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Approximate Optimization of Watt's Straight-Line Mechanism

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Abstract
The paper develops elementary formulas for the synthesis of straight-line linkages of Watt's type by optimizing a quintic which approximates the flat part of Watt's coupler sextic.

1. Watt's Sextic

To lead the end of the piston-rod of his steam-engine approximately along a rectilinear segment, James Watt in 1784 used a coupler curve generated by a 4-bar linkage $LABM$ with equal rockers $LA = MB = a$ [1, 2, 3]. The center C of the coupler $AB = 2b$ describes a sextic c which has the shape of the digit 8, if $2b < LM$ and $2|a-b| < LM < 2(a+b)$. Each one of the two branches passing through the inflection node O in the middle of the fixed base LM can serve, in a certain neighbourhood of O, to approximate a segment of a straight line. Watt is said to have been prouder of this detail of his steam-engine than of the machine as a whole.

The simplest and most natural proposal for a good approximation consists in using a linkage which in its initial position LA_0B_0M forms right angles at A_0 and B_0 (Fig. 1), as then the coupler curve c has a 5-point contact at $C_0 = O$ with the ideal straight line A_0B_0 (viz. Section 4). In principle this idea belongs to Burmester theory which asks for the closest possible contact between the ideal line and an approximating curve in a certain point. Now just in kinematics it is well-known that better results can be obtained by Chebyshev theory which aims to minimize the maximal deviation from the ideal line, by allowing the approximating curve to reach the maximal deviation as often as possible in a certain interval. Hence the flat branch of Watt's coupler sextic c should be forced to remain between two parallel lines, touching each one in two points and cutting it in a third point (Fig. 3).

An attempt in this direction has been made by Bloch[4], but does not appear quite satisfactory, as the straight border lines are replaced by hyperbolas. A decisive progress has been achieved in a recent paper of Folkeson[5], but as the author uses a computer program to calculate the coordinates of a sufficient number of points of the coupler curve, his paper contains no formulas which could serve for the synthesis of optimized Watt mechanisms.

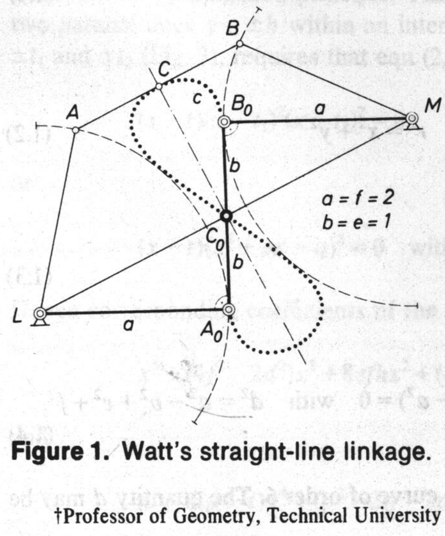

Figure 1. Watt's straight-line linkage.

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To approximate, by a Watt sextic, a segment of the x-axis of a cartesian frame $(O; x, y)$, let us locate the fixed pivots at $L(-e, f)$ and $M(e, -f)$. Then we replace the 4-bar linkage $LABM$ by a 5-bar linkage $LICJM$, whose arms $LI = MJ = b$ are parallel to the coupler AB and whose links $IC = JC = a$ are parallel to LA and MB, respectively (Fig. 2). To provide a constrained motion of the 5-bar linkage, the arms LI and MJ have to be driven simultaneously in such a way that they conserve their parallel position; this might be achieved by an auxiliary parallelogram.

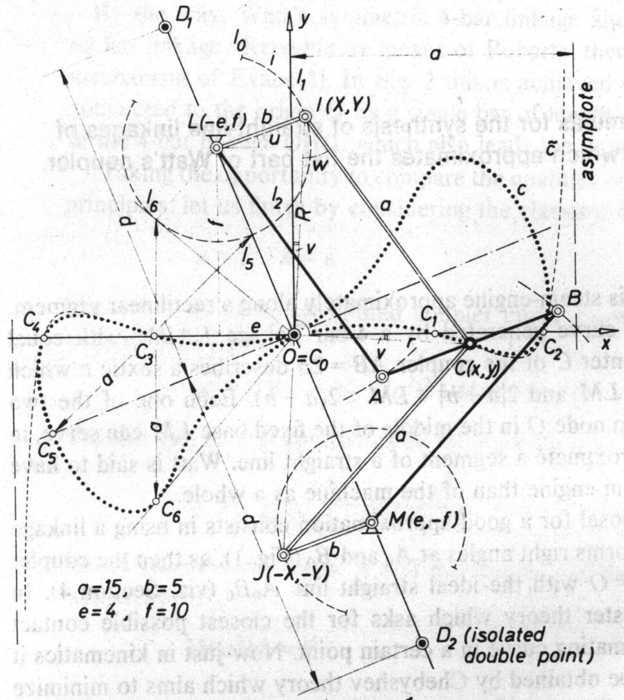

Figure 2. Different generations of Watt's sextic.

As the triangle IJC is isosceles, the radii vectores $OC = r$ and $OI = R$ are orthogonal and connected by $r^2 + R^2 = a^2$. This induces a remarkable correspondence $I \to C$ which maps the circular path i of I onto the path c of C. This planar transformation, defined by $OI \perp OC$ and $IC = a$ (Fig. 2), allows an elegant graphical construction of the coupler curve c.

Denoting the coordinates of C with x and y and those of I with X and Y, the equations of the involutary algebraic two-to-two transformation $I \leftrightarrow C$ (of degree 4) read

$$x = \lambda Y,\ y = -\lambda X \quad \text{with} \quad \lambda^2 = \frac{a^2 - R^2}{R^2}, \quad R^2 = X^2 + Y^2. \tag{1.1}$$

Applying the inverse formulas

$$X = -\mu y, \quad Y = \mu x \quad \text{with} \quad \mu^2 = \frac{a^2 - r^2}{r^2}, \quad r^2 = x^2 + y^2 \tag{1.2}$$

for the mapping of the circle i, described by

$$(X + e)^2 + (Y - f)^2 = b^2, \tag{1.3}$$

we obtain the equation of the coupler curve c

$$(x^2 + y^2)(x^2 + y^2 - d^2)^2 + 4(fx + ey)^2(x^2 + y^2 - a^2) = 0 \quad \text{with} \quad d^2 = a^2 - b^2 + e^2 + f^2. \tag{1.4}$$

This equation shows that c is a sextic, i.e. an algebraic curve of order 6. The quantity d may be interpreted as the central distance of two isolated double points of c, situated on the base line LM $(fx + ey = 0)$ which is an axis of symmetry for c.

Representing the circle i (1.3) in parametric form by

$$X = b \cos u - e, \quad Y = b \sin u + f, \tag{1.5}$$

and introducing the auxiliary angles v and w of Fig. 2 by

$$\tan v = X/Y, \quad \cos w = R/a, \tag{1.6}$$

we obtain by application of (1.1) a polar representation of c

$$x = r \cos v, \quad y = -r \sin v \quad \text{with} \quad r = a \sin w. \tag{1.7}$$

This simple procedure to calculate points of the coupler sextic is similar to that used by Folkeson[5], but perhaps somewhat more convenient.

2. Approximating Quintic

For small values of y we may neglect higher powers of y in eqn (1.4) of Watt's coupler sextic c. Thus we get a quintic $\tilde{c}$

$$x(x^2 - d^2)^2 + 4f(fx + 2ey)(x^2 - a^2) = 0. \tag{2.1}$$

It approximates the flat branch of c. The coincidence of the curves c and $\tilde{c}$ is good, as they have in common five precision points $C_0, C_1, \ldots, C_4$ on $y = 0$ (Fig. 2).

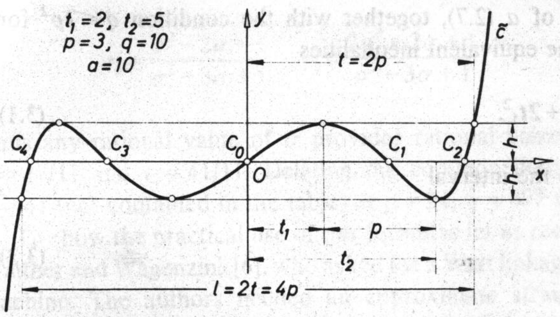

Figure 3. Optimized approximating quintic.

The rational quintic $\tilde{c}$, symmetric with respect to the origin O, is now better apt for applying Chebyshev's optimization principle. The fundamental condition that $\tilde{c}$ has to remain between two parallel lines $y = \pm h$ within an interval $-t \leqq x \leqq t$, touching them at points with abscissas $\pm t_1$ and $\pm t_2$ (Fig. 3), requires that eqn (2.1) for $y = h$ is equivalent to

$$(x - t)(x - t_1)^2(x + t_2)^2 = 0 \tag{2.2}$$

or

$$(x - t)(x^2 + px - q)^2 = 0 \quad \text{with} \quad p = t_2 - t_1, \quad q = t_1 t_2. \tag{2.3}$$

Hence corresponding coefficients of the polynomials

$$x^5 + (4f^2 - 2d^2)x^3 + 8efhx^2 + (d^4 - 4a^2f^2)x - 8a^2efh = 0 \tag{2.4}$$

and

$$x^5 + (2p - t)x^4 + (p^2 - 2q - 2pt)x^3 - (2pq + p^2t - 2qt)x^2 + (q^2 + 2pqt)x - q^2t = 0 \tag{2.5}$$

must be equal. This yields the following relations

$$t = 2p, \quad 2d^2 - 4f^2 = 3p^2 + 2q, \quad 4efh = p(q - p^2),$$
$$d^4 - 4a^2f^2 = q(4p^2 + q), \quad 4a^2efh = pq^2. \tag{2.6}$$

Choosing the values of p, q and h (all positive), we get

$$a^2 = \frac{q^2}{q - p^2}, \tag{2.7}$$

$$d^2 = \frac{q^2 + p(2p^2 - q)\sqrt{q}}{q - p^2}, \quad f^2 = \frac{1}{4}(2d^2 - 3p^2 - 2q), \tag{2.8}$$

$$e = \frac{p(q - p^2)}{4fh}, \quad b^2 = a^2 - d^2 + e^2 + f^2. \tag{2.9}$$

Provided that h is small, the coupler sextic c (1.4) with the data above will represent an almost optimal straight-line approximation, as the corresponding quintic $\tilde{c}$ (2.1) has the maximal deviation $\pm h$ from the ideal line $y = 0$ on a length of

$$l = 2t = 4p. \tag{2.10}$$

Nevertheless there has to be paid attention to some restrictive conditions.

3. Restrictive Conditions

The condition $p^2 < q$ for a real value of a (2.7), together with the condition $q < 2p^2$ for $t_2 < t = 2p$, leads with respect to (2.3) to the equivalent inequalities

$$t_2^2 - 3t_2t_1 + t_1^2 < 0 < 2t_2^2 - 5t_2t_1 + 2t_1^2. \tag{3.1}$$

Hence the quotient $t_2/t_1 > 1$ is restricted to the interval

$$2 < \frac{t_2}{t_1} < \frac{1}{2}(3 + \sqrt{5}) = 2.6180. \tag{3.2}$$

Other conditions were mentioned at the beginning of Section 1 with $2b < LM$ and $2|a - b| < LM < 2(a + b)$. The first of them, equivalent to $b^2 < e^2 + f^2$ or $a^2 < d^2$, is already satisfied because of the due choice of the sign of the square root in (2.8). The second condition, necessary and sufficient for the existence of a real triangle LAC_0, is equivalent to $2ab < |2a^2 - d^2|$ and requires (after elimination of b)

$$d^2 < 2a\sqrt{e^2 + f^2}. \tag{3.3}$$

With respect to eqns (2.8) and (2.9) this relation yields

$$e^2 > \frac{(4p^2 + q)(q - p^2)}{4q} \quad \text{or} \quad h^2 < \frac{p^2q(q - p^2)}{4f^2(4p^2 + q)} \tag{3.4}$$

and thus provides a lower bound for e and an upper bound for the wanted maximal deviation h.

4. Examples

In order to get simple illustrative examples for the theory, with integer values of t_1 and t_2, we may assume $q - p^2 = 1$, hence

$$t_1^2 - 3t_1t_2 + t_2^2 = -1. \tag{4.1}$$

Starting with a known solution of this diophantic equation, say $t_1 = t_2 = 1$, we proceed to a new solution $t_1' = t_2$, $t_2' = t_3$ satisfying analogously

$$t_2^2 - 3t_2t_3 + t_3^2 = -1. \tag{4.2}$$

The difference of eqns (4.1) and (4.2) leads to the recursion formula

$$t_3 = 3t_2 - t_1. \tag{4.3}$$

Operating in this way we obtain the following table:

t_1	t_2	p	$q = a$	d	f	$eh = p/4f$	e_{min}	h_{max}
1	2	1	2	2	0.5	0.5	0.866	0.5774
2	5	3	10	13.263	8.729	0.0859	1.072	0.0801
5	13	8	65	91.041	63.747	0.0314	1.111	0.0283
13	34	21	442	624.199	440.750	0.0119	1.117	0.0107

Consecutive values of t_1, t_2 and p belong to the well-known Fibonacci sequence connected with the golden ratio. The value of e_{min} (3.4) tends towards $\sqrt{5}/2 = 1.118$. Figure 3 is based upon the second line of the table.

Another possibility to obtain simple values consists in putting

$$t_2 - 1 = \sigma(t_1 - 1). \tag{4.4}$$

This substitution, introduced in eqn (4.1), yields the parametric representation

$$t_1 = \frac{\sigma^2 - 2\sigma + 2}{\sigma^2 - 3\sigma + 1}, \quad t_2 = \frac{2\sigma^2 - 2\sigma + 1}{\sigma^2 - 3\sigma + 1}. \tag{4.5}$$

Thus any rational value of σ provides rational values of t_1 and t_2. For instance $\sigma = 5$ gives $t_1 = 17/11$ and $t_2 = 41/11$. Deleting the common denominator 11, we arrive with $t_1 = 17$ and $t_2 = 41$ (not contained in the table) at $p = 24$, $q = 697$ and $a = 697/11$.

To show the practical use of our formulas let us reconsider—like Folkeson [5]—a problem of Walther and Wagenzink [6], who asked for a Watt linkage to be applied in a modern manufacturing machine. The authors needed an approximate straight-line motion over a length of about $l = 100$ (mm), prescribing the length $a = 150$ of the rockers. At first we have from eqn (2.10): $p = l/4 = 25$; then by solving the quadratic eqn (2.7)

$$q = \frac{a}{2}(a - \sqrt{a^2 - 4p^2}) = 75(150 - 100\sqrt{2}) = 643.3983.$$

With eqns (2.8) we proceed to

$$d = 208.3452, \quad f = 144.6147,$$

and then with eqns (2.9) to

$$eh = pq^2/4a^2f = 0.7951, \quad b^2 - e^2 = a^2 - d^2 + f^2 = 5.6995.$$

Choosing now—like the authors—the coupler length with $2b = 60$, we get

$$e = 29.9049.$$

This value satisfies the condition (3.4) and provides a maximal deviation of

$$h = 0.0266$$

for the approximating quintic $\bar{c}$. The effective deviation of the coupler sextic c might be controlled by calculating some characteristic ordinates y by means of eqns (1.1); we find in the neighbourhood of the maximum: $x_1 = 15.778$, $y_1 = 0.0267$, and in the neighbourhood of the minimum: $x_2 = 40.778$, $y_2 = -0.0265$. To guarantee the full length l of the approximation also for the sextic c it would have been better to choose p somewhat greater than $l/4$, say $p = 26$. In any case the obtained approximation is essentially better than that of the not optimized linkage in [6]; corresponding to the choice of the pivot coordinates $e = 29.692$ and $f = 145.710$ in [6], the deviation amounts there to $y_{max} = 0.077$.

By the way, Watt's symmetric 4-bar linkage always might be replaced by another double-rocker linkage, derivable by means of Roberts' theorem and sometimes called the grasshopper mechanism of Evans [3]. In Fig. 2 this is achieved directly if the midpoint H of the bar IC is connected to the origin O by a single bar of length $a/2$. Then the replacing mechanism consists in the 4-bar linkage OHIL which also leads the coupler point C along Watt's curve c.

Taking the opportunity to compare the qualities of Burmester's and Chebyshev's optimization principles, let us finish by considering the classical mechanism of Fig. 1

$$a = f, \quad b = e. \tag{4.6}$$

The eqn (1.4) of the generated coupler curve c—which now has a higher inflection point at O—takes the form

$$(x^2 + y^2)(x^2 + y^2 - 2a^2)^2 + 4(ax + by)^2(x^2 + y^2 - a^2) = 0. \tag{4.7}$$

The flat part of c is approximated by a quintic $\bar{c}$ (2.1), represented now by

$$y = \frac{x^5}{8ab(a^2 - x^2)} \approx \frac{x^5}{8a^3b}. \tag{4.8}$$

Hence the deviation error h and the length l of the approximation interval $-l/2 \leqq x \leqq l/2$ are related by

$$256a^3bh \approx l^5. \tag{4.9}$$

With the data of the numerical example above, namely $a = 150$ and $b = 30$, we would get for $l = 100$ a deviation of at least $h \approx 0.386$ (in fact $h = 0.444$), whereas the deviation $h = 0.0266$ of the optimized mechanism could be ensured only over a length of $l \approx 58.6$ (precisely $l = 58.1$). This shows very clearly, that Chebyshev's principle, being a global one, provides remarkably better results than Burmester's local principle.

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ANGENÄHERTE OPTIMIERUNG DES WATTSCHEN GERADFÜHRUNGSMECHANISMUS

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Kurzfassung - Zur angenäherten Geradführung nach Watt dient bekanntlich eine achterförmige Koppelkurve, die vom Mittelpunkt der Koppel eines gleichschenkligen Gelenkvierecks beschrieben wird. Eine genauere Untersuchung dieser symmetrischen Sextik führt zunächst auf eine neue Konstruktion derselben und liefert auch numerisch vorteilhaft auswertbare analytische Darstellungen (Abschnitt 1). Zum Zweck der Optimierung nach dem Tschebyschewschen Prinzip wird dann der flache Teil der Sextik durch eine bequemere Quintik angenähert, für welche die Bedingungen abgeleitet werden, dass sie zwei parallele Doppeltangenten aufweist (Abschnitt 2). Nach Hinweisen auf einschränkende Bedingungen für die Wahl der verfügbaren Parameter (Abschnitt 3) wird schliesslich an Hand von Beispielen die praktische Anwendung der entwickelten Formeln zur Synthese von Wattschen Geradführungen mit gewünschter Länge und Güte gezeigt (Abschnitt 4).