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MATHEMATICS

ON DEFORMABLE NINE-BAR LINKAGES WITH SIX TRIPLE JOINTS

BY

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1. If two coplanar point-triples A_1, A_2, A_3 and B_1, B_2, B_3 are connected by the nine line-segments A_iB_j we get a plane network consisting of a closed hexagon $A_1B_2A_3B_1A_2B_3$ and its three diagonals A_iB_i. A framework of this structure is in general *rigid*, even if the six knots in which the nine bars concur three by three are materialized by cylindrical joints whose axes are perpendicular to the plane of the configuration.

It is well known that this framework is *infinitesimally deformable*, if the six knots are situated on a *conic* k (which may degenerate into a line-pair) [3, 5]. It can be shown that the number of knots on the conic may be arbitrarily raised [6].

Moreover there exist two particular forms which are *continually deformable*. These remarkable plane linkages which allow a constrained motion were detected by A. C. Dixon [2] in the course of a difficult investigation devoted to the problem of jointed quadrilaterals whose corners can move on circles. The present note gives an elementary exposition of the two mechanisms in question. It is evident that the six joints of each one are always situated on a conic k, although this conic will vary in the course of the motion.

Questions of this kind have newly arisen in connection with the theory of *trilateration* [6], a modern method of geodesy which intends to determine the configuration of two finite point-sets by measuring directly all distances from the points of the first set to the points of the second set (without the use of angles).

2. The *first mechanism* of Dixon [2, § 27, lit. d] is characterized by the condition that the points A_1, A_2, A_3 and B_1, B_2, B_3 are situated on two orthogonal lines. Let us use these lines as axes of a cartesian system of coordinates x, y. Denoting the distance of $A_i(x_i, 0)$ and $B_j(0, y_j)$ with r_{ij}, we have for the quadrangle $A_1B_1A_2B_2$ (fig. 1):

(2.1) $$r_{11}^2 = x_1^2 + y_1^2,\quad r_{22}^2 = x_2^2 + y_2^2,\quad r_{12}^2 = x_1^2 + y_2^2,\quad r_{21}^2 = x_2^2 + y_1^2.$$

It follows

$$r_{11}^2 + r_{22}^2 = r_{12}^2 + r_{21}^2, \tag{2.2}$$

and thus we have the

Lemma. *A quadrangle with orthogonal diagonals is characterized by equal square sums of the two pairs of opposite sides.* [1])

Hence our jointed quadrilateral $A_1B_1A_2B_2$ can move in such a way that its corners run along the two perpendicular axes x and y. Adding now an arbitrary point A_3 on x and joining it by rods with B_1 and B_2, we see by application of the Lemma to the quadrilateral $A_1B_1A_3B_2$ that A_3 will also move on x. Repeating the argument for an arbitrary point B_3 of y which is joined by rods with A_1, A_2 and A_3, we arrive at Dixon's deformable nine-bar linkage (fig. 1).

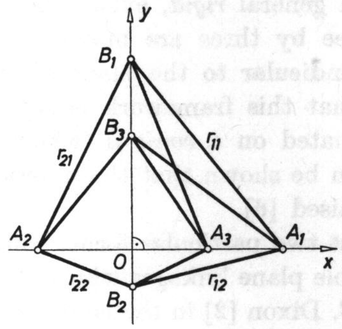

Fig. 1. Dixon's nine-bar mechanism of the first kind.

As yet more points $A_i \in x$ and $B_j \in y$ may be added, we obtain the generalized

Theorem 1. *If $m \geqq 3$ collinear points $A_1, \ldots, A_m$ and $n \geqq 3$ collinear points $B_1, \ldots, B_n$ are connected by the mn rods A_iB_j, and all knots are realized by cylindrical joints, then the so constructed linkage is continually deformable with one degree of freedom, provided that the two supporting lines are orthogonal.* [2])

3. The *second mechanism* of Dixon [2, § 28, lit. n, o, p] was originally described as follows: In one (particular) position the quadrilateral

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[1]) The Lemma is valid not only for plane quadrangles, but also for skew ones.

[2]) The Theorem holds also for the spatial linkage with skew supporting lines.

$A_2B_2A_3B_3$ is a rectangle whose circumcircle k passes through A_1 and B_1, and in this position A_1A_2 and B_1B_2 are parallel, as also A_1A_3 and B_1B_3 (fig. 2).

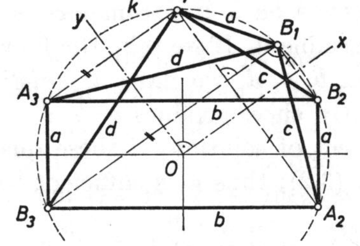

Fig. 2. Dixon's nine-bar mechanism of the second kind; initial position.

By reasons of symmetry we state $A_1B_1 = A_2B_2 = A_3B_3$, $A_1B_2 = A_2B_1$, $A_1B_3 = A_3B_1$. For abbreviation we put

(3.1) $$r_{11} = r_{22} = r_{33} = a,\ r_{23} = r_{32} = b,\ r_{12} = r_{21} = c,\ r_{13} = r_{31} = d.$$

Inspecting once more fig. 2, we see that the linkage contains *two antiparallelograms* $A_1B_1A_2B_2$ and $A_1B_1A_3B_3$ whose pairs of diagonals $A_1A_2 \| B_1B_2$ and $A_1A_3 \| B_1B_3$ have perpendicular directions; hence their mirror axes x, y form also a right angle at the center O of the circle k. Furthermore the linkage contains two quadrangles $A_1B_1A_2B_3$ and $A_1B_1A_3B_2$ with orthogonal diagonals $A_1A_2 \perp B_1B_3$ and $A_1A_3 \perp B_1B_2$, respectively. Applying the Lemma of Section 2, we find that the quantities of (3.1) are related by

(3.2) $$a^2 + b^2 = c^2 + d^2.$$

Now the *deformability* of Dixon's linkage is simply to be proved: Begin with a fixed base A_1B_1 of length a and construct two antiparallelograms $A_1B_1A_2B_2$ (side lengths a, c) and $A_1B_1A_3B_3$ (side lengths a, d) with given orthogonal directions of the diagonals. The sides A_2B_2 and A_3B_3 of common length a are parallel and thus represent opposite sides of a parallelogram $A_2B_2A_3B_3$ (fig. 3). The other sides A_2B_3 and A_3B_2 of the parallelogram have, due to the Lemma and (3.2), the common constant length b. Varying now the prescribed directions $A_1A_2 \perp A_1A_3$ we obtain a continuous one-parametric sequence of positions of the linkage.

Thus we have a more natural characterization of Dixon's second mechanism:

THEOREM 2. *The plane nine-bar linkage consisting of two antiparallelograms $A_1B_1A_2B_2$ and $A_1B_1A_3B_3$ and the parallelogram $A_2B_2A_3B_3$ is constraintly deformable. During any deformation the mirror axes of the antiparallelograms remain perpendicular and the variable triangles $A_1A_2A_3$ and $B_1B_2B_3$ conserve their right angles at A_1 and B_1.*

4. Using the mirror axes of the antiparallelograms as frame for cartesian coordinates x, y (fig. 3), we may put:

(4.1) $A_1(x, y)$, $A_2(x, -y)$, $A_3(-x, y)$; $B_1(X, Y)$, $B_2(X, -Y)$, $B_3(-X, Y)$.

This point-sextuple depends on four parameters, so that four conditions may be imposed to determine it. If we take the four conditions (3.1) with given distance values a, b, c, d, we have a *poristic problem*: It has no solution for independently chosen values a, b, c, d because of a contradiction, but an infinite set of solutions if these quantities are bound by the necessary condition (3.2); thus it is sufficient to prescribe the values a, b, c.

To determine a possible position of Dixon's linkage we have to solve the system of equations

(4.2) $(x-X)^2+(y-Y)^2=a^2,\ (x+X)^2+(y+Y)^2=b^2,\ (x-X)^2+(y+Y)^2=c^2,$

or

(4.3) $x^2+y^2+X^2+Y^2=\frac{1}{2}(a^2+b^2),\ 4xX=b^2-c^2,\ 4yY=c^2-a^2.$

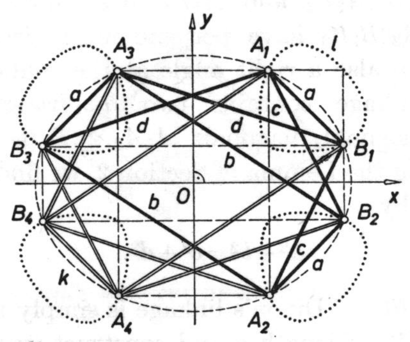

Fig. 3. Bottema's 16-bar mechanism, containing Dixon's mechanism of the second kind in general position.

As there are three equations for four variables, we get ∞^1 solutions and have an analytic proof for the deformability of the linkage.

The elimination of X, Y from (4.3) leads to the equation

(4.4) $16x^2y^2(x^2+y^2)-8(a^2+b^2)x^2y^2+(c^2-a^2)^2x^2+(b^2-c^2)^2y^2=0$

of a *circular sextic* l which represents the *common path* of all joints (4.1) during that motion of the linkage which leaves invariant the axes x and y. This sextic contains also the free intersection points $(x, \pm Y)$, $(\pm X, y)$ etc. of the antiparallelogram diagonals, consists of four equal ovals placed symmetrically with respect to the coordinate axes (fig. 3) and possesses an isolated double point at the origin O.

5. Let us now complete the configuration (4.1) by adding the points $A_4(-x, -y)$ and $B_4(-X, -Y)$. By reasons of symmetry in fig. 3 or by means of the relations (4.2) we find

(5.1) $A_4B_4=a,\ A_4B_1=B_4A_1=b,\ A_4B_3=B_4A_3=c,\ A_4B_2=B_4A_2=d.$

This means that the new points A_4, B_4 may be connected with each other and with the original sextuple (4.1) by seven additional rods of known constant lengths. Thus we arrive at a *deformable* 16*-bar linkage* with eight quadruple joints (fig. 3). It was detected by O. Bottema [1], who derived it from a remarkable 12-bar linkage with eight triple joints investigated by the author [4].—The results may be summarized in

THEOREM 3. *If all corners A_i of a first rectangle are connected by rods of constant lengths with all corners B_j of a second rectangle having the same mirror axes as the first one, there arises a plane deformable* 16*-bar linkage with eight quadruple joints. It contains* 4 *parallelograms* (*equal two by two*), 8 *antiparallelograms* (*equal two by two*), 8 *quadrilaterals with orthogonal diagonals* (*equal four by four*), *and* 16 *Dixon nine-bar linkages* (*equal four by four*). *During that motion which leaves invariant the mirror axes, all eight joints A_i, B_i of the mechanism are led* (*two by two*) *along four equal branches of a circular sextic* (4.4).

6. All plane mechanisms considered here have *spherical analoga*. The theory of the latter can be developed in quite a similar way, by interpreting the variables x and y as geographical coordinates on the unit sphere (longitude and latitude, respectively) and by replacing straight line segments by arcs of great circles.

Instead of (2.1) we have then the formulae

$$\cos r_{ij} = \cos x_i \cos y_j, \tag{6.1}$$

hence instead of (2.2) the relation

$$\cos r_{11} \cdot \cos r_{22} = \cos r_{12} \cdot \cos r_{21}, \tag{6.2}$$

characteristic for the orthogonality of the spherical diagonals of the spherical quadrangle $A_1B_1A_2B_2$.

The formulae (4.2) are to be replaced by

$$\cos a = \sin y \sin Y + \cos y \cos Y \cos (x-X) \text{ etc.} \tag{6.3}$$

It follows, analogous to (3.2),

$$\cos a + \cos b = \cos c + \cos d. \tag{6.4}$$

Hence the conditions a, b, $c = \text{const}$ induce $d = \text{const}$.

Therefore there exist spherical analoga not only for both of the Dixon nine-bar mechanisms, but also for Bottema's 16-bar linkage.

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REFERENCES

1. Bottema, O. – *Die Bahnkurven eines merkwürdigen Zwölfstabgetriebes*. Österr. Ingen. Archiv 14, 218–222 (1960).
2. Dixon, A. C. – *On certain deformable frameworks*. Mess. Math. 29, 1–21 (1899/1900).
3. Ritter, W. and C. Culmann – *Anwendungen der graphischen Statik, II*. Zürich 11f (1890).
4. Wunderlich, W. – *Ein merkwürdiges Zwölfstabgetriebe*. Österr. Ingen. Archiv 8, 224–228 (1954).
5. Wunderlich, W. – *Ebene Kinematik*. Mannheim, 143f (1970).
6. Wunderlich, W. – *Gefährliche Annahmen der Trilateration und bewegliche Fachwerke, I*. Z. angew. Math. Mech. (in print).