux ergibt sich auf diesem Weg nebenbei ein elementarer Beweis für Satz von Graves, der für gewöhnlich mit Hilfe von elliptischen Integralen geführt wird.