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RULED SURFACES WITH OSCULATING STRICTION SCROLL

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1. INTRODUCTION

On each regular generator e of a skew (nondevelopable) *ruled surface* or *"scroll"* $\Phi \subset E_3$ there is a *central point* Z, the contact point of that plane $\zeta \supset e$ which is orthogonal to the asymptotic tangent plane (touching Φ at the point at infinity of e) [5]. The tangent line $f \subset \zeta$ forming a right angle with e at Z may shortly be called the *central tangent*; it is the limit position of the common perpendicular of e and a neighbour generator e_1, when e_1 tends towards e. The locus k of all central points Z is the *striction line* of Φ. The ruled surface Φ^* consisting of all central tangents f has the same striction line k and is called the *striction scroll* of Φ (German: Striktionsband). Conversely Φ is the striction scroll of Φ^* [1], [2], [3].

Whereas in general Φ and Φ^* have a simple contact along the common striction line k, the present note deals with pairs of surfaces Φ, Φ^* *osculating* each other along k. Due to a theorem of A. Mannheim, examples for surfaces of this kind are the scrolls Φ formed by the principal normals of arbitrary curves of constant total (Lancret) curvature [7].

The *central developable* Γ, i.e., the envelope of the *central planes* ζ of Φ (and Φ^*), is circumscribed to Φ (and Φ^*) along the striction line k. Therefore the tangent t of k at a point Z and the generator t' of Γ passing through Z are conjugate tangents of Φ (and Φ^*). In the case of osculating surfaces Φ, Φ^* the generators e and f indicate the asymptotic directions at the central point Z, and thus the harmonic arrangement of e, f, t, t' means that e and f are the *bisectors* of the angle tt'.

2. DEVELOPMENT OF THE CENTRAL DEVELOPABLE

A well-known theorem of G. Darboux states that in the course of the plane development of the central developable Γ of a ruled surface Φ the line elements (Z, e) become *parallel* [1], [2], [3]. Due to the conservation of angles the same is valid for the elements (Z, f). Conversely, having a plane curve $\bar{k}$ and in each point $\bar{Z} \in \bar{k}$ a line element $(\bar{Z}, \bar{e})$ of a fixed direction, then by arbitrary bending of the plane $\bar{\Gamma} \supset \bar{k}$ into a developable surface Γ the parallel elements $(\bar{Z}, \bar{e})$ are transformed into generator elements (Z, e) of a ruled surface $\Phi(e)$, whose striction line k corresponds to the given curve $\bar{k}$.

In the present case we have in the plane $\bar{\Gamma}$ the situation of Figure 1:

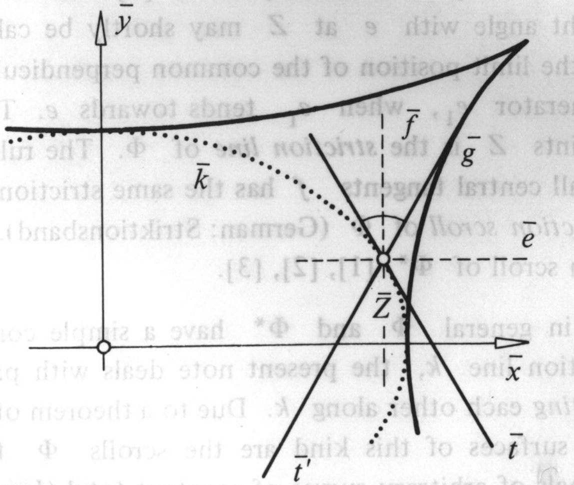

Figure 1

Four line elements are connected with each point $\bar{Z} \in \bar{k}$; two elements $(\bar{Z}, \bar{e})$ and $(\bar{Z}, \bar{f})$ in fixed orthogonal directions, and two elements $(\bar{Z}, \bar{t})$ and $(\bar{Z}, \bar{t}')$ making opposite equal angles with $\bar{e}$ (hence also with $\bar{f}$). The line $\bar{t}$ is the tangent of $\bar{k}$, and $\bar{t}'$ corresponds to the generator t' of the central developable Γ; thus the envelope $\bar{g}$ of the lines $\bar{t}'$ originates from the edge of regression g of Γ. – Considering the points at infinity of the directions $\bar{e}$ and $\bar{f}$ as the absolute points of a *pseudo-euclidean metric* (with arc element $d\bar{s}^2 = d\bar{x} \cdot d\bar{y}$, say), the lines $\bar{t}$ and $\bar{t}'$ are pseudo-orthogonal, and $\bar{g}$ may be considered as the pseudo-evolute of $\bar{k}$.

Bending now the plane $\bar{\Gamma} \supset \bar{k}$ in arbitrary way along the lines $\bar{t}'$ (preserving the straightness of these lines), we obtain a developable surface Γ. The parallel elements $(\bar{Z}, \bar{e})$ become the generator elements (Z, e) of a ruled surface $\Phi(e)$, and the elements $(\bar{Z}, \bar{f})$ the generator elements (Z, f) of the striction scroll $\Phi^*(f)$ *osculating* Φ along the common striction line k corresponding to $\bar{k}$ [7].

3. SPECIAL CASES

If the central developable Γ is an arbitrary *cone* with vertex S, the lines $\bar{t}'$ form a pencil $\bar{S} (= \bar{g})$. Consequently the plane curve $\bar{k}$ must be a pseudo-circle, i.e., an *equilateral hyperbola* with asymptotes in the directions of $\bar{e}$ and $\bar{f}$ [7].

Assuming Γ for instance as a *cone of revolution*, one arrives at explicitely describable specimens of surfaces Φ with osculating striction scroll. It can be shown that they are algebraic, if the angle of aperture 2ω of Γ is determined by a rational value of $n = \sin \omega$ [8]. The simplest example is furnished by $n = \frac{1}{2}$ $\left(\omega = \frac{\pi}{6}\right)$: Φ is then a surface of degree 7 and congruent to its striction scroll Φ^*; the striction line k is a quartic.

If the central developable Γ is an arbitrary *cylinder*, the lines $\bar{t}'$ are parallel. Therefore $\bar{k}$ is a *straight line*, pseudo-orthogonal to the direction of $\bar{t}'$. Consequently, the striction line k is a *geodesic* of the cylinder Γ, cutting its generators t' under a constant angle, and the scrolls Φ and Φ^* have rotational director cones.

Taking for Γ a *cylinder of revolution*, one obtains in general *helicoids* Φ, Φ^* osculating each other along a circular helix k. – Algebraic surfaces Φ are obtained if the central cylinder has a closed *epicycloid* or *hypocycloid* as its base. The simplest example is of degree 4. Other remarkable specimens among this family are the surfaces consisting of the principal normals of a curve of constant total curvature, where all these normals have the same inclination with respect to a fixed plane [8].

4. FINITE MODELS

In order to get finite demonstration models, we replace the plane curve $\bar{k}$ by a *polygon* $\bar{Z}_0\bar{Z}_1\bar{Z}_2 \ldots$ on a sheet of stiff paper (Figure 2).

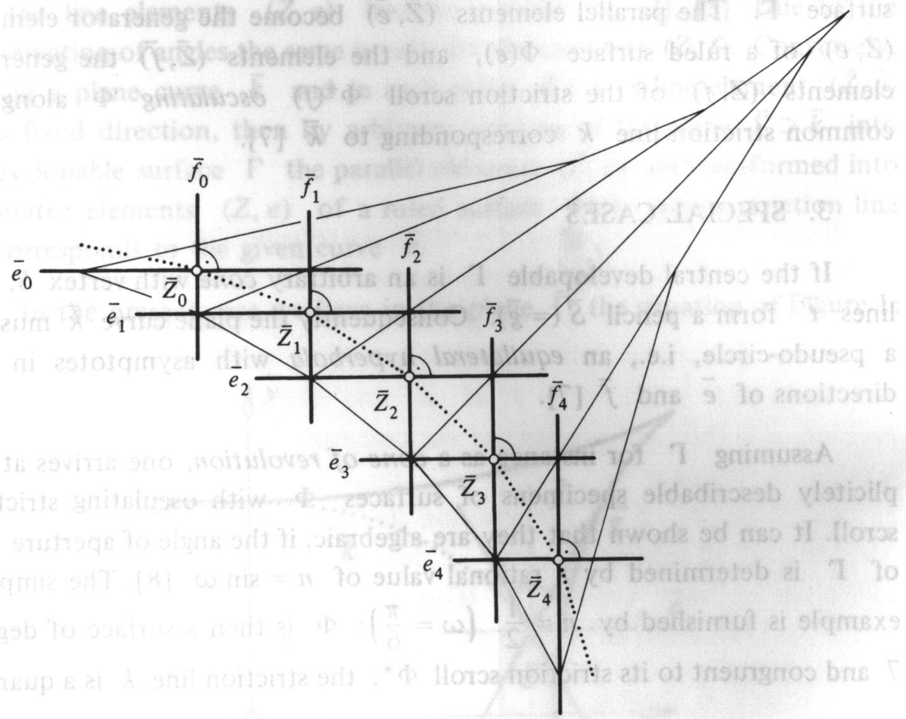

Figure 2

When parallel lines $\bar{e}_i$ and $\bar{f}_i \perp \bar{e}_i$ are drawn through the points $\bar{Z}_i$, the polygon sides $\bar{Z}_i\bar{Z}_{i+1}$ appear as diagonals of a chain of rectangles. Folding then the sheet along the other diagonals $\bar{t}'$, we obtain a polyhedron whose

edges t_i' are the prolonged sides of a spatial polygon $Z_0Z_1Z_2 \ldots$. The straight lines e_i and f_i corresponding to $\bar{e}_i$ and $\bar{f}_i$ may be materialized by thin rods or plastic stalks fastened to appropriate faces of the polyhedron. They constitute a deformable spatial configuration consisting of two systems: in any position each rod e_i of the first system meets three rods f_{i-1}, f_i, f_{i+1} of the second system and vice versa, e_i and f_i always forming a right angle. The triple incidence indicates the osculation of the ruled surfaces Φ and Φ^* represented by the discrete line sequences $\{e_i\}$ and $\{f_i\}$.

In the special cases of Section 3 the generating plane polygon is inscribed to an equilateral hyperbola or consists of collinear points. The resulting polyhedra will then become pyramids or prisms, respectively.

5. ANALYTIC TREATMENT

The appropriate analytic concept for the intrinsic ("natural") geometry of ruled surfaces, as developed by E. K r u p p a [1], [2] but introduced already by G. S a n n i a [5], connects with each generator e of a scroll Φ an *accompanying frame* consisting of three mutually orthogonal unit vectors, fastened at the central point Z: the vector e_1 of the generator e, the surface normal e_2, and the vector $e_3 = e_1 \times e_2$ indicating the central tangent f. Using the arc length s of the striction line k as natural parameter, there exist *derivation formulas* of Frenet type:

$$(1) \qquad e_1' = \kappa e_2, \quad e_2' = -\kappa e_1 + \tau e_3, \quad e_3' = -\tau e_2.$$

The coefficients $\kappa = |e_1'|$ and $\tau = e_2' e_3$ are called the *natural curvature* and the *natural torsion* of Φ; they are invariant under isometric transformations. This is valid also for the *total* or *Lancret curvature* $\lambda = |e_2'| = (\kappa^2 + \tau^2)^{\frac{1}{2}}$. An additional invariant is the *striction angle* $\sigma = \sphericalangle te$ measuring the deviation of the generator e from the striction line k: $z = z(s)$; it is determined by

$$(2) \qquad z' = e_1 \cos\sigma + e_3 \sin\sigma$$

and may be restricted to the interval $-\frac{\pi}{2} < \sigma \leqslant \frac{\pi}{2}$.

A fundamental theorem guarantees that the shape of a ruled surface Φ is uniquely determined – apart from its position in space – by the "natural equations" $\kappa = \kappa(s)$, $\tau = \tau(s)$, $\sigma = \sigma(s)$. By means of the solutions of the differential system (1) – existing for continuous functions $\kappa(s)$, $\tau(s)$, $\sigma(s)$ – the surface Φ is described by

$$(3) \qquad \boldsymbol{x} = \boldsymbol{z}(s) + r\boldsymbol{e}_1(s) \quad \text{with} \quad \boldsymbol{z} = \int (\boldsymbol{e}_1 \cos\sigma + \boldsymbol{e}_3 \sin\sigma)\, ds.$$

In similar form the striction scroll Φ^* is represented by

$$(4) \qquad \boldsymbol{y} = \boldsymbol{z}(s) + r\boldsymbol{e}_3(s).$$

The central developable Γ was introduced in Section 1 as the envelope of the central plane ζ: $(\boldsymbol{x} - \boldsymbol{z}(s))\boldsymbol{e}_2 = 0$. Hence its generator t' has the direction vector

$$(5) \qquad \boldsymbol{d} = \boldsymbol{e}_2 \times \boldsymbol{e}_2' = \tau\boldsymbol{e}_1 + \kappa\boldsymbol{e}_3,$$

known as the *Darboux vector* (which allows to collect the derivation formulas (1) by $\boldsymbol{e}_i' = \boldsymbol{d} \times \boldsymbol{e}_i$).

As pointed out in Section 1, for *osculating pairs* Φ, Φ^* the tangent lines t (2) and t' (5) are symmetric with respect to the generator $e \subset \Phi$ (or $f \subset \Phi^*$). Thus we get the necessary and sufficient *condition*

$$(6) \qquad \kappa\cos\sigma + \tau\sin\sigma = 0$$

for ruled surfaces Φ with osculating striction scroll. Using the Lancret curvature λ we have the relations

$$(7) \qquad \kappa = \pm\lambda\sin\sigma, \qquad \tau = \mp\lambda\cos\sigma.$$

The *distribution parameter* d, measuring the twist of Φ along a generator e, is given by $d = \frac{\sin\sigma}{\kappa}$; analogously the distribution parameter of Φ^* is given by $d^* = \frac{\cos\sigma}{\tau}$ [1], [2]. It follows from (7) that corresponding generators e, f of an osculating surface pair Φ, Φ^* have *opposite equal* distribution parameters:

$$(8) \qquad d = \pm\frac{1}{\lambda}, \quad d^* = \mp\frac{1}{\lambda}.$$

By means of a somewhat intricate analysis, using Taylor series expansions up to terms of the third order, it can be shown that ruled surfaces Φ with *hyperosculating* striction scroll Φ^* are characterized by the conditions (7) with $\lambda = \text{const}$ [8]. The helicoids mentioned in Section 3 are examples for hyperosculating surface pairs Φ, Φ^*, as obviously the distribution parameter $d = -d^*$ is constant.

6. OSCULATING SURFACE PAIRS WITH PRESCRIBED SPHERICAL IMAGE OF THE CENTRAL DEVELOPABLE

When the accompanying frame $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ is transposed to the origin O by the translation $Z \to O$, it performs there a *spherical motion* about the fixed center O. In the course of this motion the end points of the three radius vectors $\boldsymbol{e}_i$ describe well defined paths c_i on the unit sphere which are important for the ruled surface Φ. The parametric representation of c_i is $\boldsymbol{e}_i = \boldsymbol{e}_i(s)$; in particular c_1 is the spherical (generator) image of Φ, c_2 the spherical normal image, and c_3 the spherical image of Φ^*.

According to the elements of spherical kinematics the motion of the frame at any moment is an infinitesimal rotation about an *instantaneous axis* m whose direction is given by the Darboux vector $\boldsymbol{d}$ (5). The corresponding point on the sphere, determined by

$$(9) \qquad \boldsymbol{p} = \frac{\boldsymbol{d}}{\lambda} = \mp\boldsymbol{e}_1\cos\sigma \pm \boldsymbol{e}_3\sin\sigma,$$

is the instantaneous rotation pole P. The locus of P in the moving system, the so-called *moving polhode*, is the great circle q passing through $\boldsymbol{e}_1$ and $\boldsymbol{e}_3$. On the other hand the locus of P in the fixed system, the spherical curve p: $\boldsymbol{p} = \boldsymbol{p}(s)$, represents the *fixed polhode* and at the same time the *spherical image of the central developable* Γ. During the motion of the frame the circle q is rolling on the polhode p, the point of contact being P. Hence the spherical images c_1 and c_3 of Φ and Φ^* may be considered as *spherical involutes* of the curve p; they have the constant spherical distance $\frac{\pi}{2}$ from each other. Similarly the spherical normal image c_2 has the constant distance $\frac{\pi}{2}$ from p (Figure 3).

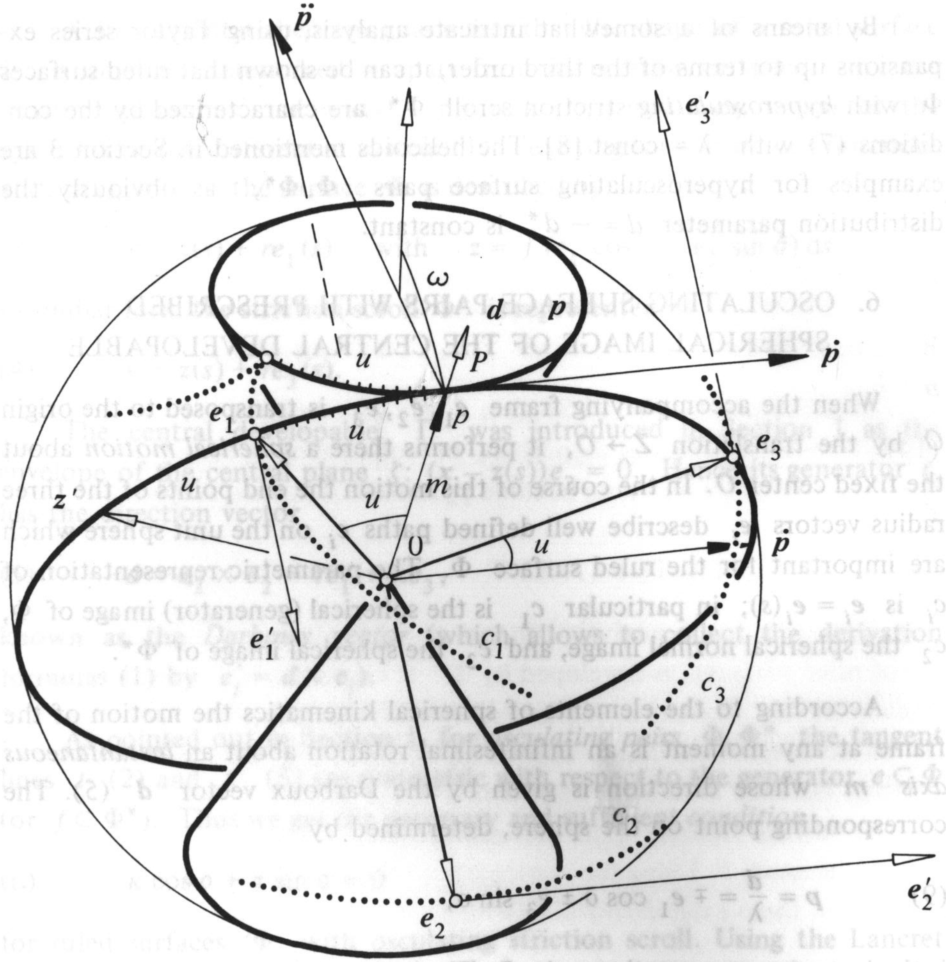

Figure 3

Since all spherical images can be derived from the curve p, it seems appropriate to prescribe this curve. Using its arc length u as parameter and denoting the derivative by a dot, we thus start from

(10) $\quad p = p(u) \quad \text{with} \quad p^2 = \dot{p}^2 = 1.$

Differentiating equation (9) we get, with attention to (1) and (6):

(11) $\quad \dot{p} = \pm\,\dot{\sigma}(e_1 \sin\sigma + e_3 \cos\sigma);$

hence $|\dot{\sigma}| = 1$ or $\sigma = c \pm u$. To fix ideas, let us assume $-\frac{\pi}{2} < \sigma < 0$.

Then without restriction $-\sigma = u$ may serve as arc length of p, the lower signs have to be taken in (6) and (11), and the moving frame is given as follows (viz. Figure 3):

(12) $\quad e_1 = p\cos u - \dot{p}\sin u, \quad e_2 = \dot{p} \times p, \quad e_3 = p \sin u + \dot{p}\cos u.$

For the derivatives we have:

(13) $\quad \dot{e}_1 = -(p + \ddot{p})\sin u, \quad \dot{e}_2 = \ddot{p} \times p, \quad \dot{e}_3 = (p + \ddot{p})\cos u.$

By aid of $p\dot{p} = \dot{p}\ddot{p} = 0$ and $p\ddot{p} = -1$ it may be controlled that indeed $p + \ddot{p}$ is parallel to e_2. Putting now

(14) $\quad p + \ddot{p} = -\mu e_2,$

the derivation formulas (13) read

(15) $\quad \dot{e}_1 = \mu \sin u\, e_2, \quad \dot{e}_2 = -\mu \sin u\, e_1 + \mu \cos u\, e_3,$
$\quad\quad \dot{e}_3 = -\mu\cos u\, e_2.$

From $\dot{e}_2 = \mu\dot{p}$ it follows that the quantity μ means the *conical curvature* of the central developable Γ (and of its director cone Op). It may be calculated by means of

(16) $\quad \mu = \dot{e}_2\dot{p} = (p, \dot{p}, \ddot{p}).$

Comparing now the derivation formulas (1) and (15) we get, because of $\dot{e}_i = \dot{s}e_i'$, the relations

(17) $\quad \dot{s}\kappa = \mu \sin u, \quad \dot{s}\tau = \mu\cos u, \quad \dot{s}\lambda = \mu.$

With respect to (2) the tangent vector of the striction line is expressed by

(18) $\quad \dot{z} = \dot{s}z' = \dfrac{\mu}{\lambda}(e_1 \cos u - e_3 \sin u) = \dfrac{\mu}{\lambda}(p\cos 2u - \dot{p}\sin 2u).$

Integration delivers the *striction line* k:

(19) $\quad z = \int (p\cos 2u - \dot{p}\sin 2u)(p, \dot{p}, \ddot{p})\,\dfrac{du}{\lambda},$

where the Lancret curvature $\lambda(u)$ yet may be arbitrarily chosen. For the parametric representations of the ruled surface Φ and its striction scroll

Φ^* the vectors e_1 in (3) or e_3 in (4) are to be taken from (12).

7. SPECIAL EXAMPLES

When we take $\lambda = \text{const}$, the ruled surface Φ will be of constant distribution parameter $d = \pm \lambda^{-1}$ – a *"constantly twisted scroll"*; because of (8) the (hyperosculating) striction scroll Φ^* will share this property.

On the other hand $\mu = \text{const}$ will yield surfaces Φ with a central developable Γ of *constant conical curvature*. The spherical image p of Γ, being a circle, may be determined by

$$p = \left(n \cos \frac{u}{n}, n \sin \frac{u}{n}, m\right), \tag{20}$$

with constants $m = \cos \omega > 0$, $n = \sin \omega > 0$, where 2ω denotes the angle of aperture of the rotational director cone Op of Γ. The limit case $m = 0$ $(n = 1)$ has to be excluded, as then Γ would degenerate to a plane.

Surfaces Φ possessing both of the mentioned properties are explicitly representable. Their striction line k (19) is described by

$$z_1 = \frac{m}{2\lambda} \left(\frac{1+n}{1-2n} \sin \frac{1-2n}{n} u - \frac{1-n}{1+2n} \sin \frac{1+2n}{n} u\right),$$

$$z_2 = -\frac{m}{2\lambda} \left(\frac{1+n}{1-2n} \cos \frac{1-2n}{n} u - \frac{1-n}{1+2n} \cos \frac{1+2n}{n} u\right), \tag{21}$$

$$z_3 = \frac{m^2}{2n\lambda} \sin 2u,$$

provided $n \neq \frac{1}{2}$. Surfaces of this kind belong to a wider family of scrolls distinguished by $\lambda = \text{const}$, $\mu = \text{const}$ and a striction line k with constant geodesic curvature $\gamma (= -\sigma')$; hence the plane development $\bar{k}$ of k (viz. Section 2) is a circle. Those remarkable surfaces were studied by G. Pillwein [4]; in general they are not osculated by their striction scroll. – For rational values $n \neq \frac{1}{2}$ the striction line k (21) is *algebraic*, and so is the ruled surface Φ, too. The simplest case is obtained with $n = \frac{1}{4}$: The surface Φ, congruent with its striction scroll Φ^*, is of degree 6, the striction line is of order 5, and the central developable Γ has class 4. The excluded case $n = \frac{1}{2}$ leads to a transcendental surface.

The limit case $n = 0$ $(m = 1)$ concerns surfaces with a *central cylinder* Γ. Here we have $p = (0, 0, 1)$; $\dot{p} = o$ yields, because of (11) and in accord with Section 3, $\sigma = \text{const}$. Putting

$$e_1 = (a \cos v, a \sin v, b),$$

$$e_2 = (-\sin v, \cos v, 0), \tag{22}$$

$$e_3 = (-b \cos v, -b \sin v, a)$$

with not vanishing constants $a = -\sin \sigma$, $b = \cos \sigma$, and determining the central plane ζ: $z e_2 + q = 0$ by means of its support function $q(v)$, we get the following parametric representation of the striction line k:

$$z_1 = q \sin v + \dot{q} \cos v,$$

$$z_2 = -q \cos v + \dot{q} \sin v, \tag{23}$$

$$z_3 = \frac{b^2 - a^2}{2ab} (\dot{q} + \int q \, dv).$$

The helicoidal surfaces Φ of Section 3 are found with $q = \text{const}$. More interesting examples are obtained with $q = c \cdot \sin nv$ $(0 < n \neq 1)$; here the base of the central cylinder Γ is an epicycloid (for $n < 1$) or a hypocycloid (for $n > 1$), and the striction line k is a helix on a rotational quadric. Rational values of n lead to *algebraic surfaces* again; the simplest specimen is a quartic scroll $\left(n = \frac{1}{2}, \sigma = \pm \frac{\pi}{6}\right)$. – It is still an open question whether there exist cubic ruled surfaces with osculating striction scroll.

In order to approach the case of surfaces Φ with a *central cone* Γ, we put $z = rp$ with $r = r(u)$. Comparison of the tangent vector $\dot{z} = \dot{r}p + r\dot{p}$ with its expression in (18) leads to the differential equation $\frac{\dot{r}}{r} = -\cot 2u$ and thus to

$$r^2 \sin 2u = a^2 = \text{const}, \tag{24}$$

in accord with the remark in Section 3 that the plane development $\bar{k}$ of the striction line k is an equilateral hyperbola. The cone itself may still be arbitrarily chosen.

Taking for instance a *cone of revolution*, $\boldsymbol{p}$ may be expressed as in equation (20). Then the striction line k is described by

$$z_1 = nr \cos \frac{u}{n},$$

$$(25) \qquad z_2 = nr \sin \frac{u}{n},$$

$$z_3 = nr \quad \text{with} \quad r = \frac{a}{\sqrt{\sin 2u}}.$$

For rational values of n $(0 < n < 1)$ we get *algebraic* curves and surfaces again. The simplest example yields with $n = \frac{1}{2}$: Φ and Φ^* are congruent scrolls of degree 7 with a striction line of order 4.

The question for constantly twisted surfaces with osculating striction scroll and a central cone does not allow an elementary solution. – A more detailed exposition of the present topic is to be found in [8].

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