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Mechanisms Related to Poncelet's Closure Theorem

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Dedicated to Prof. O. Bottema, on the occasion of his 80th birthday

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Abstract

A closed equilateral polygon, whose vertices alternatively lie on two fixed circles, is moveable: The vertices can move along the circles without change of the common length of all sides. This phenomenon leads to a remarkable family of overconstrained linkages and is of interest for certain ball-bearings. In the case of coincident or parallel circle planes an algorithm is developed which allows the side length of such polygons to be calculated.

1. Introduction

LET k, l be two coplanar circles with centers M, N and radii a, b, respectively; the central distance MN is denoted by c. Starting from a point A_1 on k, we construct, with a given side length d, an equilateral zig-zag polygon $A_1B_1A_2B_2\ldots$ having its vertices A_i on k and B_i on l. If this polygon closes after $2n$ steps ($A_{n+1} = A_1$), without being doubly covered, then there exist a continuous set of such polygons: a closed $2n$-gon will always be obtained, wherever the starting point on k may have been chosen.

This remarkable fact was first discovered by Bottema[1]. Astonishing enough, the closure phenomenon holds also for circles k, l with an arbitrary position in space. In its general extent this closure theorem was stated by Black and Howland[2]; they confirmed it by an analytic reasoning based on a certain differential equation. An adequate proof was then given by Hohenberg[3], who used an appropriate (2, 2)-correspondence and thus pointed out the algebraic character of the problem; he considered also equilateral zig-zags between a circle and a straight line[4] and between two lines[5]. In a short note[6] the author showed by means of descriptive geometry how the problem might be lead back to the classical closure theorem of Poncelet (1822), mentioned already by Bottema. A simplified procedure will be presented here.

2. Proof of the Closure Theorem in the Planar Case

Using Cartesian coordinates x, y in the common plane π of the circle pair and locating the centers at $M(0, 0)$ and $N(c, 0)$, the circle k may be represented in parametric form by

$$x = a \cdot \cos u, \quad y = a \cdot \sin u, \tag{2.1}$$

whereas the circle l will be described by its equation

$$(x - c)^2 + y^2 = b^2. \tag{2.2}$$

An auxiliary circle s with radius d and centered at a point A on k has the equation

$$(x - a \cos u)^2 + (y - a \sin u)^2 = d^2. \tag{2.3}$$

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The circles l and s intersect in two points $B, \bar{B}$ situated on the common chord or radical axis g, whose equation is obtained by forming the difference of (2.2) and (2.3)

$$(a \cos u - c)x + a \sin u \cdot y = e^2 = \frac{1}{2}(a^2 + b^2 - c^2 - d^2). \tag{2.4}$$

With varying parameter u the point A moves along the circle k and the corresponding line g generates an envelope h (Fig. 1). The contact point H of g and h is determined by the derivative of (2.4) with respect to u, i.e. (after division by a)

$$-\sin u \cdot x + \cos u \cdot y = 0. \tag{2.5}$$

From (2.4) and (2.5) we obtain the parametric representation of h

$$X = \frac{e^2 \cos u}{a - c \cos u}, \qquad y = \frac{e^2 \sin u}{a - c \cos u}. \tag{2.6}$$

The equivalent polar equation

$$r = \frac{e^2}{a - c \cos u} \tag{2.7}$$

shows that the envelope h is a conic with one focus at the origin M, and the numerical eccentricity c/a.

Applying this result to the original problem, we see that the secondary polygon $B_1B_2B_3\ldots$ is inscribed in the circle l and circumscribed to the conic h. Now the famous poristic theorem of Poncelet states that a plane polygon, inscribed in one conic and circumscribed to another, closes either ever or never ("aut semper aut numquam")—independently of the starting point and always after the same number of steps. This means in our case: If there exists a (non degenerate) closed n-gon $B_1B_2..B_n$, whose vertices lie on the circle l and whose sides touch the conic h, then we may start from any point of l and, successively drawing tangents to h, we always shall obtain a closed n-gon inscribed in l and circumscribed to h. It is evident that each of the polygons $B_1B_2\ldots B_n$ easily can be completed to an equilateral $2n$-gon $A_1B_1\ldots A_nB_n$ which has its vertices alternatively on the circles k and l.

Expressing the closure phenomenon in other words, we may say: If in the course of a first

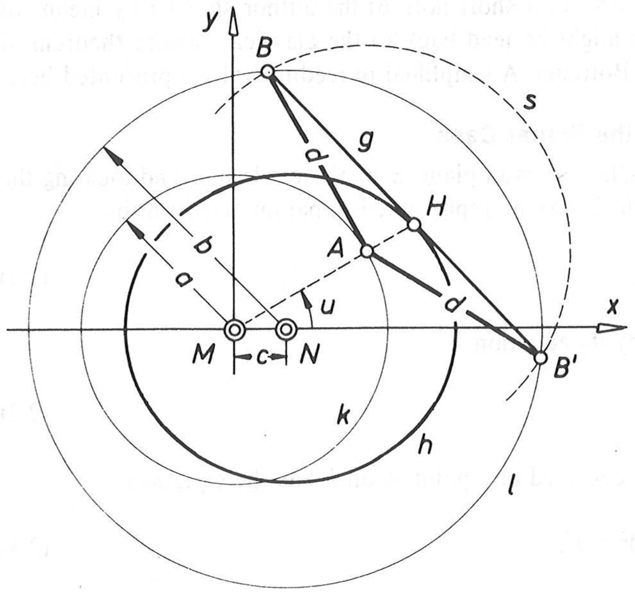


Figure 1.

trial the zig-zag polygon does not close after $2n$ steps, it would be in vain to try again with another starting point.

There still remains the problem to determine the appropriate side length d for given quantities a, b, c and a prescribed number $2n$ of steps. Having once found a fitting value of d, the polygon will close independently of the starting point. Hence the polygon can move along the circle pair, changing only its angles.

Figure 2 shows a number of Poncelet triangles belonging to the equilateral hexagon of Fig. 6.

3. General Case

Let us now consider the general case of two arbitrary circles k, l in space, supporting the vertices of an equilateral zig-zag polygon $A_1B_1A_2B_2\dots$ with side length d. We introduce the sphere Λ which passes through l and has its center in the plane π of k.† The two polygon sides issuing from an arbitrary point A of k have their endpoints $B, \bar{B}$ on l and on the auxiliary sphere Σ with center A and radius d. Consequently they lie in the radical plane γ of Λ and Σ which contains the intersection circle of these spheres. As γ is perpendicular to π, the orthogonal projection onto π leads just to the configuration of Fig. 1 (where the circle l is to be replaced by the great circle of Λ in π). Applying now the result of Section 2, we see that the base line g of γ envelopes a conic h when A varies along the circle k.

This means for our problem that the orthogonal projection $B_1'B_2'\dots$ of the secondary polygon $B_1B_2\dots$ is circumscribed to the conic h and inscribed in that ellipse l' which appears as the image of the circle l. Consequently, due to Poncelet, we have the closure phenomenon again, at first for the image polygon $B_1'B_2'..B_n'$, then for the polygon $B_1B_2..B_n$ itself, and finally for the equilateral polygon $A_1B_1..A_nB_n$. Thus the vertices of a closed equilateral polygon indeed can move along the circles k and l.

4. Applications

A plane linkage consisting of $2n$ bars represented by the sides of a closed polygon $A_1B_1..A_nB_n$, and with additional n bars connecting all joints A_i to a fixed pivot M, and another n bars connecting all B_i to a second pivot N, in general will be rigid: Together with the base MN we have $4n+1$ links and $2n+2$ joints; hence, due to Grübler's formula, the formal degree of freedom has the value $f=0$. —If, however, the mentioned polygon is equilateral (side

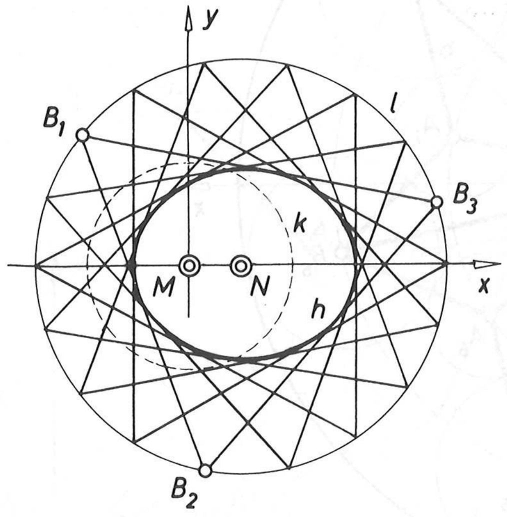


Figure 2.

†The exceptional case of orthogonal circle planes needs a somewhat modified treatment, as then the sphere Λ would degenerate into a plane.

length d), all spokes MA_i have equal length a and also all spokes NB_i have equal length b (Fig. 3), then, due to Bottema's theorem, the linkage will allow a constrained motion, and thus the true degree of freedom is $f' = 1$ [3].

The said linkage contains two series of kites: n kites $MA_iB_iA_{i+1}$ with side lengths a and d, and n kites $NB_iA_{i+1}B_{i+1}$ with side lengths b and d. The linkage also may be considered as an aggregate of $2n$ four-bars MA_iB_iN and $MA_{i+1}B_iN$ with common base MN. Provided that all these four-bars are double-cranks (as in Fig. 3), the linkage can perform a complete revolution. Mainly this case will be considered in the following developments; the respective conditions read (with positive values of a, b, c, d)

$$b - a > c, \quad a + b \pm c > d. \tag{4.1}$$

A series of equal balls with radius r centered at the vertices A_i of a Bottema polygon $A_1B_1 \ldots A_nB_n$, and a second series of equal balls with radius r' about the centers B_i will constitute a twin bearing if $r + r' = d$, as then neighbouring balls remain in contact during the motion of the polygon (Fig. 3).

A slight modification of both of the mechanisms (the spoke linkage and the twin ball-bearing) is obtained by arranging the circles k and l in parallel planes. Although the equilateral zig-zag polygon is now a spatial one, the closure theorem still holds (due to Section 3, or simply because of the fact that the orthogonal projection $A_1B'_1 \ldots A_nB'_n$ of the polygon onto the plane π of k is an equilateral polygon again, with side length $d' = \sqrt{(d^2 - p^2)}$, where p is the distance of the circle planes). The joints of the spoke linkage may still be cylindrical hinges, and the corresponding twin bearings can be derived in the same way as before; Fig. 3 may be considered as a cross-section through the contact points of the balls. Thrust bearings with coaxal ball series are well-known, but it is a remarkable statement that devices of this kind are

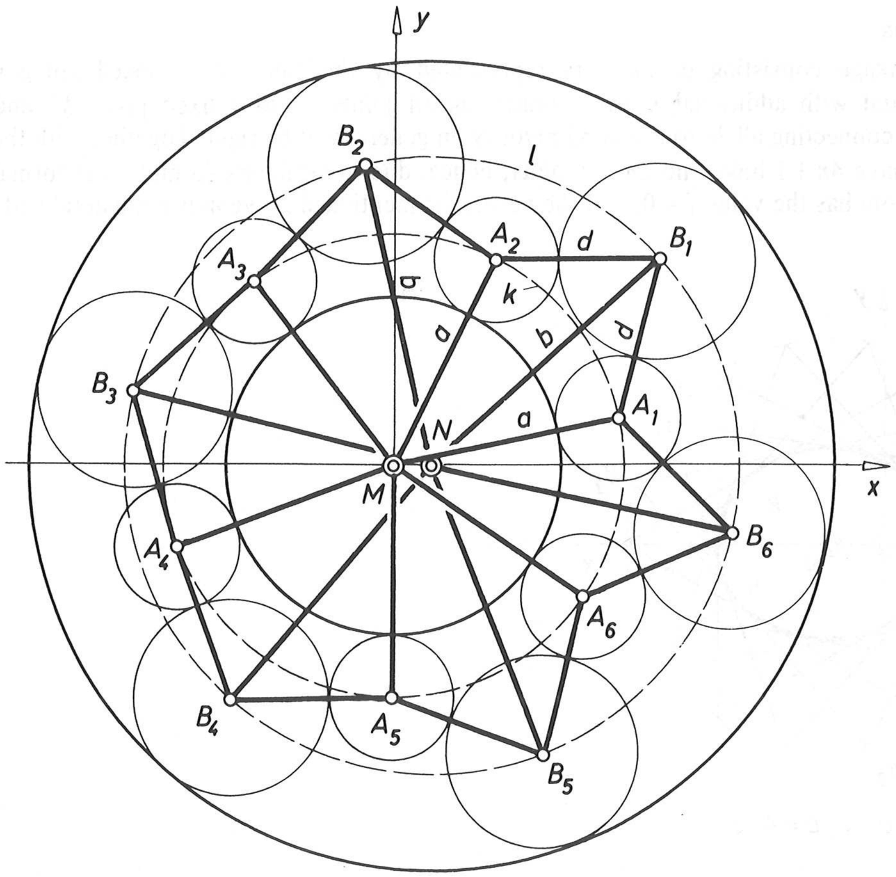


Figure 3.

working also when the ring axes are parallel or even skew. This unexpected phenomenon was the motive for the investigations of H.C. and B. Howland [2].

Spoke linkages with cylindrical joints and twin ball-bearings with intersecting axes may be derived from circle pairs k, l situated on a common sphere or on concentric spheres.

5. An Algorithm

Returning to the simple case of coplanar circle pairs k, l we have to state that the postponed problem of establishing closure conditions, relating the quantities a, b, c, d for prescribed values of n, is still open. In algebraical form such conditions were explicitely given by Hohenberg [3] for $n = 2, 3, 4$ and 5 only; his result for $n = 6$ has not been published. For greater values of n an iterative process may serve to find the numerical value of d, when a, b and c are known.

Using the parametric representation (2.1) for the circle k and an analogous representation for l (Fig. 4), the points A_i and B_i are determined by

$$A_i{:}\quad x_i = a\cos u_i, \qquad y_i = a\sin u_i;$$
$$B_i{:}\quad x_i' = b\cos v_i + c, \qquad y_i' = b\sin v_i. \tag{5.1}$$

The distance formulas for consecutive segments A_iB_i, B_iA_{i+1} and $A_{i+1}B_{i+1}$ (all of length d) after simple transformations furnish the equations

$$ac\cos u_i - bc\cos v_i + ab\cos(u_i - v_i) = f^2 = \frac{1}{2}(a^2 + b^2 + c^2 - d^2),$$
$$ac\cos u_{i+1} - bc\cos v_i + ab\cos(u_{i+1} - v_i) = f^2, \tag{5.2}$$
$$ac\cos u_{i+1} - bc\cos v_{i+1} + ab\cos(u_{i+1} - v_{i+1}) = f^2.$$

Forming differences and paying attention to $u_{i+1} \neq u_i$ and $v_{i+1} \neq v_i$ we get the relations

$$c\sin\varphi_i + b\sin(\varphi_i - v_i) = 0 \quad \text{with} \quad \varphi_i = \frac{1}{2}(u_i + u_{i+1}),$$

$$c\sin\psi_i + a\sin(u_{i+1} - \psi_i) = 0 \quad \text{with} \quad \psi_i = \frac{1}{2}(v_i + v_{i+1}). \tag{5.3}$$

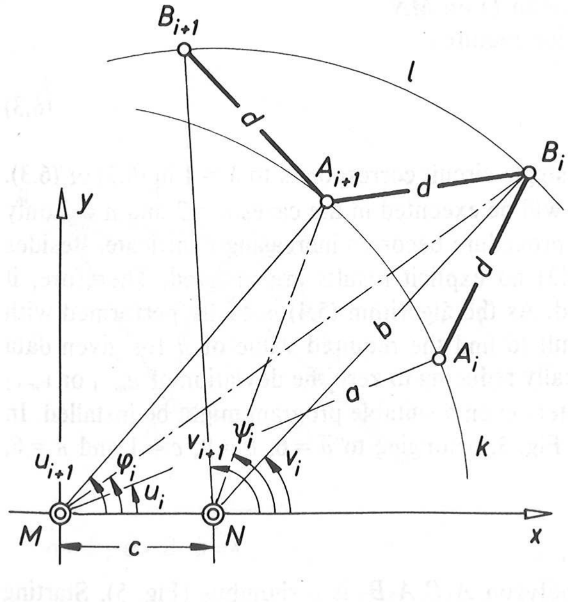


Figure 4.

They finally lead to appropriate recursion formulas

$$\tan \varphi_i = \frac{b \sin v_i}{b \cos v_i + c}, \quad u_{i+1} = 2\varphi_i - u_i;$$

$$\tan \psi_i = \frac{a \sin u_{i+1}}{a \cos u_{i+1} - c}, \quad v_{i+1} = 2\psi_i - v_i. \tag{5.4}$$

Starting with a chosen initial value u_1 we calculate v_1 by means of the first eqn (5.2) for $i = 1$. As soon as we have decided between the two possible values of v_1, the recursion formulas (5.4) successively and without ambiguity furnish φ_1, u_2, ψ_1, v_2 etc. The corresponding polygon $A_1B_1A_2B_2\ldots$ closes after $2n$ steps (with $A_{n+1} = A_1$), if

$$u_{n+1} = u_1 + 2\lambda\pi \quad (\lambda = \text{integer}). \tag{5.5}$$

By elimination of all angles which occur in the equations used one could arrive (after hard work) at an algebraic closure condition containing only the lengths a, b, c, d. For $n > 2$ more than one condition may be found, as not all possible values of λ in (5.5) lead to the same result. Hohenberg[3] proceeds in a similar way, using recursion formulas for the abscissas x_i of A_i and x'_i for B_i.

6. Closure Conditions

After all it will be sufficient to execute the algorithm of Section 5 with the special starting values

$$u_1 = 0, \quad v_1 = \arccos \frac{f^2 - ac}{(a - c)b}. \tag{6.1}$$

With respect to the axial symmetry of the corresponding polygon the closure condition (5.5) and the elimination process may be considerably simplified. For n even ($n = 2m$) the condition reduces to

$$u_{m+1} = \lambda\pi. \tag{6.2}$$

A corollary of Poncelet's theorem states that in this case the diagonals A_iA_{i+m} (indices modulo n) of the moving polygon constantly pass through a fixed point P on the line MN; similarly all diagonals B_iB_{i+m} pass through another fixed point Q on MN.

For n odd ($n = 2m + 1$) the closure condition requires

$$u_{m+1} = \lambda\pi. \tag{6.3}$$

Under the assumption (4.2) closure after one single circuit corresponds to $\lambda = 1$ in (6.2) or (6.3).

The above-mentioned elimination process will be executed in the cases $n = 2$ and $n = 3$ only (Sections 7 and 8). For larger values of n the procedure becomes increasingly intricate. Besides Hohenberg's closure conditions for $n = 2$–5[3] no explicit results are at hand. Therefore, it seems necessary to use the numerical method. As the algorithm (5.4) is easily performed with the aid of a pocket computer, it is not difficult to find the required value of d for given data a, b, c by means of trail and error, systematically reducing to zero the deviation of u_{m+1} or v_{m+1} from the prescribed value π. In better computers even a suitable program might be installed. In this way the side length of the dodecagon in Fig. 3, belonging to $a = 6$, $b = 8$, $c = 1$ and $n = 6$, was found to be $d = 4.2088$.

7. Equilateral Quadrangles

In the simplest case, $n = 2$ ($m = 1$), the polygon $A_1B_1A_2B_2$ is a rhombus (Fig. 5). Starting with $u_1 = 0$, we have to attain $u_2 = \pi$; see (6.2). From (5.4) we get $\varphi_1 = \pi/2$ and $\cos v_1 = -c/b$.

Comparing this value with that of (6.1), we obtain the closure condition $c^2 = f^2$ or

$$a^2 + b^2 = c^2 + d^2, \tag{7.1}$$

consonant with the result of Bottema[1] and Hohenberg[3].

Figure 5 shows the spoke linkage of Section 4, determined by $a = 2$, $b = 4$, $c = 1$ and $d = \sqrt{19} = 4.3589$. The linkage consists of 9 bars and is a combination of two Peaucellier inversors which map each of the circles k and l onto itself. According with a remark in Section 6 the diagonals A_1A_2 and B_1B_2 of the moving rhombus pass through the fixed points $P = N$ and $Q = M$, respectively. These points are the centers of the inversions $A_1 \leftrightarrow A_2$ and $B_1 \leftrightarrow B_2$. By the way, the mechanism is a particular case of an overconstrained nine-bar linkage with six triple joints which are distributed three by three over two orthogonal lines[7].

8. Equilateral Hexagons

In the case $n = 3$ ($m = 1$) of closed equilateral hexagons $A_1B_1A_2B_2A_3B_3$ the initial value $u_1 = 0$ requires $v_2 = \pi$; see (6.3). From the first and the third of the relations (5.2) we get

$$\cos v_1 = \frac{f^2 - ac}{(a-c)b}, \quad \cos u_2 = \frac{f^2 - bc}{a(c-b)}. \tag{8.1}$$

Instead of the second eqn (5.2) we take the first recursion formula (5.4); it says, with respect to (8.1)

$$\tan\frac{u_2}{2} = \frac{(a-c)b}{f^2 - c^2}\sin v_1. \tag{8.2}$$

By means of

$$\sin v_1 = \frac{w}{(a-c)b} \quad \text{with} \quad w^2 = (a-c)^2b^2 - (f^2 - ac)^2 \tag{8.3}$$

we find

$$\tan\frac{u_2}{2} = \frac{w}{f^2 - c^2}, \tag{8.4}$$

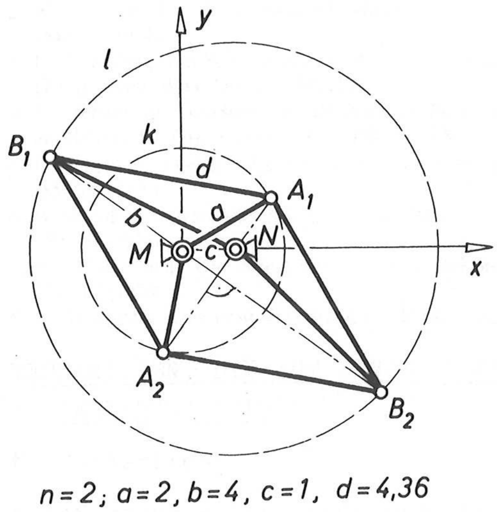


Figure 5.

hence

$$\cos u_2 = \frac{(f^2 - c^2)^2 - w^2}{(f^2 - c^2)^2 + w^2}. \tag{8.5}$$

Comparing the expressions for $\cos u_2$ in (8.1) and (8.5), we obtain a preliminary closure condition in the form

$$(f^2 + ab - ac - bc)(f^2 - c^2)^2 + (f^2 - ab + ac - bc)w^2 = 0$$
$$\text{with} \quad w^2 = (f^2 + ab - ac - bc)(ab + ac - bc - f2). \tag{8.6}$$

Division by the common factor

$$f^2 + ab - ac - bc = \frac{1}{2}(a + b - c + d)(a + b - c - d) \tag{8.7}$$

excludes only the trivially degenerating solutions $a + b - c \pm d = 0$. The remaining definitive closure condition may be written in the form

$$(ab - c^2)(ab - d^2) + ab(a - b)^2 = 0; \tag{8.8}$$

it is equivalent to formulas of Bottema[1] and Hohenberg[3].

Figure 6 shows a corresponding spoke linkage determined by $a = 2$, $b = 4$, $c = 1$ and $d = \sqrt{(88/7)} = 3.5456$. Changing the sign of a, we get another hexagon with side length $\bar{d} = \sqrt{(24)} = 4.8990$; it is supported by the same circle pair, but closes after two circuits ($\varkappa = 2$).

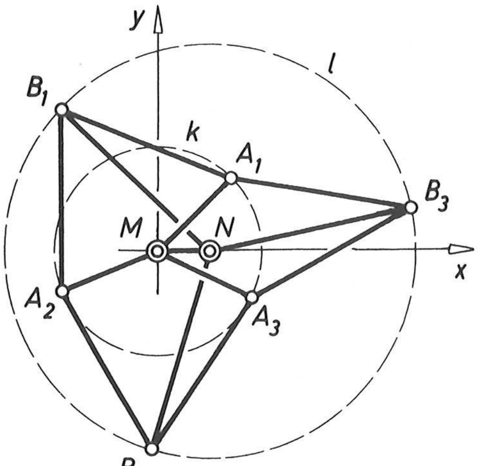
A peculiar situation arises from the supposition $a = b$: In this case we see from the closure condition (8.8) that either

$$a = b = c, \quad d \quad \text{arbitrary} \tag{8.9}$$

or

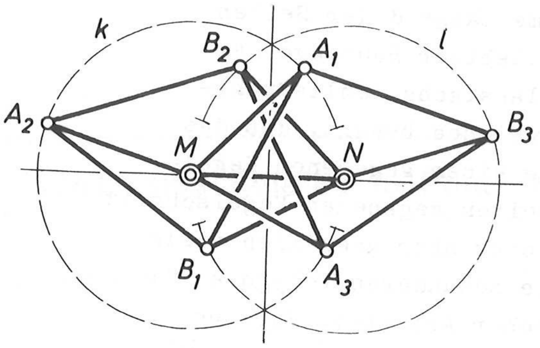
$$a = b = d, \quad c \quad \text{arbitrary}. \tag{8.10}$$

The first possibility, pointed out by Hohenberg[8] and illustrated in Fig. 7, means that under the assumption $a = b = c$ any equilateral polygon will close after six steps, whatever the side



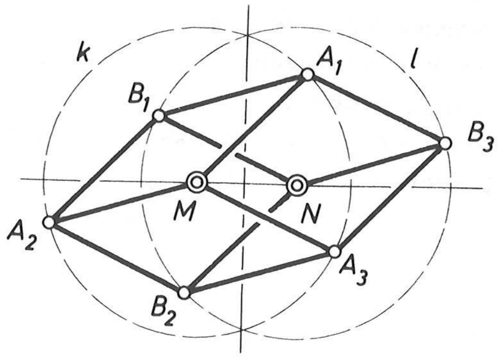
$n = 3;\ a = 2,\ b = 4,\ c = 1,\ d = 3{,}55$

Figure 6.



$n=3;\ a=b=c=3,\ d=4$

Figure 7.



$n=3;\ a=b=d=3,\ c=2$

Figure 8.

length d may be. So in this case there do not exist closed equilateral polygons with number of sides other than six.

The second possibility $a = b = d$ leads to spoke linkages looking like the parallel projection of a cube (Fig. 8), and the base length c may be altered. After deleting the base MN, the mechanism will have the degree of freedom $f' = 2$.

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MECHANISMEN, DIE ZUM PONCELETSCHEN SCHLIESSUNGSSATZ IN BEZIEHUNG STEHEN

W. Wunderlich

Kurzfassung - Ein geschlossenes gleichseitiges Polygon $A_1B_1..A_nB_n$, dessen Ecken abwechselnd auf zwei Kreisen k und l liegen, ist beweglich: Seine Ecken können auf den Kreisen

wandern, ohne daß sich die gemeinsame Länge d der Seiten ändert. Dieses Phänomen, das bei beliebiger Raumlage der Kreise besteht, läßt sich auf das klassische Schließungstheorem von Poncelet zurückführen, welches besagt, daß das ebene Problem, n-Ecke zu finden, die einem gegebenen Kegelschnitt eingeschrieben und einem zweiten gegebenen Kegelschnitt umgeschrieben sind, entweder keine oder aber unendlich viele Lösungen hat. Es zeigt sich, daß die sekundären Polygone $A_1A_2..A_n$ und $B_1B_2..B_n$ von Ponceletscher Art sind. Für den schon von Bottema betrachteten Fall komplanarer Kreise k und l wird ein Algorithmus mit trigonometrischen Rekursionsformeln (5.4) entwickelt, der die numerische Ermittlung der Seitenlänge d aus den Kreisdaten a (Radius von k), b (Radius von l) und c (Zentralabstand von k und l) bei vorgeschriebener Seitenzahl 2n gestattet. Abschließend werden die Annahmen n = 2 und n = 3 ausführlicher besprochen. Die Beweglichkeit der genannten Zickzack-Polygone führt auf bemerkenswerte Gelenkmechanismen und ist für Zwillings-Kugellager von Bedeutung.