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Projective Invariance of Shaky Structures

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With 5 Figures

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Summary

A framework consisting of rigid rods which are connected in freely moveable knots, in general is stable if the number of knots is sufficiently large. In exceptional cases, however, the rodwork may allow an infinitesimal deformation. Due to a theorem of Liebmann, this apparently metric property of existing shakiness in fact is a projective one, as it does not vanish if the structure is transformed by an affine or projective collineation. The paper presents a new analytic proof of this remarkable phenomenon. The developments are applicable also to polyhedra with rigid plates and to closed chains of rigid links.

1. Introduction

Let us consider a framework consisting of n rigid rods connected at m knots. The connections are assumed to be freely moveable; that means that the joints are cylindrical for plane linkages and spherical for spatial structures.

The rod lengths impose n conditions on the $2m$ or $3m$ knot coordinates. Taking into account the number of displacement variables — 3 in the plane, 6 in space — we see that the degree of freedom with respect to deformability of the structure is

$$f_2 = 2m - n - 3 \quad \text{or} \quad f_3 = 3m - n - 6, \tag{1.1}$$

respectively. Consequently the rodwork will be moveable if $f > 0$, and (in general) rigid if $f \leqq 0$.

In the latter case the shape of the framework is determined by the rod lengths, assuming the topology of the structure to be known. One may freely choose 3 or 6 coordinates, respectively; the remaining $2m - 3$ or $3m - 6$ coordinates can be calculated from $n \geqq 2m - 3$ or $n \geqq 3m - 6$ distance relations, provided they do not contain any contradictions.

As these equations are quadratic, they may lead to more than one solution. This means that the framework considered may admit several different configurations. If two (real) configurations do not differ too much from each other, it will be possible, if the rods actually have some elasticity, to force the structure to "snap" from one position to the other one. Think for instance of a very flat tetrahedron ($m = 4$, $n = 6$; $f_3 = 0$): Fixing a base triangle the tetrahedron may snap from the original form to the mirror image with respect to the base plane.

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When, in a limit case, two neighbour positions of a rodwork coincide, we have a "shaky" structure with infinitesimal deformability. As example take a tetrahewron with coplanar vertices: If three of them are fixed, the fourth will still allow a small displacement perpendicular to the fixed plane; although its extent is infinitesimal, it is in practice quite perceptible. — Such exceptional structures, to be avoided for reasons of stability, have been studied by engineers and mathematicians [2], [8], [9], [10], [11], [15], [16], [17]. Their application to problems in geodesy were recently indicated by the author [13].

An unusual phenomenon occurs when the given system of distance relations, in spite of $f \leqq 0$, possesses a continuous set of solutions. In this case we have a finitely moveable linkage, an "overconstrained mechanism". Take for instance two equal triangles $A_1A_2A_3$ and $B_1B_2B_3$ in the plane, connected by three bars A_iB_i of equal length and direction ($m = 6$, $n = 9$); deviating from the formal value $f_2 = 0$ the true degree of freedom is $f_2' = 1$, as the parallel cranks A_iB_i can perform simultaneous rotations about the fixed pivots A_i. — Classical examples of overconstrained spatial mechanism are the "isogram" (hinged skew parallelogram) of G. T. Bennett [1], [4 (p. 193ff.)], [11] ($m = 8$, $n = 20$; $f_3 = -2$, $f_3' = 1$), various closed hinged chains of more than four links [4 (p. 315ff.)], [7] and the hinged octahedra of R. Bricard [4], [10] ($m = 6$, $n = 12$; $f_3 = 0$, $f_3' = 1$). Another deformable rodwork, important in geodesy [14], consists of ten rods A_iB_j connecting four coplanar points A_i with six arbitrary points B_j in space ($m = 10$, $n = 24$; $f_3 = 0$, $f_3' = 2$).

Obviously the (finite or infinitesimal) deformability of a framework seems to be a metric property, as it depends on particular lengths of the rods. However, the shakiness of a structure is in fact a projective property. This means that a shaky structure, transformed by arbitrary linear transformations, will always produce (at least) shaky structures again. Existing finite moveability, however, in general will get lost under such transformations and reduced to shakiness only. — The astonishing fact of the projective invariance of shakiness was firstly proved by H. Liebmann [9], but on the assumption that there exists at least one triangle in the framework. A new, somewhat more natural proof without restriction will be presented here. The systematic use of velocity diagrams ("hodographs"), appealing to the mind of the technician, will allow a quantitative judgment of the infinitesimal deformations.

2. Hodographs

Let the position of a framework knot X with respect to a convenient fixed system be described by non-homogeneous Cartesian coordinates x_i ($i = 1, 2$ in the plane, $i = 1, 2, 3$ in space). Assembling them in a column matrix $\boldsymbol{x}$, the length s of a rod XY is determined by

$$s^2 = \sum (x_i - y_i)^2 = (\boldsymbol{x} - \boldsymbol{y})^T (\boldsymbol{x} - \boldsymbol{y}), \tag{2.1}$$

where the symbol T denotes transposition.

In the course of a displacement of the knots their coordinates are to be considered as functions of a time variable t. The actual velocity vector of the knot X is given by the derivative $\dot{\boldsymbol{x}} = d\boldsymbol{x}/dt$ which defines a point $\dot{X}$. The set of all points

$\dot{X}, \dot{Y}, \ldots$ constitutes a velocity diagram, shortly called the "hodograph". If the distance $s = XY$ is constant or at least stationary, we have $\dot{s} = 0$ or, corresponding to (2.1):

$$(\boldsymbol{x} - \boldsymbol{y})^T (\dot{\boldsymbol{x}} - \dot{\boldsymbol{y}}) = 0. \tag{2.2}$$

Hence the line segment $\dot{X}\dot{Y}$ of the hodograph is orthogonal to the rod axis XY. In other words (Fig. 1): The velocity vectors of the ends of a rod have equal components in the rod axis.

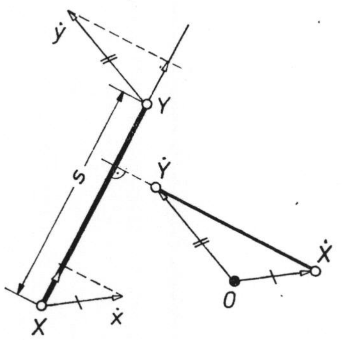

Fig. 1. Velocity diagram of a rigid rod

If a spatial rodwork is rigid, then, according to a fundamental theorem of kinematics, its displacement as a whole at every instant may be considered as a helicoidal motion. As such an instantaneous screw motion is composed of an instantaneous rotation about a well defined axis and an instantaneous translation along that axis, the hodograph will be contained in a plane perpendicular to the instantaneous axis; moreover the shape of the hodograph will be directly similar to the orthogonal projection of the framework onto the hodograph plane.

On the other hand, if there exists a three-dimensional hodograph fulfilling all orthogonality conditions (2.2), or a plane hodograph that is not similar to the orthogonal projection of the framework, this indicates (at least infinitesimal) deformability. Infinitesimal displacements of the knots $X, Y, \ldots$, in direction and extent corresponding to the position vectors of the hodograph points $\dot{X}, \dot{Y}, \ldots$, will cause variations of the rod lengths XY etc. of higher order only.

For a plane structure the condition for infinitesimal deformability in its plane is the existence of a plane hodograph which satisfies all orthogonality relations $XY \perp \dot{X}\dot{Y}$, without being similar to the framework.

Let us consider, as an example, the rodwork of Fig. 2: It consists of 16 bars A_iB_j connecting the vertices of two rectangles $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$ with common medians ($m = 8$, $n = 16$; $f_2 = -3$). As indicated by O. Bottema [3], [12], this plane structure is a moveable overconstrained linkage ($f_3{}' = 1$). Denoting the knot positions by

$$\begin{aligned} &A_1(u_1, u_2),\ A_2(-u_1, u_2),\ A_3(-u_1, -u_2),\ A_4(u_1, -u_2); \\ &B_1(v_1, v_2),\ B_2(-v_1, v_2),\ B_3(-v_1, -v_2),\ B_4(v_1, -v_2) \end{aligned} \tag{2.3}$$

and the rod lengths by

$$A_iB_i = a, \quad A_iB_{i+1} = b, \quad A_iB_{i+2} = c, \quad A_iB_{i+3} = d \tag{2.4}$$

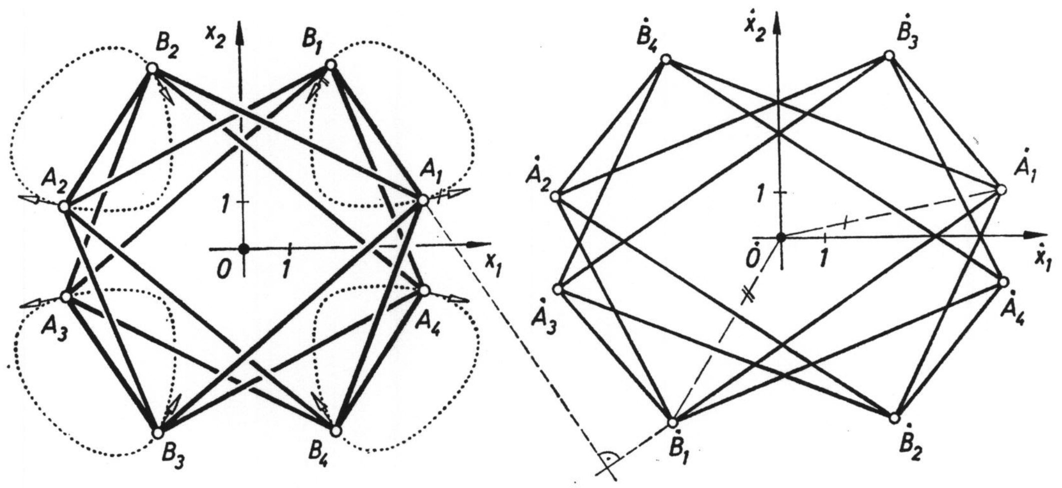

Fig. 2. Bottema's moveable 16-bar linkage with 8 quadruple knots

(indices modulo 4), it can be shown that $a^2 + c^2 = b^2 + d^2$ and

$$\begin{aligned} &8u_1^2u_2^2(2u_1^2 + 2u_2^2 - a^2 - c^2) + (c^2 - b^2)^2 u_1^2 + (b^2 - a^2)^2 u_2^2 \\ &= 8v_1^2v_2^2(2v_1^2 + 2v_2^2 - a^2 - c^2) + (c^2 - b^2)^2 v_1^2 + (b^2 - a^2)^2 v_2^2 = 0. \end{aligned} \tag{2.5}$$

This means that, fixing the coordinate axes, all eight knots will describe paths which belong to a single sextic consisting of four equal ovals (Fig. 2). — In any configuration of the linkage, the hodograph defined by the points $\dot A_1(\dot u_1, \dot u_2)$, $\dot B_1(\dot v_1, \dot v_2)$ etc. will show the same structure as the rodwork. The governing equations (2.2) prove to be equivalent to the linear system

$$\begin{aligned} v_1\dot u_1 \qquad\qquad + u_1\dot v_1 \qquad\qquad &= 0, \\ v_2\dot u_2 \qquad\qquad + u_2\dot v_2 &= 0, \\ u_1\dot u_1 + u_2\dot u_2 + v_1\dot v_1 + v_2\dot v_2 &= 0 \end{aligned} \tag{2.6}$$

which leads to the solution

$$\dot u_1 : \dot u_2 : \dot v_1 : \dot v_2 = u_1(u_2^2 - v_2^2) : u_2(v_1^2 - u_1^2) : v_1(v_2^2 - u_2^2) : v_2(u_1^2 - v_1^2). \tag{2.7}$$

Fig. 2 is based upon the values $u_1 = 4$, $u_2 = 1$, $v_1 = 2$, $v_2 = 4$ ($a^2 = 13$, $b^2 = 45$, $c^2 = 61$, $d^2 = 29$); the hodograph on the right is constructed with $\dot u_1 = 5$, $\dot u_2 = 1$, $\dot v_1 = -5/2$, $\dot v_2 = -4$, whereas on the left the velocity vectors, indicating the infinitesimal displacement of the knots, are represented to a reduced scale.

It may be remarked that the orthogonality relationship between a shaky framework and its deformation hodograph is of involutory character. Consequently the hodograph, if not degenerate, represents a second shaky structure of the same kind, and the first framework serves as the corresponding hodograph. Figures 2, 3 und 5 illustrate the situation for plane structures.

3. Affine Invariance of Shakiness

The study of the effect induced by an affine transformation applied to a framework is not restricted by assuming the origin O to be fixed, as a translocation $O \to O'$ may be compensated by the translation $O' \to O$. Thus it is admissible to set up the transformation $X \to X'$ by

$$x_i' = \sum_k a_{ik} x_k \quad \text{or} \quad \boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}, \tag{3.1}$$

where $\boldsymbol{A} = (a_{ik})$ denotes a non-singular (2×2)-matrix in the plane case and a (3×3)-matrix in the spatial case.

Now it is easy to find a corresponding affine transformation

$$\dot{\boldsymbol{x}}' = \boldsymbol{B}\dot{\boldsymbol{x}} \tag{3.2}$$

of the hodograph, in such a manner that the orthogonality relationships (2.2) are conserved. Because

$$(\boldsymbol{x}' - \boldsymbol{y}')^T (\dot{\boldsymbol{x}}' - \dot{\boldsymbol{y}}') = (\boldsymbol{x} - \boldsymbol{y})^T \boldsymbol{A}^T\boldsymbol{B}(\dot{\boldsymbol{x}} - \dot{\boldsymbol{y}}), \tag{3.3}$$

it is sufficient to take simply

$$\boldsymbol{B} = (\boldsymbol{A}^T)^{-1} = (\boldsymbol{A}^{-1})^T. \tag{3.4}$$

Consequently the shakiness of a structure is invariant under arbitrary affine transformations. — To complete the proof it should be pointed out that a shaky structure cannot become rigid after transformation. Now, if the deformation hodograph of a spatial rodwork is three-dimensional (as in general), the transformed hodograph will not be plane, as $\boldsymbol{B} = (\boldsymbol{A}^{-1})^T$ is regular, and therefore the new structure certainly is not rigid. If, however, the first hodograph was already plane (which is possible also in the spatial case, see [17]), this argument fails.

Hence, especially in the plane case, another chain of reasoning is necessary, showing for instance that a rigid structure always stays rigid. Without restriction we may assume that one knot is fixed at the origin and that the axis of the instantaneous displacement (which now is a rotational one) coincides with the x_3-axis of the coordinate system; the hodograph is then contained in the plane $x_3 = 0$ and defined by

$$\dot{\boldsymbol{x}} = \lambda \boldsymbol{D}\boldsymbol{x} \quad \text{with} \quad \boldsymbol{D} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; \lambda \neq 0. \tag{3.5}$$

Furthermore we modify the applied affine transformation by adding a suitable similarity transformation in such a way that each point of the x_3-axis is fixed; thus we have then the matrices

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{B} = \begin{pmatrix} a_{22} & -a_{21} & 0 \\ -a_{12} & a_{11} & 0 \\ 0 & 0 & b_{33} \end{pmatrix}, \tag{3.6}$$

where $b_{33} = a_{11}a_{22} - a_{12}a_{21} \neq 0$. Hence it follows after a little algebra:

$$\dot{\boldsymbol{x}}' = \lambda \boldsymbol{B}\boldsymbol{D}\boldsymbol{x} = \lambda \boldsymbol{B}\boldsymbol{D}\boldsymbol{A}^{-1}\boldsymbol{x}' = \lambda \boldsymbol{B}\boldsymbol{D}\boldsymbol{B}^T\boldsymbol{x}' = \lambda b_{33}\boldsymbol{D}\boldsymbol{x}'. \tag{3.7}$$

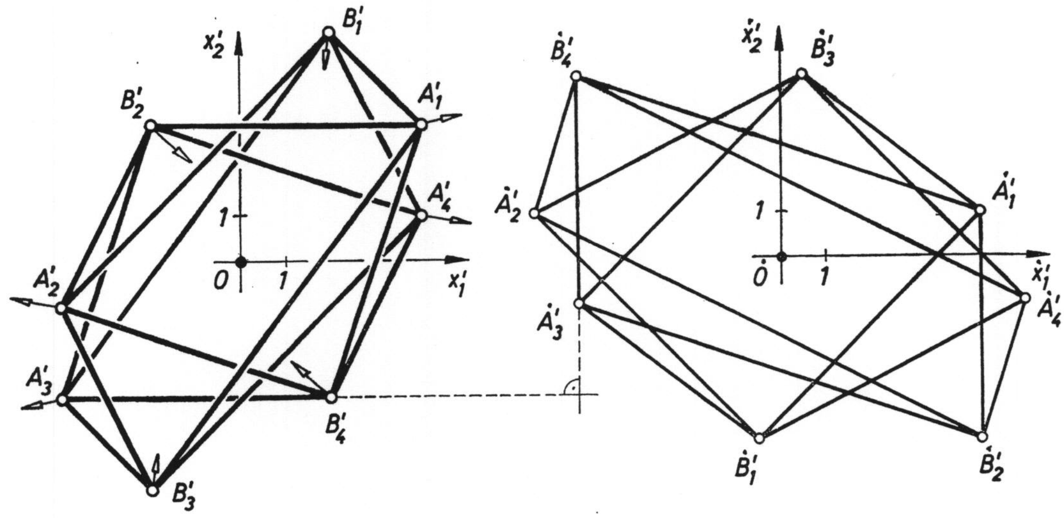

Fig. 3. Shaky 16-bar linkage (Parallel projection of the structure in Fig. 2)

This means that the new hodograph, situated again in the plane $x_3 = 0$, is similar to the orthogonal projection of the new rodwork onto that plane. Consequently the displacement is only an infinitesimal rotation (about the x_3-axis) and the new structure is indeed rigid.

To illustrate the formulas let us apply the affine transformation

$$x_1' = x_1, \qquad x_2' = \frac{1}{2}x_1 + x_2, \qquad A = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{pmatrix} \tag{3.8}$$

to Bottema's linkage in its position of Fig. 2, where the knots have the coordinates $A_1(4, 1)$, $B_1(2, 4)$ etc. The transformed rodwork is defined by the points $A_1'(4, 3)$, $B_1'(2, 5)$, $A_2'(-4, -1)$, $B_2'(-2, 3)$ and their mirror images A_3', B_3', A_4', B_4' with respect to the origin O (Fig. 3). The original hodograph with the vertices $\dot{A}_1(5, 1)$, $\dot{B}_1(-11/2, -4)$ etc. is then transformed by the matrix $B = (A^{-1})^T$, i.e.

$$B = \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{pmatrix}: \qquad \dot{x}_1' = \dot{x}_1 - \frac{1}{2}\dot{x}_2, \qquad \dot{x}_2' = \dot{x}_2. \tag{3.9}$$

Hence the vertices of the new hodograph are $\dot{A}_1'(9/2, 1)$, $\dot{B}_1'(-1/2, -4)$, $\dot{A}_2'(-11/2, 1)$, $\dot{B}_2'(9/2, -4)$ and their mirror images with respect to the origin $\dot{O}$; their position vectors determine the velocity vectors of the new framework knots. In contrast to the original linkage the derived framework is only infinitesimally deformable.

4. Projective Invariance of Shakiness

As the proof of the invariance of shakiness under projective collineations is more difficult, the present paper will develop the proof in extenso for plane structures only. First of all it is appropriate to make use of homogeneous coordinates; a proper point X is then determined by the ratios of three numbers $x_0 : x_1 : x_2$ with $x_0 \neq 0$, where x_1/x_0 and x_2/x_0 give the ordinary (non-homogeneous) Cartesian coordinates. A triplet $x_0 : x_1 : x_2$ with $x_0 = 0$ defines the point at infinity with the direction vector (x_1, x_2).

A collineation (homography) $X \to X'$ is now described by

$$x_i' = \sum_{k=0}^{2} a_{ik} x_k \qquad (i = 0, 1, 2). \tag{4.1}$$

To avoid degenerate mappings, let $\det(a_{ik}) = A \neq 0$. It is fruitless to search for a corresponding projective mapping $\dot{X} \to \dot{X}'$ of the hodograph, hoping that it might preserve the orthogonality relationships $XY \perp \dot{X}\dot{Y}$ which now read

$$\sum_{i=1} (x_0 y_i - y_0 x_i)(\dot{x}_0 \dot{y}_i - \dot{y}_0 \dot{x}_i) = 0. \tag{4.2}$$

Nevertheless, it will be possible to reach the aim by means of another transformation $\dot{X} \to \dot{X}'$, but this one will also depend on X.

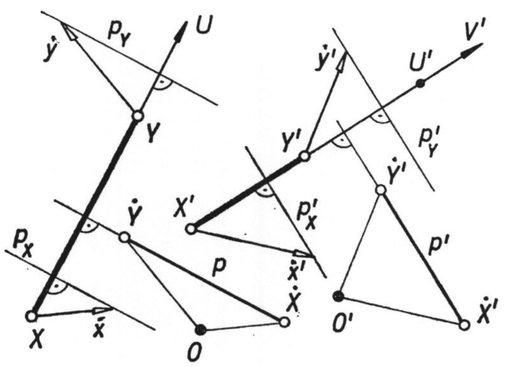

Fig. 4

For this purpose consider a framework rod issuing from the knot $X(x_i)$; its direction is indicated by an improper point $U(u_i)$ with $u_0 = 0$ (Fig. 4). The corresponding rod in the transformed framework contains the knot $X'(x_i')$ and the vanishing point $U'(u_i')$ defined by

$$u_i' = \sum_{k=0}^{2} a_{ik} u_k \qquad (i = 0, 1, 2). \tag{4.3}$$

Its point at infinity, $V'(v_i')$ with $v_0' = 0$, is obtainable by a linear combination $v_i' = \lambda x_i' - \mu u_i'$ with $\lambda : \mu = u_0' : x_0' = \sum a_{0k} u_k : \sum a_{0j} x_j$; hence we have (for $i = 1, 2$):

$$v_i' = u_0' x_i' - x_0' u_i' = \sum_{k=1}^{2} \alpha_{ik} u_k \quad \text{with} \quad \alpha_{ik} = \sum_{j=0}^{2} (a_{0k} a_{ij} - a_{0j} a_{ik})\, x_j. \tag{4.4}$$

It may be remarked that the linear transformation formulas (4.4) describe the projectivity relating the corresponding ray pencils X and X' as induced by the collineation (4.1).

Now let us consider all possible velocity vectors fastened at the knot X and having the same component along the rod axis XU. Their end points lie on a perpendicular $p_X \perp XU$ (Fig. 4), and so the corresponding hodograph points $\dot{X}$

lie on a line $p \parallel p_X$ representable by

$$\sum_{k=1}^{2} u_k \dot{x}_k = c \cdot \dot{x}_0 \quad \text{with} \quad c = \text{const}. \tag{4.5}$$

Due to the rigidity assumption, the same perpendicular p relates to any other point Y of XU. In order to have equal properties for the transformed framework and its hodograph, we require that the corresponding rod axis $X'U' = X'V'$ be related to a perpendicular $p' \perp X'V'$ (Fig. 4), described by

$$\sum_{i=1}^{2} v_i' \dot{x}_i' = \sum_{i,k=1}^{2} \alpha_{ik} u_k \dot{x}_i' = c' \cdot \dot{x}_0'. \tag{4.6}$$

As this property must hold for all rods issuing from any knot X, or in other words for arbitrary directions $u_1 : u_2$, comparison of (4.5) with (4.6) leads to the conditions

$$\varrho \dot{x}_k = \sum_{i=1}^{2} \alpha_{ik} x_i' \quad \text{or} \quad \varrho' \dot{x}_i' = \sum_{k=1}^{2} \beta_{ik} \dot{x}_k \qquad (i, k = 1, 2), \tag{4.7}$$

where β_{ik} denotes the cofactor of α_{ik} in the (2×2)-matrix (α_{ik}). Evaluating the definitions (4.4) we get

$$\begin{aligned} \beta_{11} &= \alpha_{22} = A_{10}x_1 - A_{11}x_0, & \beta_{12} &= -\alpha_{21} = A_{10}x_2 - A_{12}x_0, \\ \beta_{21} &= -\alpha_{12} = A_{20}x_1 - A_{21}x_0, & \beta_{22} &= \alpha_{11} = A_{20}x_2 - A_{22}x_0, \end{aligned} \tag{4.8}$$

where A_{ik} denotes the cofactor of a_{ik} in the (3×3)-matrix (a_{ik}).

Introducing the expressions (4.7) of $\dot{x}_i'$ into Eqs. (4.6), we have

$$\varrho' c' \dot{x}_0' = \sum_{1}^{2} \alpha_{ik} u_k \beta_{il} \dot{x}_l = \Delta \cdot \sum_{1}^{2} \delta_{kl} u_k \dot{x}_l = \Delta \cdot \sum_{1}^{2} u_k \dot{x}_k = \Delta c \dot{x}_0, \tag{4.9}$$

where δ_{kl} is the Kronecker symbol and $\Delta = \det(\alpha_{ik})$. This determinant has the value

$$\Delta = \sum_{k=1}^{2} \alpha_{1k}\beta_{1k} = \sum_{j,k=0}^{2} (a_{0k}a_{1j} - a_{0j}a_{1k})(A_{10}x_k - A_{1k}x_0)\, x_j = \sum_{j=0}^{2} b_j x_j \tag{4.10}$$

with

$$\begin{aligned} b_j &= A_{10}\left(a_{1j} \sum_{0}^{2} a_{0k}x_k - a_{0j} \sum_{0}^{2} a_{1k}x_k\right) - x_0\left(a_{1j} \sum_{0}^{2} a_{0k}A_{1k} - a_{0j} \sum_{0}^{2} a_{1k}A_{1k}\right) \\ &= A_{10}(a_{1j}x_0' - a_{0j}x_1') + A a_{0j} x_0 \quad \text{and} \quad A = \det(a_{ik}) \neq 0. \end{aligned} \tag{4.11}$$

This leads to

$$\Delta = A_{10}(x_0'x_1' - x_1'x_0') + A x_0 x_0' = A x_0 x_0', \tag{4.12}$$

and over (4.9) to

$$\varrho' c' \dot{x}_0' = A c x_0 x_0' \dot{x}_0. \tag{4.13}$$

As we are operating with homogeneous coordinates, it is admissible to put $x_0 = \dot{x}_0 = 1$ and $\varrho' = -1$. Moreover choosing $\dot{x}_0' = x_0'$ we state that the quantity c' in (4.6) is really a constant having the value $c' = -Ac$. With all these appointments the required hodograph transformation $\dot{X} \to \dot{X}'$ reads, according to (4.1) and (4.7):

$$\begin{aligned} \dot{x}_0' &= x_0' = \sum_{k=0}^{2} a_{0k} x_k \qquad (x_0 = 1), \\ \dot{x}_i' &= \sum_{k=1}^{2} (A_{ik} x_0 - A_{i0} x_k)\, \dot{x}_k \qquad (i = 1, 2;\ \dot{x}_0 = 1). \end{aligned} \tag{4.14}$$

Its existence secures the projective invariance of shakiness for plane rodworks of the kind considered.

For spatial frameworks the proof runs in a similar way. A certain difficulty arises only at step (4.8), as the evaluation of the cofactors β_{ik} in the (3×3)-matrix (α_{ik}) is rather laborious. Nevertheless, there exists again a suitable hodograph transformation $\dot{X} \to \dot{X}'$; it is described by equations of the form (4.14), but with indices i and k going up to 3 [18]. — These equations are identical with formulas derived by N. W. Efimow [6, p. 61], but concerning infinitesimal isometric deformations of smooth surfaces. When such a surface is successively approximated by frameworks with triangular faces (see Section 5), it seems evident that our formulas will hold also in the limit case. Conversely, however, Efimow's deductions, based upon the displacements of rigid surface elements, are not transferable to rodworks.

In order to illustrate the formulas (4.1) and (4.14) by a numerical example, let us apply the collineation

$$\begin{aligned} x_0' &= 2x_0 - x_2, \\ x_1' &= x_1, \\ x_2' &= 2x_2, \end{aligned} \qquad (a_{ik}) = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \qquad (A_{ik}) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 1 & 0 & 2 \end{pmatrix} \tag{4.15}$$

to Bottema's linkage in the position of Fig. 2. The original knots $A_1(1:4:1)$, $B_1(1:2:4)$ etc. are then replaced by $A_1'(1:4:2)$, $B_1'(1:-1:-4)$, $A_3'(1:-4/3 : -2/3)$, $B_3'(1:-1/3:-2/3)$ and their mirror images A_2', B_2', A_4', B_4' with respect to the axis $x_1' = 0$ (Fig. 5). The hodograph transformation (4.14) reads

$$\dot{x}_0' = 2x_0 - x_2, \qquad \dot{x}_1' = 4\dot{x}_1, \qquad \dot{x}_2' = 2\dot{x}_2 - x_1\dot{x}_1 - x_2\dot{x}_2. \tag{4.16}$$

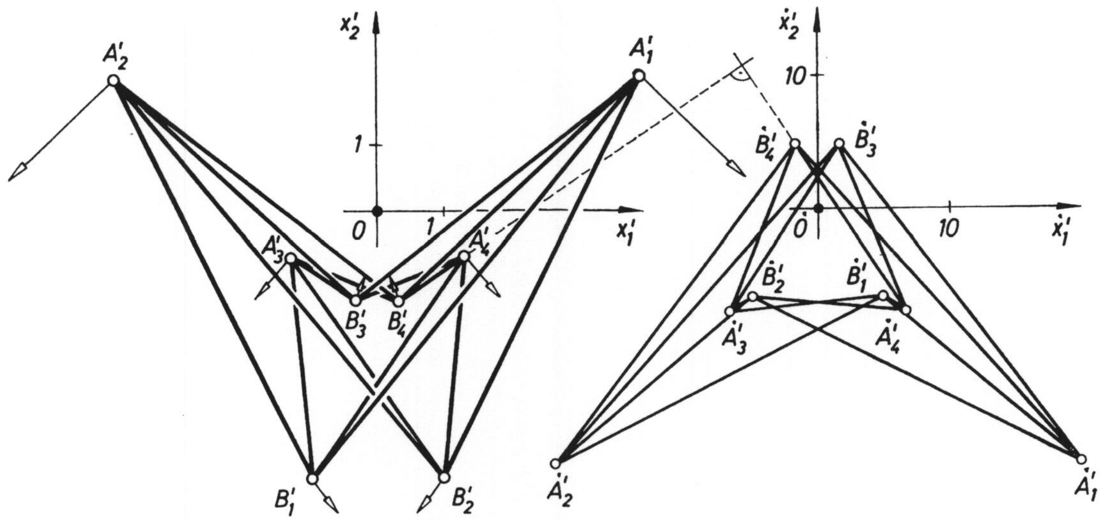

Fig. 5. Shaky 16-bar linkage (Central projection of the structure in Fig. 2)

The new hodograph points, derived from $\dot{A}_1(1:5:1)$, $\dot{B}_1(1:-5/2:-4)$ etc., are now $\dot{A}_1'(1:20:-19)$, $\dot{B}_1'(1:5:-13/2)$, $\dot{A}_3'(1:-20/3:-23/3)$, $\dot{B}_3'(1:5/3:29/6)$ together with the mirror images with respect to the axis $\dot{x}_1' = 0$. The two part diagrams of Fig. 5 are plotted with differnt scales.

5. Polyhedra and Closed Chains

The foregoing results, stating the invariance of the shakiness of rodworks under affine and projective transformations, may easily be extended to other structures, as polyhedra and closed chains.

A polyhedron, consisting of rigid plates which are connected by hinges along common edges, is rigid if it is convex (A. Cauchy, 1813). Non-convex polyhedra however may snap [8], [10], [15], [16], be shaky [2], [8], [10], [15], [17] or even be finitely moveable, e.g. the famous open octahedra of R. Bricard [4], [10] or the recently discovered closed polyhedra of R. Connelly [5] and K. Steffen.

If all faces of a polyhedron are triangles, an equivalent framework model may be constructed by materializing the edges as rods connected by spherical joints at the vertices. If there are faces with $r > 3$ vertices, each one can be replaced by a (sufficiently flat) r-sided pyramid, whose base has to be stiffened by $r - 3$ additional diagonal rods. Thus in any case we obtain an equivalent framework appropriate for the application of our results on rodworks. — For instance shaky octahedra as considered by W. Blaschke [2] and the author [10] are projectively characterizable structures. Similarly the shaky antiprisms, dipyramids, dodecahedra and icosahedra as indicated by M. Goldberg [8] and the author [15], [17], mostly by metrically particular examples, will be transformed by arbitrary affine or projective collineations into more general shaky structures of the same topological kind.

Similar developments are possible for closed hinged chains of rigid spatial systems, as each link may be represented by a tetrahedron with opposite edges on the two hinge axes belonging to the link [7]. Slight modifications are necessary, if the axes are not skew (as in general) or if there are links connected by means of spherical joints. In every case the chain can be replaced by an equivalent framework. Hence a shaky or moveable chain, subjected to an arbitrary affine or projective transformation, always will lead again to an at least shaky chain. — A closed chain of four links, for instance, in general is rigid ($m = 8, n = 20; f_3 = -2$). It is shaky, if the four hinge axes are skew generators of a quadric [4, p. 315]; it is continuously moveable in the case of Bennett's isogram linkage [1], [7], apart from the trivial cases of parallel or concurrent axes (plane or spherical four-bar linkages).

After all we have the final statement: If a jointed rodwork or any other equivalent structure admits an infinitesimal deformation, this property is a projective one, as an arbitrary affine or projective transformation in turn generates an analogous structure. Corresponding velocity diagrams for the new structures may be derived from the original hodographs by means of the formulas (3.2) and (4.14), respectively.

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