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Spatial Tractrices of the Circle.

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Sunto – *Si studiano, nello spazio euclideo, le trattrici di un cerchio (raggio m, segmento generatore di lunghezza l), cioè le traiettorie ortogonali del sistema delle sfere unitarie aventi il centro sul cerchio dato. Si trova che le trattrici sono in generale curve sferiche, piane nel caso limite. Le trattrici spaziali del tipo ellittico* ($0 < m < 1$) *sono collineari (o inverse) ad eliche sferiche con l'angolo di inclinazione* $\alpha = \arccos m$; *si hanno esempi algebrici relativi a valori razionali di* $n = \operatorname{sen} \alpha$. *Nel caso iperbolico* ($m > 1$) *le trattrici sferiche sono collineari ad eliche tracciate su un iperboloide di rotazione a due falde. Le trattrici del tipo parabolico sono inverse dell'evolvente di un cerchio. La proiezione di una trattrice spaziale dal centro del cerchio direttore sopra un piano parallelo a quello del cerchio stesso è una epicicloide nel caso ellittico, una paracicloide nel caso iperbolico e l'evolvente del cerchio nel caso parabolico. Si danno semplici rappresentazioni parametriche delle trattrici e interpretazioni non-euclidee di esse.*

1. – Introduction.

When a line segment AB of constant length 1 is moved in such a way that its initial point A is lead along a curve a, whereas its endpoint B is constrained to proceed always in the direction BA, describes a path b, called a *tractrix* of a. These tractrices b are the *orthogonal trajectories of the system of unit spheres* σ centered in the points of the guide curve a.

For instance the (common) tractrix b of the straight line a is a plane curve [**3**; II, 188], well-known as the meridian of E. Beltrami's pseudosphere [**1**; I, 14]. The plane tractrices of the circle have been studied in connection with certain helices on quadrics of revolution by the author [4], who has given parametric representations more convenient than those to be found in G. LORIA [**3**; II, 195] and due to S. RICCATI.

The present note has the aim to determine the *spatial tractrices of a circle* a. Evidently the developments can be restricted to a space of three dimensions; it is supposed to be euclidean and real, although it will be occasionally extended to the complex domain.

There will be distinguished three cases, corresponding to the (real) value m of the circle radius and denoted as « elliptic » $(0 < m < 1)$, « hyperbolic » $(m > 1)$, and « parabolic » $(m = 1)$.

2. – Elliptic case.

Let the guide circle a be given, in cartesian coordinates, by

$$x^2 + y^2 = m^2 , \quad z = 0 . \tag{2.1}$$

The system of unit spheres σ having their centers on a define an enveloping surface Φ, in our case $(m^2 < 1)$ a *torus* of spindle type with two real conical points $C(-n, 0, 0)$ and $D(n, 0, 0)$, where

$$m^2 + n^2 = 1 , \quad n > 0 . \tag{2.2}$$

We now apply the *inversion* (transformation by reciprocal radii) $P \to P^*$ with center C and power $2n^2$. It is described by

$$x^* = 2n^2x/Q , \quad y^* = 2n^2y/Q , \quad z^* + n = 2n^2(z + n)/Q \quad \text{with } Q = x^2 + y^2 + (z + n)^2 \tag{2.3}$$

and changes the family of spheres σ, passing all through C and D, into the family of planes σ^* tangent to a *cone of revolution* Φ^*. This cone (Fig. 1) has the same axis of revolution as the circle a, its vertex at the origin $D^* = O$ and the angle of aperture 2α with

$$\cos\alpha = m , \quad \sin\alpha = n . \tag{2.4}$$

The equation of Φ^* reads

$$m^2(x^{*2} + y^{*2}) = n^2z^{*2} . \tag{2.5}$$

Any tractrix b of the circle a (2.1), characterized in Section 1 as orthogonal trajectory of the sphere system $\{\sigma\}$, is transformed by the angle-preserving inversion (2.3) into an orthogonal trajectory b^* of the family of tangent planes σ^* of the cone Φ^* (2.5). Consequently b^* is a *spherical curve* supported by a sphere Σ^* with center O. As Σ^*, orthogonal to Φ^*, is inversive to a sphere Σ orthogonal to the torus Φ and therefore passing through the imaginary circle

$$x^2 + y^2 + n^2 = 0 , \quad z = 0 , \tag{2.6}$$

we have:

THEOREM 1. – *The spatial tractrices of a circle are spherical curves. For a fixed length of the generating line segment they lie on spheres belonging to a pencil.*

The theorem is valid also in the hyperbolic and in the parabolic case, only that then the pencil of spheres is not hyperbolic as in the elliptic case, but elliptic or parabolic, respectively.

The spherical curve b^*, situated on a sphere Σ^*

$$x^{*2} + y^{*2} + z^{*2} = c^2 , \tag{2.7}$$

can be generated in the following kinematical way: Take a circular disk with center O and radius c and let its plane σ^* roll on the circular cone Φ^*; then any point fixed on the border of the disk will describe an orthogonal trajectory of all positions of the rolling plane. It follows that such a path b^* is a *helix* on the sphere Σ^* and has the constant angle of inclination α, determined by (2.4). This curve may be considered as a *spherical involute* of the circle-pair common to Φ^* and Σ^*.

Choosing a suitable point of generation, the trajectory b^* can be parametrically represented by

$$\begin{aligned} x^* &= c(n \cos nu \cos u + \sin nu \sin u) , \\ y^* &= c(n \cos nu \sin u - \sin nu \cos u) , \\ z^* &= -cm \cos nu . \end{aligned} \tag{2.8}$$

In fact this curve lies on the sphere Σ^* (2.7) and its tangents have the direction

$$\dot{x}^* : \dot{y}^* : \dot{z}^* = m \cos u : m \sin u : n , \tag{2.9}$$

and thus the inclination angle α.

For rational values of $n = \sin\alpha$ the helix b^* (2.8) is closed and *algebraic*. Writing $n = \mu/\nu < 1$ as reduced fraction (μ, ν positive integers without common factor) and introducing the complex parameter $w = \exp(iu/\nu)$, we find that b^* is rational and of *order* $N = 2(\mu + \nu)$. Such a curve b^* consists of 2μ elementary arcs, congruent to the primitive arc $0 \leqq u \leqq \pi/n$ and fastened together in 2μ cusps which are distributed alternatively over the circles $z^* = \pm mc$ with the common radius nc (Fig. 1).

Now we have to transform back the spherical helix b^* (2.8) by means of the inversion (2.3) with exchanged asterisks. Due to the fact that for points on the sphere Σ^* (2.7) we have

$$Q^* = x^{*2} + y^{*2} + (z^* + n)^2 = c^2 + n^2 + 2nz^* \,, \tag{2.10}$$

the mapping of the sphere Σ^* onto the corresponding sphere Σ can be considered as a *perspective collineation*

$$x = 2n^2 x^*/Q^* \,, \qquad y = 2n^2 y^*/Q^* \,, \qquad z + n = 2n^2(z^* + n)/Q^* \tag{2.11}$$

Its center is the point $C(0, 0, -n)$ and its fixed plane β is given by

$$z = z^* = f = (n^2 - c^2)/2n \,. \tag{2.12}$$

For commodity we introduce homogeneous coordinates $x_0 : x_1 : : x_2 : x_3 = 1 : x : y : z$. Using equations (2.8), (2.10) and (2.11) we obtain —in the elliptic case—the following parametric representation of the *spatial tractrix* b of the circle a (2.1):

$$\begin{aligned} x_0 &= n^2 + c^2 - 2mnc \cos nu \,, \\ x_1 &= 2n^2 c(n \cos nu \cos u + \sin nu \sin u) \,, \\ x_2 &= 2n^2 c(n \cos nu \sin u - \sin nu \cos u) \,, \\ x_3 &= n(n^2 - c^2) \,. \end{aligned} \tag{2.13}$$

Excluding the case $c^2 = n^2$ which leads to a plane tractrix of the circle a, the tractrix b (2.13) is situated on the *sphere* Σ:

$$(n^2 - c^2)[x_1^2 + x_2^2 + (x_3 + nx_0)^2] = 4n^3 x_0 x_3 \,. \tag{2.14}$$

Its radius r and its central distance d, connected by the relation $d^2 - r^2 = n^2$, are given by

$$r = 2n^2 c/(n^2 - c^2) \,, \qquad d = n(n^2 + c^2)/(n^2 - c^2) \,. \tag{2.15}$$

As the spherical helix b^* (2.8) is an orthogonal trajectory of a rotational system of (great) circles of Σ^*, the tractrix b (2.13) is an orthogonal trajectory of a rotational system of (small) circles on the sphere Σ. This means that b can be considered as a *spherical tractrix of a circle* in the sense of the geometry on a sphere. Under this point of view these curves occur already in [5].—Summarizing we have

THEOREM 2. – *In the elliptic case the spatial tractrices of a circle are collinear to spherical helices. In the sense of spherical geometry they are identical with spherical tractrices of a circle.*

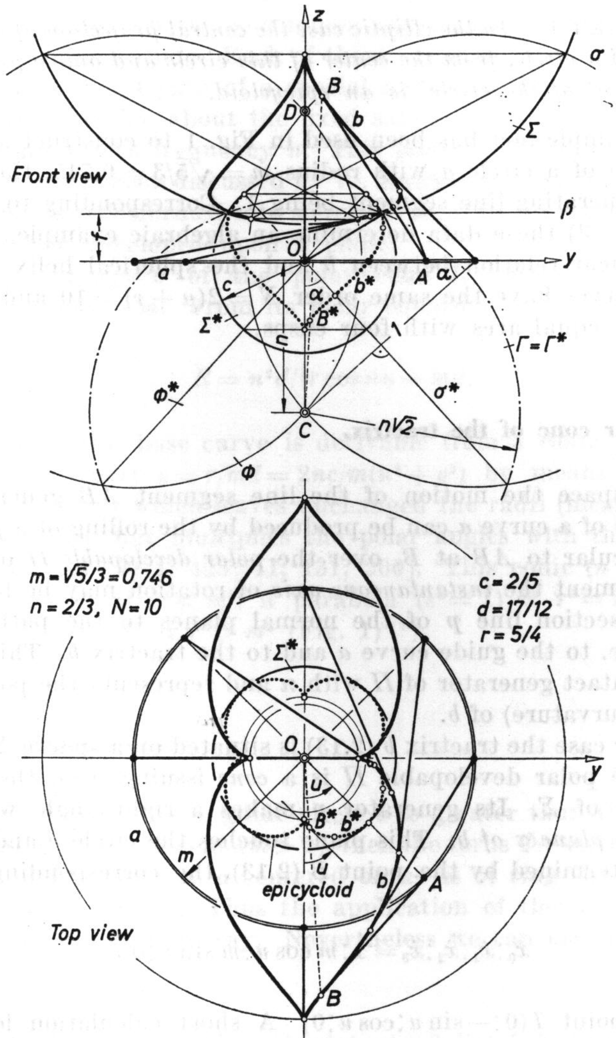

Fig. 1.

A well-known theorem, due to A. ENNEPER and easily verified by means of (2.8), states that the orthogonal projection of a spherical helix b^* onto a base plane $z = \text{const}$ is an *epicycloid*. If we use for base plane the fixed plane β (2.12) of the collineation (2.11), we see that the result of this projection is the same as the result of the central projection of the tractrix b from the origin O onto that plane β. Thus we have

THEOREM 3. – *In the elliptic case the central projection of a spatial tractrix of a circle, from the center of this circle and onto a plane parallel to that of the circle, is an epicycloid.*

This simple fact has been used in Fig. 1 to construct a spatial tractrix b of a circle a with radius $m = \sqrt{5}/3 = 0.745$, the length of the generating line segment being 1. Corresponding to $n = 2/3$ ($\mu = 2, \nu = 3$) these data determine an algebraic example. Due to the collinear relation between b and the spherical helix b^* both of the curves have the same order $N = 2(\mu + \nu) = 10$ and consist of $2\mu = 4$ equal arcs with four cusps.

3. – Polar cone of the tractrix.

In 3-space the motion of the line segment AB generating a tractrix b of a curve a can be produced by the rolling of a *plane* π, perpendicular to AB at B, over the *polar developable* Π of b. At every moment the *instantaneous axis* of rotation may be found as the intersection line p of the normal planes to the paths of A and B, i.e. to the guide curve a and to the tractrix b. This line p is the contact generator of Π with π and represents the polar axis (axis of curvature) of b.

In our case the tractrix b (2.13) is situated on a sphere Σ (2.14), hence the polar developable Π is a *cone* issuing from the center $M(0, 0, d)$ of Σ. Its generator p makes a right angle with the *osculating plane* τ *of* b. This plane touches the circle a and therefore is determined by the point B (2.13), the corresponding guide point A

$$\dot{x}_0 : \dot{x}_1 : \dot{x}_2 : \dot{x}_3 = 1 : m \cos u : m \sin u : 0 \,, \tag{3.1}$$

and the point $T(0 : -\sin u : \cos u : 0)$ A short calculation leads to the following equation of $\tau = ABT$:

$$-mn^2 x_0 + n^2 \cos u \cdot x_1 + n^2 \sin u \cdot x_2 + (md - r \cos nu) x_3 = 0 \,. \tag{3.2}$$

Now the *axis* p—perpendicular to τ (3.2) and passing through $M(1:0:0:d)$—is described by

$$(3.3)\quad x_0 = v\,,\ x_1 = n^2\cos u\,,\ x_2 = n^2\sin u\,,\ x_3 = (m+v)d - r\cos nu\,.$$

This is also a parametric representation (in u and v) of the *polar cone* Π. The parameter curves $v = \text{const}$ are situated on cylinders of revolution with the common axis z, and homothetic with M as center of similitude. Each of them—take for instance the curve $v = -m$—is point-path of a special motion which is combined of a uniform rotation about the fixed axis z and a harmonic oscillation along z with frequency n. This motion (« harmonischer Umschwung ») has been discussed by W. KAUTNY [2]. Such a trajectory $v = \text{const}$ is transformed into a sine curve, if its supporting circular cyclinder is developed upon a plane.

The *base* $x_3 = 0$ of the polar cone Π (3.3) is defined by $v = (r/d)\cos nu - m$. From its polar equation

$$(3.4)\qquad R = n^2 d/(r\cos nu - md)$$

we see that this base curve is derivable from a *conic* with numerical eccentricity $\varepsilon = r/md = 2nc/m(n^2 + c^2)$ by means of a « fan transformation » which leaves unchanged the radii (measured from the origin O), but multiplies the polar angles with the constant factor $n^{-1} > 1$ [**3**; I, 423; II, 131, 200]. This conic ($n = 1$) is an ellipse ($\varepsilon^2 < 1$) if $r^2 < m^2$, a parabola ($\varepsilon^2 = 1$) if $r^2 = m^2$, and a hyperbola ($\varepsilon^2 > 1$) if $r^2 > m^2$ (Fig. 1) (*).

4. – Hyperbolic case.

If the radius of the guide circle a is greater than the tractrix-generating unit segment ($m > 1$), then the torus Φ—envelope of the family of unit spheres σ centered on a—is of ring shape and has no real conic points. Thus the application of the inversion (2.3) is not possible in real way. Nevertheless we can use some results

(*) In the latter case the real points at infinity of the base curve (3.4) correspond to the real inflection points of the orthogonal projection of the tractrix b onto the plane of a; the inflection tangents touch the guide circle a.

of Section 2, mainly Theorem 1. In fact, as there exists a pencil of *spheres* Σ

$$x^2 + y^2 + z^2 - 2dz = n^2 = m^2 - 1 > 0 \,, \tag{4.1}$$

making right angles with all spheres σ, each orthogonal trajectory b of the family $\{\sigma\}$ will lie on one of the spheres Σ (including the plane $z = 0$).

Instead of the mapping (2.11) we apply now the *harmonic homology* $P \to P^*$, defined by

$$x = nx^*/z^* \,, \qquad y = ny^*/z^* \,, \qquad z = n^2/z^* \,. \tag{4.2}$$

This involutory perspective collineation—with center $H(0, 0, -n)$ and fixed plane γ: $z = z^* = n$—changes the sphere Σ (4.1) into the *hyperboloid of revolution* Σ^*

$$x^{*2} + y^{*2} - z^{*2} - 2dz^* + n^2 = 0 \,; \tag{4.3}$$

it is of two sheets, with right angle of aperture, and has its center at $I(0, 0, -d)$. A tractrix b on Σ, whose tangents meet the circle a (2.1), is transformed by the homology (4.2) into a curve b^* on Σ^*, whose tangents meet the circle at infinity a^* which is determined by the cone of direction

$$n^2(x^{*2} + y^{*2}) = m^2 z^{*2} \,. \tag{4.4}$$

This means that b^* is a *helix* with angle of inclination $\alpha = \arctan(n/m)$.

Such a helix b^* can be represented in the following form analogous to (2.8):

$$\begin{aligned} x^* &= c(n \operatorname{ch} nu \cos u + \operatorname{sh} nu \sin u) \,, \\ y^* &= c(n \operatorname{ch} nu \sin u - \operatorname{sh} nu \cos u) \,, \\ z^* &= \pm cm \operatorname{ch} nu - d \,. \end{aligned} \tag{4.5}$$

Indeed this curve lies on the hyperboloid Σ^* (4.3) if we put $c^2 = d^2 + n^2$, and its tangents have the direction

$$\dot{x}^* : \dot{y}^* : \dot{z}^* = m \cos u : m \sin u : \pm n \,, \tag{4.6}$$

according with (4.4). Transforming now back the helix b^* (4.5) by means of the homology (4.2), we arrive at the following parametric representation (in homogeneous coordinates again) of the required *tractrix* b in the hyperbolic case:

$$\begin{aligned} x_0 &= \pm cm \operatorname{ch} nu - d\,, \\ x_1 &= cn(n \operatorname{ch} nu \cos u + \operatorname{sh} nu \sin u)\,, \\ x_2 &= cn(n \operatorname{ch} nu \sin u - \operatorname{sh} nu \cos u)\,, \\ x_3 &= n^2\,. \end{aligned} \tag{4.7}$$

Each of these curves is situated on a sphere Σ (4.1) with radius $r = c$ and central distance $d = \sqrt{c^2 - n^2}$ from the origin, has one real cusp (for $u = 0$) and approaches asymptotically (with $u \to \pm\infty$) the circle $x^2 + y^2 = n^2$, $z = 0$. This circle is the base of the elliptic pencil of spheres (4.1) and is the unique algebraic real tractrix of the circle a (2.1).

The orthogonal projection of the helix b^* (4.5) onto the fixed plane γ ($z = n$) of the homology (4.2) is a so-called *paracycloid* [**3**; II, 120]. Because of its invariance under the homology it is identical with the central projection of the tractrix b from the origin O. Thus we have

THEOREM 4. – *In the hyperbolic case the spatial tractrices of a circle are collinear to helices on a hyperboloid of revolution of two sheets. Their central projections from the center of the circle onto a plane parallel to that of the circle are paracycloids.*

The *polar cone* of the tractrix b (4.7) has its vertex in the center $M(0, 0, d)$ of the sphere Σ (4.1) and can be obtained in the same way as in Section 3. Its base in the plane $z = 0$ has the polar equation

$$R = n^2 d / (\pm md - c \operatorname{ch} nu)\,. \tag{4.8}$$

5. – Non-euclidean interpretations.

The essential role of the spheres σ and Σ in the foregoing developments suggests interpretation of the results in the so-called *conformal model* of a non-euclidean space. This model, due to H. POINCARÉ, W. KILLING, G. DARBOUX and J. WELLSTEIN, is based upon an «absolute» euclidean sphere Ω; non-euclidean

« planes » are then represented by euclidean spheres orthogonal to Ω, « lines » by euclidean circles orthogonal to Ω, and « points » by couples of points inversive with respect to Ω [1; II, 40]. The measurement of angles is throughout euclidean; isometric transformations are all spatial Moebius mappings which leave invariant the absolute sphere Ω. If the radius of Ω is real, the geometry is *hyperbolic*, if the radius is purely imaginary, the geometry is *elliptic*.

Within the pencil of spheres orthogonal to the family of spheres σ we let correspond to any (real) sphere Σ

$$x^2 + y^2 + z^2 \pm n^2 = 2dz \tag{5.1}$$

the orthogonal sphere Ω

$$x^2 + y^2 + z^2 \pm n^2 = \pm 2n^2 z/d\,. \tag{5.2}$$

The upper sign is to be taken in the elliptic case (Section 2), the lower sign in the hyperbolic case (Section 4). Consequently all spheres σ and the single sphere Σ are to be considered as « planes » in the sense of the non-euclidean geometry based upon Ω; this geometry is elliptic in the first case, and hyperbolic in the second case. Thus the torus Φ enveloping the sphere family $\{\sigma\}$ may be interpreted as a « cone of revolution » with the vertex (C, D). The inversive euclidean circle-pair $(q, \bar{q})$ common to Φ and Σ is the conformal representation of a non-euclidean « circle ».

Each « plane » σ cuts the « plane » Σ in a « line » t tangent to the « circle » $(q, \bar{q})$ and represented by a euclidean circle orthogonal to Ω. As our *tractrix* b—(2.13) or (4.7)—is an orthogonal trajectory of the circle family $\{t\}$, it can be considered as a *non-euclidean involute of the circle* $(q, \bar{q})$. Thus we have

THEOREM 5. – *Any euclidean spatial tractrix of a circle, if it is of elliptic or hyperbolic type, may be interpreted as conformal representant of a non-euclidean involute of a circle, in the sense of elliptic or hyperbolic geometry respectively.*

There is yet another possibility of non-euclidean interpretation. If we consider as abolute the sphere Ω_0

$$x^2 + y^2 + z^2 = \pm n^2 \tag{5.3}$$

—the upper sign belonging again to the elliptic case (Section 2), the lower sign to the hyperbolic case (Section 4)—, all spheres Σ, making right angles with Ω_0, represent « planes ». The torus Φ is now to be considered as a *Clifford surface*, i.e. a surface of constant distance from the axis z [**1**; II, 55]. Each sphere σ is one component of the conformal representant of a non-euclidean « sphere » which touches the Clifford surface Φ along a meridian; the second component $\bar{\sigma}$, inversive to σ with respect to Ω_0, belongs also to the set $\{\sigma\}$ and is identical with the sphere opposite to σ. The « centers » of the « spheres » $(\sigma, \bar{\sigma})$ are situated on a « line » l, represented by the circle

$$x^2 + y^2 \pm n^2 = 0 , \quad z = 0 . \tag{5.4}$$

Therefore any (plane or spatial) euclidean tractrix b of the circle a (2.1), being an orthogonal trajectory of the family of « equal spheres » $(\sigma, \bar{\sigma})$ which are centered on a « straight line » l, may be considered as a *non-euclidean tractrix of a line*. All these curves are « plane » and—for a fixed length of the generating line segment—« congruent » to each other. The tractrices on two different « planes » Σ_1, Σ_2 can be interchanged by « reflection » and therefore are inversive to each other in euclidean sense.

THEOREM 6. – *All euclidean tractrices of a circle, if they are of elliptic or hyperbolic type, may be interpreted as conformal representants of non-euclidean tractrices of a straight line. In the elliptic case the geometry is hyperbolic, and the guide line is ideal, in the hyperbolic case the geometry is elliptic.*

6. – Parabolic case.

When the generating line segment AB is equal to the radius of the guide circle a $(m = 1)$, the unit spheres σ touch the z-axis at the origin O. The conic points C and D of the enveloping torus Φ are confounded with the center O which is now a biplanar double point.

The *inversion* $P \to P^*$ with center O and power 2, described by

$$x^* = 2x/Q , \quad y^* = 2y/Q , \quad z^* = 2z/Q \quad \text{with} \quad Q = x^2 + y^2 + z^2 , \tag{6.1}$$

changes the spheres σ into planes σ^* tangent to the unit *cylinder of revolution* Φ^*: $x^2 + y^2 = 1$. Any tractrix b of the considered type, being an orthogonal trajectory of the sphere family $\{\sigma\}$, is therefore transformed into an orthogonal trajectory b^* of the set $\{\sigma^*\}$, i.e. an *involute of a circle* with radius 1:

$$x^* = \cos u + u \sin u , \quad y^* = \sin u - u \cos u , \quad z^* = c . \tag{6.2}$$

Transforming back this curve by means of (6.1), we obtain the following parametric representation (in homogeneous coordinates) of the *tractrix b*:

$$(6.3)\qquad \begin{aligned} x_0 &= 1 + c^2 + u^2 , \\ x_1 &= 2(\cos u + u \sin u) , \\ x_2 &= 2(\sin u - u \cos u) , \\ x_3 &= 2c . \end{aligned}$$

For $c \neq 0$ this curve is situated on the *sphere* Σ with the equation

$$(6.4)\qquad c(x_1^2 + x_2^2 + x_3^2) = 2x_0 x_3 ;$$

it has the radius $r = 1/c$ and touches the plane $x_3 = 0$ of the circle a at the center O. The tractrix b (6.3) has one cusp (for $u = 0$) and an asymptotic point at the origin O. The central projection of the tractrix b from O onto the plane $z = x_3/x_0 = c$ is identical with the circle involute b^* (6.2).

THEOREM 7. – *In the parabolic case any spatial tractrix of a circle is inversive to the involute of a circle, and collinear to a helix on a paraboloid of revolution.*

The proof of the second part of the theorem would follow the way in Section 4.

The *polar cone* of the tractrix b (6.3) has its vertex in the center $M(0, 0, 1/c)$ of the supporting sphere Σ (6.4). Its base curve in the plane $z = 0$ is to be found as in Section 3; the polar equation of this spiral reads:

$$(6.5)\qquad R = 2/(1 - c^2 - u^2) .$$

BIBLIOGRAPHY

[1] E. BOMPIANI, *Metriche non-euclidee*, Roma, 1951/52.

[2] W. KAUTNY, *Zur Geometrie des harmonischen Umschwungs*, Monatsh. Math., **60** (1956), pp. 66-82.

[3] G. LORIA - K. SCHÜTTE, *Spezielle algebraische und transzendente ebene Kurven*, Leipzig/Berlin, 2 Aufl., 1911.

[4] W. WUNDERLICH, *Über die Schleppkurven des Kreises*, Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. IIa, **156** (1948), pp. 155-173.

[5] W. WUNDERLICH, *Beispiele für das Auftreten projektiver Böschungslinien auf Quadriken*, Mat. Tidsskr. B., **1951** (1951), pp. 9-26.

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