Reprinted from the MATHEMATICS MAGAZINE
Vol. 52, No. 4, September, 1979
p. 235

Snapping and Shaky Antiprisms

WALTER WUNDERLICH
Technical University
Vienna, Austria

In his interesting article [1] Michael Goldberg recently presented a number of interesting examples of multi-stable polyhedra. A polyhedron of this kind has the remarkable property of possessing two or more distinct forms with the same development, i.e., with the same faces in equal arrangement; the difference consists only in the corresponding dihedral angles along the edges. If two forms are not too different, a model made of rigid plates which are hinged at their edges may "snap" from one position to the other with a slight temporary elastic strain. If the two positions coincide, the structure is called "shaky". Theoretically, it would allow only an infinitesimal deformation, but in fact a corresponding model is not at all stable and shows a perceptible but limited mobility.

After mentioning the author's snapping octahedra [2], Goldberg suggested generalizing those triangular antiprisms by replacing the base triangles by other rigid polygons situated in parallel planes. In what follows we describe how to construct correctly such models having n-gons as bases.

Let $A_1A_2\ldots A_n (n \geqslant 3)$ be an arbitrary (not necessarily simple) rigid n-gon in the plane $z=h-s\geqslant 0$, h and s being numerical constants. By applying a screw motion about the z-axis, composed of a turn with angle 2σ and a translation of length $2s$, the polygon is transferred into a new position $A_1'A_2'\ldots A_n'$ in the plane $z'=h+s$. The coordinates x_i, y_i of any vertex A_i change to

$$x_i'=x_i\cos 2\sigma-y_i\sin 2\sigma,\quad y_i'=x_i\sin 2\sigma+y_i\cos 2\sigma. \tag{1}$$

The plane of symmetry of the chord A_iA_i', i.e., the plane which is perpendicular to and bisecting the segment A_iA_i', is defined by

$$(x-x_i)^2+(y-y_i)^2+(z-h+s)^2=(x-x_i')^2+(y-y_i')^2+(z-h-s)^2$$

The intersections $b_i (i=1,2,\ldots,n)$ of these planes with a suitable plane which we call the base plane are the sides of a polygon $B_1B_2\ldots B_n (B_iB_{i+1}=b_i$, where $B_{n+1}=B_1)$. This n-gon may serve as base of a snapping polyhedron, provided the quantities σ and s are sufficiently small. The additional edges, connecting each point A_i with B_i and B_{i+1}, conserve their lengths when A_i passes to A_i', as $d(A_i,b_i)=d(A_i',b_i)$ where $d(X,Y)$ means the distance from X to Y.

If in particular we use the base plane $z=0$, the equation of the line b_i reads, after introduction of the expressions (1) and application of some well-known formulas of trigonometry, as follows:

$$-(x_i \sin\sigma + y_i \cos\sigma)x + (x_i \cos\sigma - y_i \sin\sigma)y = sh/\sin\sigma. \tag{3}$$

To arrive at a simple graphical construction, we apply to the orthogonal projection of A_i onto the base plane a rotation with angle $90° + \sigma$ about the origin O. The obtained point A_i^* has the coordinates

$$x_i^* = -x_i \sin\sigma - y_i \cos\sigma, \quad y_i^* = x_i \cos\sigma - y_i \sin\sigma. \tag{4}$$

Hence equation (3) is equivalent to

$$x_i^* x + y_i^* y = sh/\sin\sigma = r^2, \tag{5}$$

and this relation means that b_i is the polar line of A_i^* with respect to a circle with radius r and center O. The corresponding construction is shown in FIGURE 1 and applied in FIGURE 2 to a quadrangle ($n=4$).

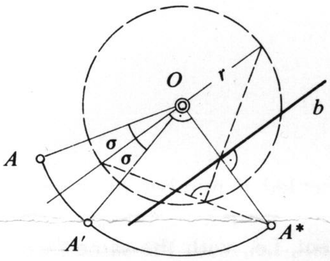

FIGURE 1

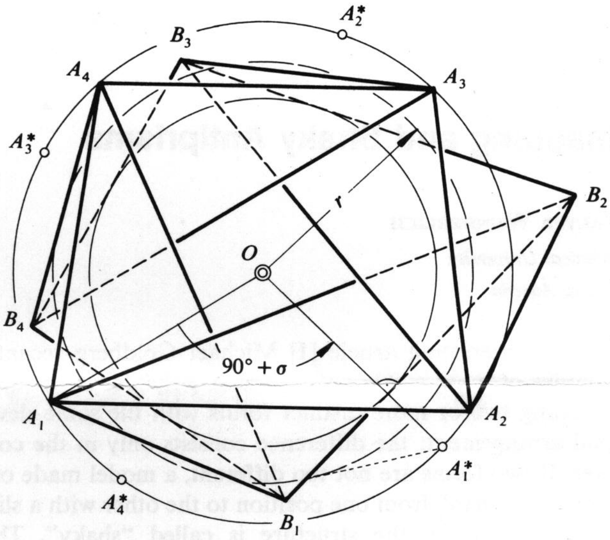

FIGURE 2

Having found a fitting pair of end faces $A_1A_2 \ldots A_n$ and $B_1B_2 \ldots B_n$ by means of the rule of FIGURE 1, the median height h of the snapping antiprism might still be changed within a certain range, as the construction depends only on r and σ; thus a prescribed value of h will be obtained by taking $s = r^2 \sin\sigma / h$.

In the case $h=s$ the first form of the antiprism is collapsed into a plane. The limit case $\sigma = s = 0$ with $\lim(s/\sigma) = c \neq 0$ leads to coinciding positions and therefore to shaky antiprisms. The construction of FIGURE 1, to be performed with a circle of radius $r = ch$, is applicable without difficulty. Due to the polarity the ray OB_i will be parallel to $A_{i-1}A_i$.

If, as in FIGURE 2, the polygon $A_1A_2 \ldots A_n$ is inscribed in a circle centered on the z-axis, the base polygon $B_1B_2 \ldots B_n$ will be circumscribed about a circle of the same kind and vice versa. (In FIGURE 2, where the (removed) upper face is a trapezoid, the base is a kite.) Consequently snapping and shaky antiprisms with similar or congruent end faces with coaxial circumcircles—as the triangular specimen in [2]—exist only if the polygons are regular.

References

[1] M. Goldberg, Unstable polyhedral structures, this MAGAZINE, 51 (1978) 165–170.

[2] W. Wunderlich, Starre, kippende, wackelige und bewegliche Achtflache, Elem. Math., 20 (1965) 25–32.