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SELF-OSCULATING COUPLER CURVES

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Abstract—Coupler curves, generated by a planar four-bar linkage and possessing the peculiarity of self-osculation, are generally discussed. Condition formulas characterizing appropriate linkages are developed.

1. INTRODUCTION

Let $LABM$ be a planar four-bar linkage with fixed pivots L, M and moving joints A, B. Any point C, rigidly connected with the coupler AB, traces, in the course of the motion of the mechanism, a so-called *coupler curve* (Fig. 1). This is, in general, a *tricircular sextic*, i.e. an algebraic curve of sixth order which has two imaginary triple points at the circular points I, J at infinity. The corresponding tangents (asymptotes) intersect in three real "singular foci", namely L, M and N.

The famous theorem of S. Roberts states that the same coupler curve k may be generated in a similar way by two other four-bars having their pivots in L, N or M, N. The pivot triangle LMN and the coupler triangle ABC are directly similar.

Besides the triple points I and J which count for three double points each, the coupler sextic k has *three proper double points* D_1, D_2, D_3, situated on the "*focal circle*" f which passes through L, M, N (Fig. 1), provided these points are not collinear (Section 7). A fourth double point D_0 arises only in the case of a flattening four-bar.

In a recent paper[1] K. H. Hunt and J. E. Kimbrell considered the phenomenon of coalescing double points $D_1 = D_2 = D_3$ ($\neq D_0$). In such a case the coupler curve k *osculates* with itself at a point S, having there the focal circle f as common circle of curvature for both of the branches of k. The authors gave a simple construction of linkages generating *symmetrical coupler curves* of this kind, but doubted the existence of unsymmetrical self-osculating coupler curves.

The present note will show that in fact *unsymmetrical coupler sextics with self-osculation* do exist. If the pivot triangle LMN is given, only *three osculation points* S, S', S'' are possible. They form a well-determined *equilateral triangle* inscribed in the focal circle f, and to each of them belongs a *one-parametric family of self-osculating coupler curves*. —A final remark discusses the *limit case* of collinear coupler points A, B, C. In this case the focal circle f degenerates into the base line LMN, the osculation point S is uniquely determined (at the centre of gravity of the point triple L, M, N). The corresponding coupler curves are symmetrical with respect to the common inflection tangent f at the *inflection point* S.

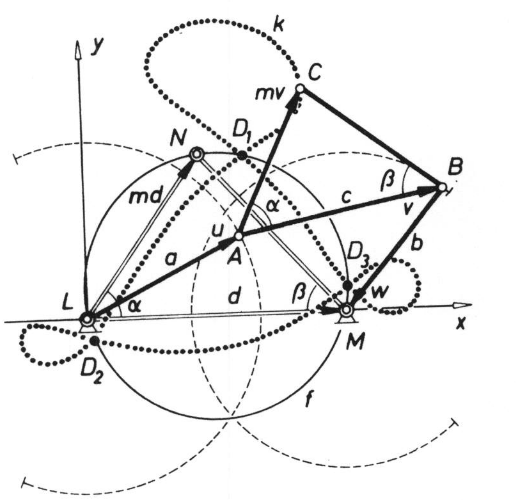

Fig. 1. Coupler sextic generated by a four-bar linkage.

2. ANALYTIC CONCEPT

Starting from a Cartesian coordinate system (L; x, y) whose x-axis passes through M, we pass to *isotropic coordinates*

$$z = x + iy, \quad \bar{z} = x - iy \quad (\text{with } i^2 = -1). \tag{2.1}$$

With real quantities x, y the complex number z determines a vector represented by the line segment from the origin $L(0, 0)$ to the real point $P(x, y)$ ([3], p. 31).

Denoting the side lengths of the quadrilateral $LABM$ with $\overline{LA} = a$, $\overline{MB} = b$, $\overline{AB} = c$, $\overline{LM} = d$ (all values real and positive) and the corresponding vectors with complex numbers $LA = u$, $AB = v$, $BM = w$, $LM = d$ ($= \bar{d}$), we have (Fig. 1):

$$u + v + w = d, \quad |u| = a, \quad |v| = c, \quad |w| = b. \tag{2.2}$$

The position vector LC of the coupler point C is then described by

$$z = u + mv \text{ with } |m| = \overline{AC} : \overline{AB}, \quad \arg m = \alpha = \sphericalangle BAC. \tag{2.3}$$

The relations (2.2) and (2.3) with variables u, v, w and

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constants a, b, c, d, m give a simple analytic representation of the coupler curve k ([3], p. 67).

Introducing the conjugates we get from $u\bar{u} = a^2$ and $w\bar{w} = b^2$ with

$$u = z - mv, \quad w = z - d + nv \quad (n = m - 1) \tag{2.4}$$

two linear equations for v and $\bar{v}$. Substitution of the calculated values in $v\bar{v} = c^2$ leads at last to the following *equation of the coupler curve* in isotropic coordinates z, $\bar{z}$ ([3], p. 68):

$$[\bar{n}(z-d)P - \bar{m}zQ]\cdot[n(\bar{z}-d)P - m\bar{z}Q] + c^2R^2 = 0 \text{ with}$$
$$P = z\bar{z} + m\bar{m}c^2 - a^2, \quad Q = (z-d)(\bar{z}-d) + n\bar{n}c^2 - b^2, \tag{2.5}$$
$$R = (\bar{m} - m)z\bar{z} + (\bar{m}nz - m\bar{n}\bar{z})d, \quad n = m - 1.$$

The equation $R = 0$ describes a circle, as $z\bar{z} = x^2 + y^2$. It is easy to confirm that this circle passes through the pivots $L(z = \bar{z} = 0)$, $M(z = \bar{z} = d)$ and $N(z = md, \ \bar{z} = \bar{m}d)$. Hence $R = 0$ is the *focal circle f*, provided $m \neq \bar{m}$. In the special case $m = \bar{m}$ the points A, B, C are collinear, and so L, M, N too; here $R = 0$ represents the base line $z = \bar{z}$ $(y = 0)$.

3. DOUBLE POINTS OF THE COUPLER CURVE

To find the three (ordinary) double points of the coupler curve k (2.5), we have to cut it with the focal circle f:

$$R \equiv (\bar{m} - m)z\bar{z} + (\bar{m}nz - m\bar{n}\bar{z})d = 0. \tag{3.1}$$

This leads to the equation

$$\bar{n}(z-d)P - \bar{m}zQ = 0 \tag{3.2}$$

or the conjugate (which would furnish the same result). In order to parametrize the circle f (3.1), we put $z/\bar{z} = s$ and obtain—under the assumption $m \neq \bar{m}$—:

$$z = \frac{\bar{m}ns - m\bar{n}}{m - \bar{m}}d, \quad \bar{z} = \frac{\bar{m}ns - m\bar{n}}{(m - \bar{m})s}d. \tag{3.3}$$

With these expressions eqn (3.2) reduces to

$$n\bar{n}(s-1)P = (\bar{m}ns - m\bar{n})Q. \tag{3.4}$$

After developing the quantities P and Q from (2.5) we arrive at a *cubic equation* of the form

$$Es^3 - Fs^2 + \bar{F}s - \bar{E} = 0 \text{ with}$$
$$E = \bar{m}^2n^2\bar{n}d^2, \tag{3.5}$$
$$F = \bar{m}n\bar{n}(mn + \bar{m}n + m\bar{n})d^2 + (m - \bar{m})^2n(\bar{m}\bar{n}c^2 + \bar{m}b^2 - \bar{n}a^2).$$

Each one of the three roots s_j $(j = 1, 2, 3)$ determines by means of the formulas (3.3) the isotropic coordinates z_j, $\bar{z}_j$ of a double point D_j of the coupler curve.

In the case of three real and distinct double points D_1, D_2, D_3 (Fig. 1) the parameters s_j satisfy the relation

$$s_1s_2s_3 = \bar{E}/E = (m/\bar{m})^2(\bar{n}/n). \tag{3.6}$$

Denoting the argument of the complex number z_j by σ_j $(= \sphericalangle MLD_j)$, we have $\arg s_j = \arg(z_j/\bar{z}_j) = 2\sigma_j$. With attention to $\arg m = \alpha = \sphericalangle BAC = \sphericalangle MLN$ and $\arg n = \pi - \beta = \pi - \sphericalangle CBA = \pi - \sphericalangle NML$ it follows from eqn (3.6):

$$2(\sigma_1 + \sigma_2 + \sigma_3) = 4\alpha + 2\beta \pmod{2\pi}. \tag{3.7}$$

As $2\sigma_j$ measures the circle arc MD_j on f, 2α the arc MN, and 2β the arc NL, eqn (3.7) means (modulo the circumference):

$$\Sigma \operatorname{arc} MD_j = \operatorname{arc} MN + \operatorname{arc} ML. \tag{3.8}$$

Replacing the initial point of arc measuring, M, by any other point P of f, it holds also

$$\Sigma \operatorname{arc} PD_j = \operatorname{arc} PL + \operatorname{arc} PM + \operatorname{arc} PN. \tag{3.9}$$

This result, compatible with a theorem of A. Cayley, provides certain information about the distribution of the double points on the focal circle.

4. SELF-OSCULATING COUPLER CURVES

Coupler curves with self-osculation are distinguished by the property that all three double points D_j coalesce in a single point S on the focal circle f. In this case the cubic equation (3.5) must have the form

$$(\lambda s - \mu)^3 \equiv \lambda^3s^3 - 3\lambda^2\mu s^2 + 3\lambda\mu^2s - \mu^3 = 0. \tag{4.1}$$

Comparison of the coefficients requires

$$E:F:\bar{F}:\bar{E} = \lambda^3 : 3\lambda^2\mu : 3\lambda\mu^2 : \mu^3. \tag{4.2}$$

The triple root $s = \mu/\lambda$ is determined, in accordance with eqn (3.6) by

$$s^3 = \bar{E}/E = (m/\bar{m})^2(\bar{n}/n). \tag{4.3}$$

This equation has three solutions s, s', s'', differing by the factor $\epsilon = \frac{1}{2}(-1 + i\sqrt{3})$ or $\epsilon^2 = 1/\epsilon = \bar{\epsilon}$.

The numerical evaluation may be performed by means of eqn (3.7) and gives

$$s = e^{2i\sigma} = \cos 2\sigma + i \sin 2\sigma \text{ with } \sigma = \frac{1}{3}(2\alpha + \beta) \pmod{\pi/3}. \tag{4.4}$$

As $\sigma = \sphericalangle MLS$, we see (Fig. 2): *For a given base triangle LMN only three osculation points S, S′, S″ are at our disposal; they form an equilateral triangle, inscribed in the circumcircle f of LMN.* Making use of eqn (3.9) with $P = S, S', S''$, we can add: *Each one of the three osculation points has vanishing sum of arc distances from the pivots L, M, N.*

After having decided upon one of the three possible values of s which satisfy eqn (4.3), we have still to take into account the rest of eqn (4.2). Thus we obtain the essential *condition for self-osculation*:

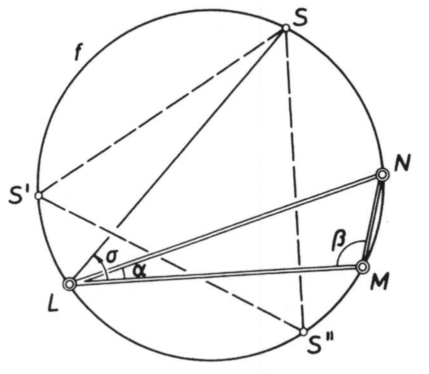

Fig. 2. Distribution of foci L, M, N and self-osculation points S, S', S'' on the focal circle f.

$$F = 3\,s\,E, \tag{4.5}$$

where E and F are to be taken from the definitions in (3.5). This complex condition (equivalent to its conjugate) induces two real conditions, corresponding to the real and the imaginary parts. These (necessary and sufficient) conditions are linear in a^2, b^2, c^2, d^2. Hence: *For each one of the three disposable osculation points S, S', S'' on the focal circle there exists a one-parametric family of self-osculating coupler curves.*

3. EXAMPLE

Prescribing for instance $m = i$, we have $n = i - 1$, $\alpha = 90°$, $\beta = 45°$, and the quantities E, F of eqn (3.5) are

$$E = 2(1-i)d^2,$$
$$F = 8a^2 - 4(1+i)b^2 + 8ic^2 - 2(1+i)d^2. \tag{5.1}$$

The key equation (4.3) reads $s^3 = i$ and has the roots

$$s = -i, \quad s' = \frac{1}{2}(\sqrt{3}+i), \quad s'' = \frac{1}{2}(i - \sqrt{3}). \tag{5.2}$$

First case. With $s = -i$ the osculation point S (3.3) coincides with the pivot L $(z = 0)$. Condition (4.5) requires

$$2a^2 - (1+i)b^2 + 2ic^2 + (1+i)d^2 = 0, \tag{5.3}$$

hence

$$2a^2 - b^2 + d^2 = 0, \quad -b^2 + 2c^2 + d^2 = 0. \tag{5.4}$$

If we choose the values of c and d, we find

$$a = c, \quad b = \sqrt{(2c^2 + d^2)}. \tag{5.5}$$

By means of Roberts' theorem it is easy to state that the obtained coupler curves are *symmetrical* ones. Figure 3 shows a particular example; the curve belongs to the same family as that of Fig. 7(a) in [1].

Second case. Using the value s' of (5.2), we find for the osculation point S' the coordinates $x = \frac{1}{4}(3+\sqrt{3})d$, $y = \frac{1}{4}(3-\sqrt{3})d$. Condition (4.5) reads

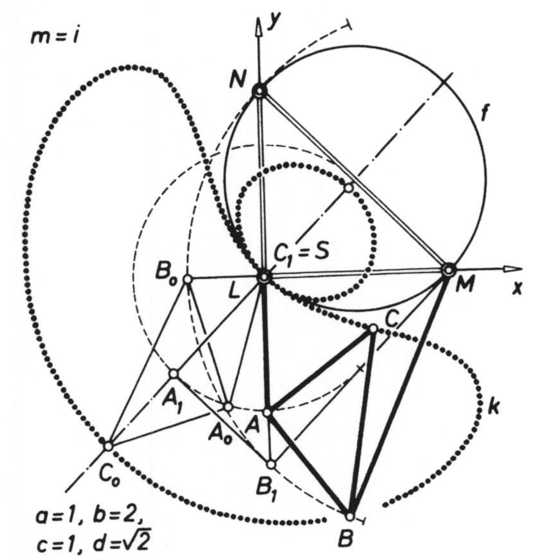

Fig. 3. Symmetrical self-osculating coupler curve.

$$8a^2 - 4(1+i)b^2 + 8ic^2 - [(5+3\sqrt{3}) + (3\sqrt{3}-5)i]d^2 = 0 \tag{5.6}$$

and is to satisfy by

$$a^2 = c^2 + 3\sqrt{(3)}\,d^2/4, \quad b^2 = 2c^2 + (3\sqrt{3}-5)\,d^2/4. \tag{5.7}$$

The coupler curves of this family are *not symmetrical*, as there does not exist an axial symmetry which would permute the singular foci L, M, N and leave S' fixed. The particular example in Fig. 4 possesses an additional double point D_0, as (by chance) the four-bar is flattenable $(a + d = b + c)$.

The *third family* of self-osculating coupler curves, corresponding to the value s'' in (5.2), proves to be congruent to family 2; this is again a consequence of Roberts' theorem.

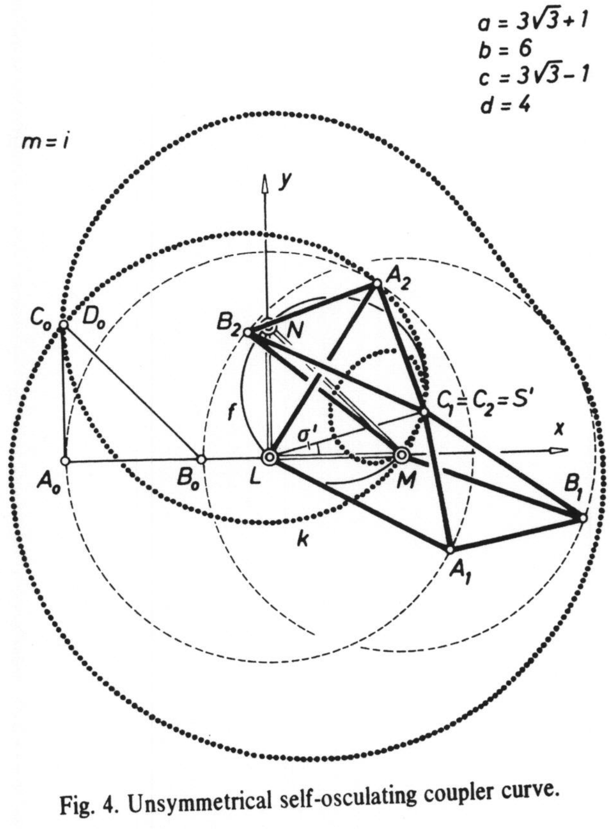

Fig. 4. Unsymmetrical self-osculating coupler curve.

6. ISOSCELES COUPLER TRIANGLES

In order to carry on developments in the simple case of an *isosceles* coupler triangle ABC (and base triangle LMN), we may assume $\alpha = \beta \neq 0$ and put

$$m = \frac{1}{2}(1 + ih) \text{ with } h = \tan\alpha \neq 0. \tag{6.1}$$

With respect to $n = m - 1 = -\bar{m}$ we obtain from eqn (3.5):

$$E = -m\bar{m}^4 d^2,\quad F/\bar{m} = h^2(ma^2 + \bar{m}b^2 - m\bar{m}c^2) - m\bar{m}(m^2 + m\bar{m} + \bar{m}^2)d^2. \tag{6.2}$$

Equation (4.3) then gives the three possible values of the osculation parameter

$$s = m/\bar{m},\quad s' = \epsilon s,\quad s'' = \bar{\epsilon}s \quad \text{with } \epsilon = \frac{1}{2}(-1 + i\sqrt{3}). \tag{6.3}$$

Symmetrical case. Using the value $s = m/\bar{m}$, we find that the osculation point S coincides with the focus N. Condition (4.5) reduces to $F + 3m^2\bar{m}^3d^2 = 0$ and requires

$$ma^2 + \bar{m}b^2 = m\bar{m}(c^2 - d^2), \tag{6.4}$$

hence

$$a^2 = b^2 = \frac{1 + h^2}{4}(c^2 - d^2) = \frac{c^2 - d^2}{4\cos^2\alpha}. \tag{6.5}$$

As the arms LA and MB have equal length $a = b$, the coupler curves are *symmetrical* and belong to the kind considered by Hunt and Kimbrell [1]. They were furtively mentioned already by Müller [2], whose corresponding formula is equivalent with (6.5).

Example: Taking for instance $h = 1$ ($\alpha = \beta = 45°$), we would obtain the same coupler curves as in Chapter 5, only generated by other four-bars. In particular the curve of Fig. 3 would be found with $a = b = \sqrt{2}$, $c = 2\sqrt{2}$, $d = 2$ (see also Fig. 3 in [1]).

Unsymmetrical case. Operating with the value $s' = \epsilon m/\bar{m}$, the condition (3.5) leads over $F + 3\epsilon m^2\bar{m}^3d^2 = 0$ to

$$h^2(ma^2 + \bar{m}b^2 - m\bar{m}c^2) = Km\bar{m}d^2 \text{ with } K = \frac{1}{8}[(9 + h^2) - 3\sqrt{(3)}(1 + h^2)i]. \tag{6.6}$$

Separation of imaginary and real parts gives the relations

$$b^2 - a^2 = pd^2,\quad a^2 + b^2 - \frac{1}{2}(1 + h^2)c^2 = qd^2,$$

where

$$p = \frac{3\sqrt{(3)}(1 + h^2)^2}{16h^3},\quad q = \frac{(1 + h^2)(9 + h^2)}{16h^2} \tag{6.7}$$

Choosing the values of c and d we finally obtain the formulas

$$a^2 = \frac{1 + h^2}{4}\left[c^2 + \frac{1}{8}\left(1 - \frac{\sqrt{3}}{h}\right)^3 d^2\right],\quad b^2 = \frac{1 + h^2}{4}\left[c^2 + \frac{1}{8}\left(1 + \frac{\sqrt{3}}{h}\right)^3 d^2\right]. \tag{6.8}$$

The corresponding coupler curves are *not symmetrical.* They are congruent with those which would be obtained with the third possibility in (6.3): $s'' = \bar{\epsilon}m/\bar{m}$; in this case the formulas (6.8) for a and b interchange.

Examples: Prescribing for instance $\alpha = \beta = 30°$, we have to take $h = 1/\sqrt{3}$ and formulas (6.8) reduce to

$$a^2 = \frac{1}{3}(c^2 - d^2),\quad b^2 = \frac{1}{3}(c^2 + 8d^2). \tag{6.9}$$

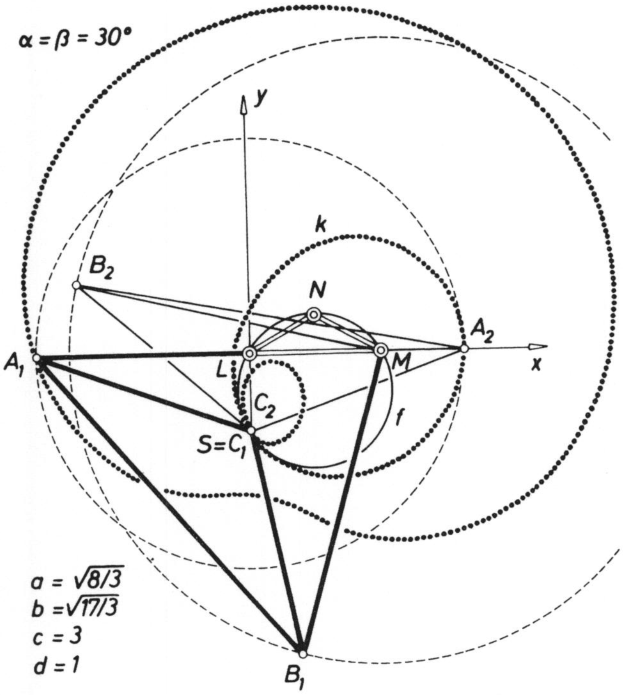

Fig. 5. Unsymmetrical self-osculating coupler sextic.

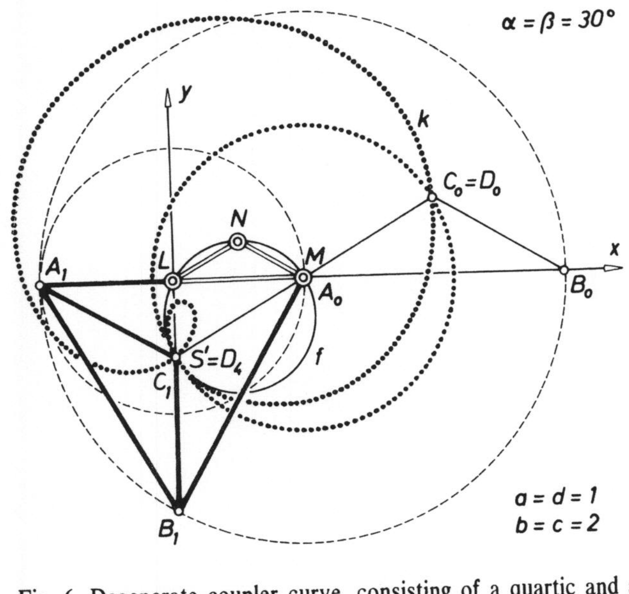

Fig. 6. Degenerate coupler curve, consisting of a quartic and a circle. Four of the five double points coincide.

Figure 5 illustrates the choice $c = 3$, $d = 1$ ($a = \sqrt{(8/3)}$, $b = \sqrt{(17/3)}$). The coupler curve consists of two parts. —The choice $c = 2$, $d = 1$ ($a = 1$, $b = 2$) leads to a kite. In this case the coupler sextic splits into a circle with center M and a bicurcular quartic with a double point at S', where it is touched by the circle (Fig. 6). Here no self-osculation takes place, although $S' = D_1 = D_2 = D_3$ and yet a fourth (extraordinary) double point coincides with S' ($x = 0$, $y = -i/\sqrt{3}$).

7. SELF-OSCULATING WATT CURVES

In order to complete the investigation, we shall discuss also the case of *collinear foci* L, M, N, until now excluded by the assumption $m \neq \bar{m}$. In the present limit case $m = \bar{m}$ ($\neq 0$, $\neq 1$) is a real number and the coupler points A, B, C lie on a line. The coupler curve k, traced by C and sometimes denoted as a "Watt curve", is symmetrical with respect to the base line LMN. This line is to be considered as the limit form of the focal circle f, whose equation (3.1) reduces to $z = \bar{z}$ ($y = 0$).

To determine the double points of k, obviously lying on the focal axis f, we use the abscissa x as parameter on f. Substitution of $z = \bar{z} = x$ in eqn (3.2) leads to the cubic equation

$$n(x-d)(x^2 + m^2c^2 - a^2) - mx[(x-d)^2 + n^2c^2 - b^2] = 0. \tag{7.1}$$

It reads, after arrangement,

$$x^3 - Ex^2 + Fx - G = 0 \tag{7.2}$$

with

$$E = (m+1)d,\ F = na^2 - mb^2 - mnc^2 + md,$$
$$G = n(a^2 - m^2c^2)d.$$

The roots x_j ($j = 1, 2, 3$) localize the three double points D_j of the coupler curve. From the relation

$$x_1 + x_2 + x_3 = E = (m+1)d = \overline{LM} + \overline{LN} \tag{7.3}$$

we conclude that both of the point triples D_1, D_2, D_3 and L, M, N have the same centre of gravity. This statement gives certain information about the distribution of the double points.

In the case of coincidence of all double points D_j in one point S on f, we have *self-osculation* of the coupler curve k. Two branches of k, mirror images with respect to the focal axis f, intersect each other at the common *inflection point* S and have there the common tangent f. In this case eqn (7.2) will be equivalent with $(x - s)^3 = 0$, where s denotes the abscissa of S, the centre of gravity of the focal triple L, M, N. Comparison of corresponding coefficients gives the relations

$$3s = E,\ 3s^2 = F,\ s^3 = G \tag{7.4}$$

which lead (by elimination of s) to the essential *conditions*

$$3F = E^2, \quad 27G = E^3. \tag{7.5}$$

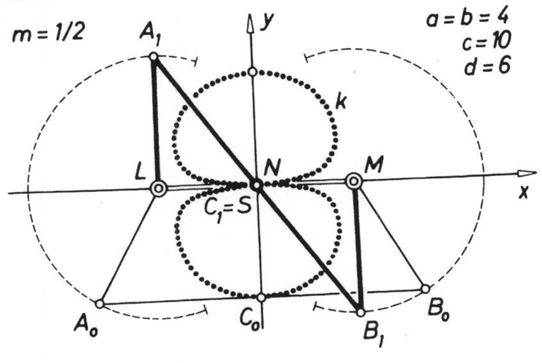

Fig. 7. Self-osculating Watt curve with two axes of symmetry.

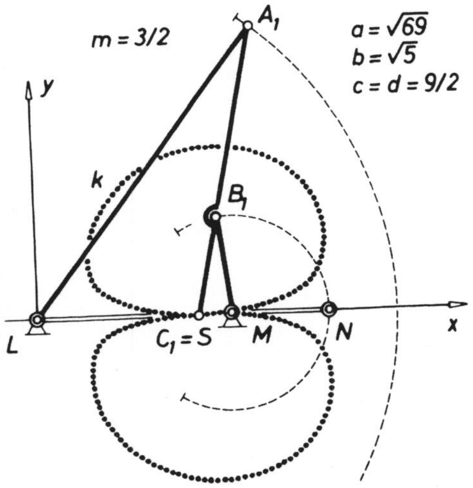

Fig. 8. Self-osculating Watt curve with a single symmetry.

By means of the definitions (7.2) we finally obtain two relations linear in a^2, b^2, c^2, d^2:

$$\begin{aligned} na^2 - mb^2 - mnc^2 &= (m^2 - m + 1)d^2/3, \\ na^2 - m^2nc^2 &= (m+1)^3 d^2/27. \end{aligned} \tag{7.6}$$

Prescribing now c, d and m, the arm lengths a and b are determined by the formulas

$$a^2 = m^2c^2 + \frac{(m+1)^3}{27n}d^2, \quad b^2 = n^2c^2 + \frac{(n-1)^3}{27m}d^2. \tag{7.7}$$

Examples: Equal arm lengths $a = b$ will be obtained with $m = 1/2$ ($n = -1/2$). In this case formulas (7.7) reduce to

$$a^2 = b^2 = \frac{1}{4}(c^2 - d^2) \tag{7.8}$$

and the coupler curve possesses two axes of symmetry (Fig. 7). On the contrary, for $m \neq 1/2$ and $m \neq \pm 2$, there exists only a single symmetry with respect to the focal line $f = LMN$ (Fig. 8).

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SELBSTOSKULIERENDE KOPPELKURVEN

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Kurzfassung - Die drei gewöhnlichen Doppelpunkte einer ebenen Koppelsextik k liegen bekanntlich auf dem Umkreis f des Lagerdreiecks LMN, das zum Koppeldreieck ABC ähnlich ist. Selbstoskulation von k tritt auf, wenn die drei Doppelpunkte an einer Stelle S von f zusammenrücken, und f ist dann Krümmungskreis für zwei Zweige von k. Es zeigt sich, daß bei Vorgabe von LMN nur drei Punkte von f in Betracht kommen, wo Selbstoskulation stattfinden kann. Diese Punkte S,S',S" bilden ein gleichseitiges Dreieck, und zu jedem von ihnen gehört eine einparametrige Familie von selbstoskulierenden Koppelkurven. Für die Abmessungen der betreffenden Gelenkvierecke werden Formeln abgeleitet. Auch der Grenzfall kollinearer Lagerpunkte L,M,N wird untersucht.