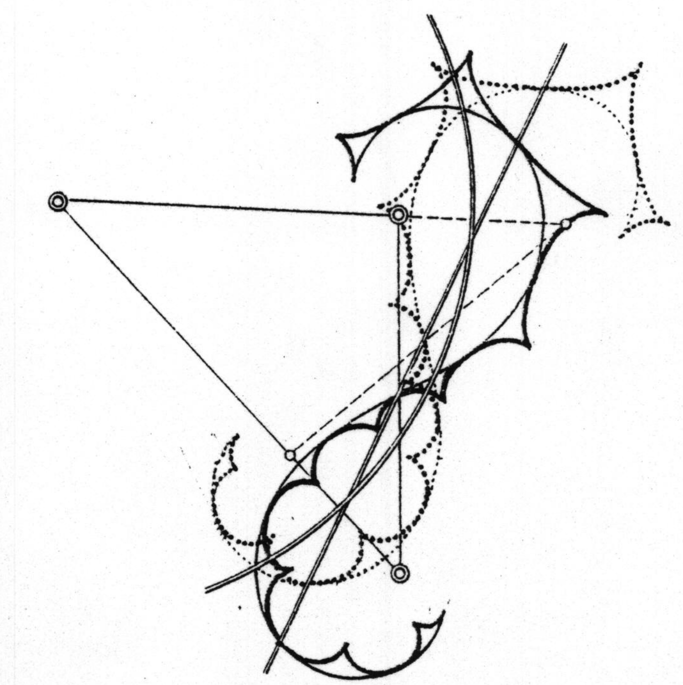
W. Wunderlich

CONGRUENT-INVERSE CURVE PAIRS



CONGRUENT-INVERSE CURVE PAIRS

A contribution to the International Open Competition 1983 in Geometry of Science Software Systems Inc., Los Angeles

By

WALTER WUNDERLICH

Emeritus Professor of Geometry
Technical University of Vienna
A-1040 Wien, Gusshausstrasse 27, Austria

CONTENTS

December 1982

I. FUNDAMENTALS

1. Introduction

The proposed problem concerns in first line the determination of plane curves q which by a certain inversion ι are mapped onto a congruent copy q'. When σ denotes the isometric mapping which brings q' back to q, then q is obviously invariant under the product mapping $\mu = \sigma\iota$. This mapping μ, composed of first ι and then σ, is well-known as "Moebius mapping". Attacking the problem from this point of view, it can be systematically and completely solved.

In this preparatory Chapter I the needed facts about inversion are summarized and then applied to construct self-inverse curves $q = q'$. Furthermore, the concept of the Moebius transformation -- conformal and circle-preserving like inversion -- is briefly discussed.

The method applied afterwards is analytic and makes use of the Gaussian (or Argand) plane of complex numbers, as this is the best tool for conformal mappings. Two main cases have to be distinguished, depending on the kind of the isometric transplacement σ. If σ is an opposite isometry, then μ is a directly conformal mapping (Chapter II). If σ is directly congruent, then μ is indirectly conformal (Chapter III). Provided μ has a real fixed point, an auxiliary inversion κ from this center transforms μ into a linear mapping μ^*, i.e. a similarity or an isometry. Thus the original problem is lead back to the simpler question for curves q^* invariant under μ^*. There is only one subcase (Section 13) which needs an individual treatment, because μ has no real fixed point. By means of an appropriate inversion κ the Moebius mapping μ in this case can be transformed into a special one, μ^*, which consists of an inversion and a rotation about the inversion center (Section 12).

In this way the problem in all cases is led back to the question for curves q^* which are invariant under a simple transformation μ^* (similarity, isometry, cyclic Moebius mapping). This question leads to simple functional equations, solvable by certain periodic functions. As the form of such a periodic function may be arbitrarily chosen within a period interval, infinitely many explicit solutions are obtainable. They enlarge unlimitedly the small number of known examples. Special attention is paid to algebraic solutions; most of the numerous illustration figures show algebraic examples. The typical periodic structure of the solution curves q is due to the fact that an elementary arc of q is repeated by successive application of the automorphism μ.

A supplementary Chapter IV demonstrates the use of stereographic projection and of so-called fan transformations for deriving additional solutions.

2. Inversion

For better understanding it may be welcome to recall the definition and some fundamental properties of the inversion, in spite of the circumstance that this classical transformation, important in higher geometry and applied mathematics, is excellently treated in many lecture books, e.g. [2].

Planar inversion with respect (or "in") a real circle j (center O, radius $a>0$) is a one-to-one point mappingwhich associates to each point $P \neq O$ its image point P', defined by the following rule: P, P' and O are on the same line, and the central distances $OP = r$ and $OP' = r'$ are reciprocal, if a is considered as length unit ($rr' = a^2$). Hence, in polar coordinates r, ϕ with origin O, the inversion $\iota: P \to P'$ is described by

$$(2.1) \qquad r' = a^2/r, \quad \phi' = \phi.$$

In cartesian coordinates $x = r\cos\phi$, $y = r\sin\phi$ the transformation formulas read

$$(2.2) \qquad x' = \frac{a^2x}{x^2+y^2}, \quad y' = \frac{a^2y}{x^2+y^2}.$$

Fig.1 shows a simple construction of inverse point pairs. Evidently, each point of the *"inversion circle"* j ($r = a$) is *fixed*, i.e. mapped onto itself ($C = C'$ in Fig.2).

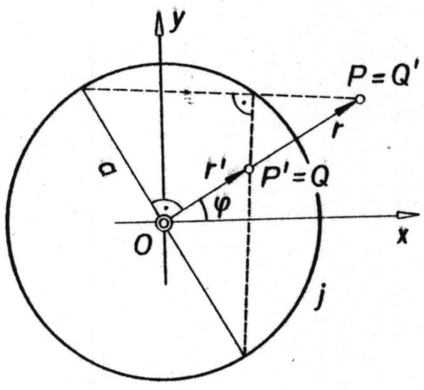


Fig.1: Inversion

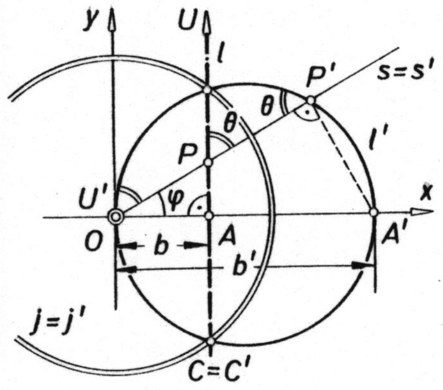


Fig.2: Inversion of a line

The most important (and well-known) *properties* of the inversion are the following:

(i) Inversion is an *involutory mapping:* If P' is the image of P, then $Q = P'$ has its image Q' at P (Fig.1). In other words: Repetition of an inversion ι produces the identity (ι^2 = "1").

(ii) Inversion is a *quadratic mapping.* When a variable point P moves along a straight line l, not passing through O, its image point P' describes a circle l'. Choosing the x-axis perpendicular to l (Fig.2), the polar equation of l is $r \cos\phi = b \neq 0$; due to (2.1), the polar equation of l' reads $r' = b' \cos\phi$ with $b' = a^2/b$ and characterizes l' as a *circle* passing through O (Fig.2). Therefore it is convenient to consider O as image U' of the point at infinity, U, of l. -- A line s passing through the inversion center O is mapped onto itself ($s = s'$, Fig.2).

(iii) Inversion is a *conformal mapping*, i.e. angle-preserving. The angle of two curves q_1, q_2 at a common point $P \neq O$ is reproduced at P', where it reappears as the angle of the image curves q_1', q_2'. This fact is an almost immediate consequence of the reproduction of the angle θ made by a line l and a radial ray s (Fig.2). It should be noticed, however, that only the size of an angle is preserved, whereas its sense of orientation is inverted: inversion is "indirectly conformal".

(iv) *Orthogonal circles* of the inversion circle j are *invariant*. Due to the power theorem, such a circle o is mapped onto itself, with interchanging of the parts inside and outside of j ($o = o'$ in Fig.3).

(v) A circle h, meeting the inversion circle j in opposite points, is mapped onto a congruent circle $h' \simeq h$ (Fig.3). This is again a consequence of the power theorem.

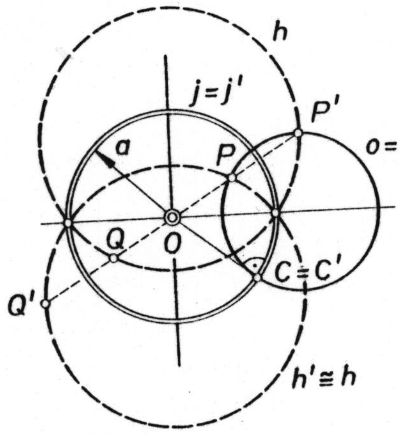


Fig.3: Self-inverse and congruent-inverse circles

(vi) Inversion is a *circle transformation*, i.e. cercle-preserving: any circle k, not passing through O, is transformed into a circle k' again. A simple proof consists in applying a dilatation $\tilde{r} = \lambda r$ onto an orthogonal circle o of j (iv) or onto a circle h (v); the image k' of k is then related with $o' = o$ or $h' \simeq h$ by the dilatation $\tilde{r}' = r'/\lambda$, hence also a circle. -- A circle which passes through O is mapped onto a line, due to (i) and (ii); therefore it is convenient to consider lines as limit forms of circles.

(vii) *Invariance of inverseness:* If two points P_1, P_2 are inverse with respect to a circle k, their images P_1', P_2', obtained by another inversion ι, are inverse with respect to the image circle k'. For a proof consider two auxiliary circles $o, \bar{o}$ passing through P_1 and P_2: as they are orthogonal to k, their images $o', \bar{o}'$ are orthogonal circles of k', hence the statement is true. -- *Corollary:* If the center O of the inversion ι is on k, then k' is a line, and P_1', P_2' are *mirror images* with respect to k'.

Other properties of the inversion are revealable only in the complex projective plane and shall not be discussed in detail. They concern, for instance, the rôle of *"isotropic lines"* $x \pm iy =$ $=$ const and their points at infinity, the *"absolute circular points"* I and J (in common to all circles of the plane and therefore invariant under isometries and similarities). These points show a similar behavior under inversion as the center O. Each one of the three *exceptional points* O,I,J has no definite image, but a whole range of possible image points: The image of O might be any point at infinity, the image of I any point on the isotropic line OI, and the image of J any point on OJ. Conversely each point at infinity $U \neq I,J$ has the image $U' = O$ (Fig.2), each point $V \neq O,I$ on the line OI has the image $V' = I$, and each point $W \neq O,J$ on OJ has the image $W' = J$. These facts may be confirmed by means of equs. (2.2).

the exceptional (or "principal") points O,I,J are important for judging the order and certain singularities of the inverse image q' of an *algebraic curve* q. If, for instance, q is a curve of order n and contains none of the exceptional points, then its image q' is a curve of order $n' = 2n$ (due to the quadratic character of the inversion) and possesses n-fold points at O, I, and J. If however a (real) algebraic curve q has an s_0-fold point at O and s_1-fold points at I and J, the order of q' reduces to $n' = 2n - s_0 - 2s_1$, O has the multiplicits $n - 2s_1$, whereas both I and J have the multiplicity $n - s_0$. Properties (ii) and (vi) are now seen from a higher point of view.

3. Self-inverse Curves

In general, a curve q and its image q'. obtained by an inversion $\iota: P \to P'$, are completely different. However, it may occur under particular circumstances that q and q' coincide. The simplest example, apart from the inversion circle $j = j'$, is offered with the orthogonal circles $o = o'$, as mentioned in Section 2 (iv). Such *self-inverse curves* $q = q'$ are called, with a traditional term, "*anallagmatic*" [3, p.40].

Now, if an anallagmatic curve $q = q'$ is transformed by means of an additional inversion κ: $P \to P^*$, whose center K is situated on j (Fig.4), then, due to the corollary of property (vii), its image $q^* = q'^*$ will have the image line j^* of j as *mirror axis*. Conversely we have

<u>Theorem 1:</u> *Anallagmatic curves $q = q'$ are obtained by inverting an arbitrary curve q^* which is symmetrical with respect to an axis j^*, provided the inversion center is not on j^*.*

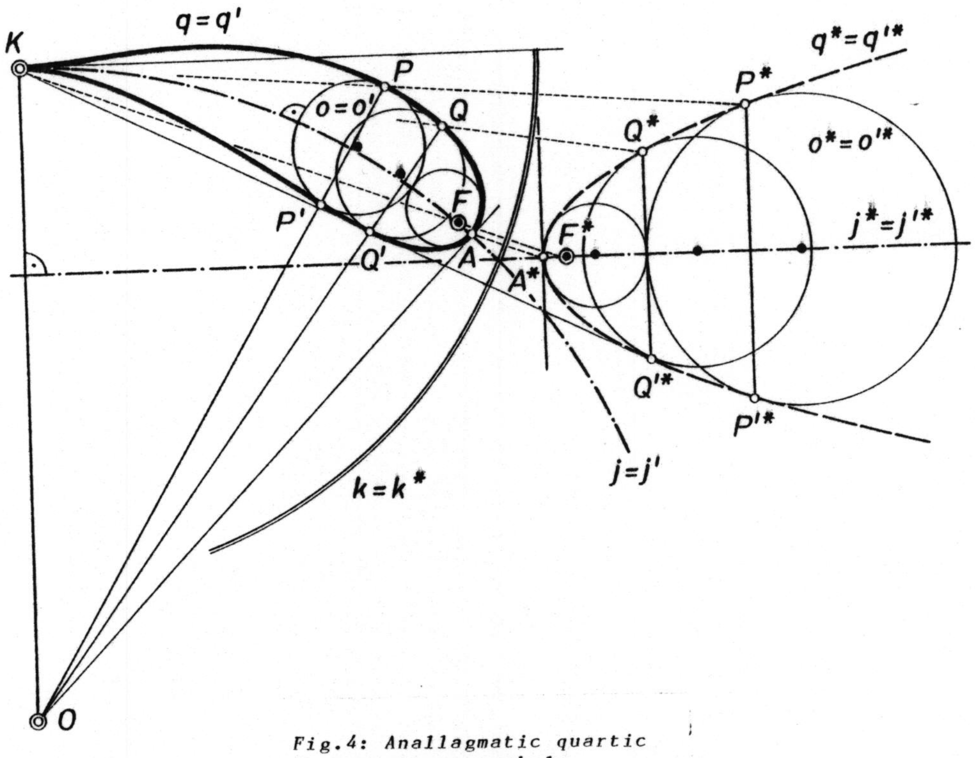


Fig.4: Anallagmatic quartic inverse to a parabola

Take, for instance, as initial curve q^* a common *parabola*, then -- due to the final remark in Section 2 -- the inverse image q is an algebraic curve of order 4, having a cusp at the inversion center K and moreover (imaginary) double points at the absolute circular points I and J: it is a *cusped "bicircular quartic"*, self-inverse with respect to the image circle j of the parabola axis j^*. Slim curves of the kind shown in Fig.4 (K outside of q^*) and also in [1, p.367] once were suggested as streamline profiles (N.JOUKOWSKY, 1910).

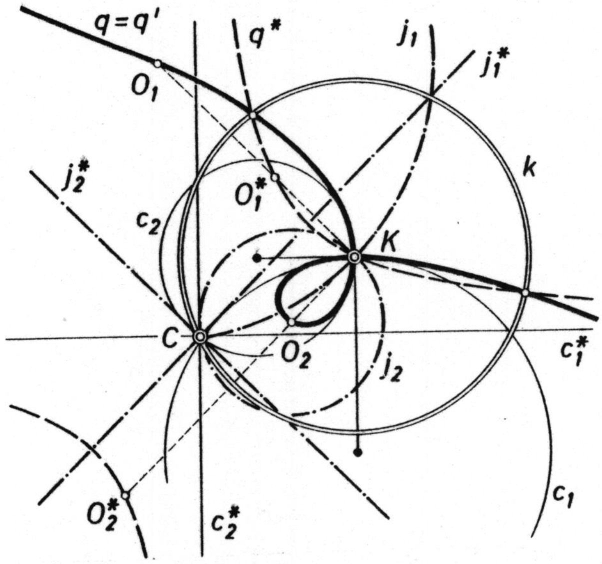


Fig.5: Doubly anallagmatic cubic inverse to a hyperbola

A *central conic* q (different from a circle) has two mirror axes and therefore generates a *doubly anallagmatic curve* q, provided the inversion center K is not situated on one or both of the axes. If q^* is a hyperbola, q has a real node at K (Figs. 5,6); if q^* is an ellipse, K is an isolates double point of q (real, but with imaginary tangents, Fig.11). Supposed the conic q^* doesnot pass through the inversion center K, then its inverse image *q* is a ***bicircular quartic***. On the other hand, if *K* is on q^*, then *q* is a ***(mono)circular cubic***; Fig.5 shows for example a so-called ***strophoid*** [3, p.37], derived from an equilateral hyperbola. -- Supposing the inversion center *K* on (only) one axis of the conic q^*, the derived curve *q* has the same mirror axis and admits only a sole (not degenerate) self-inversion. In particular, if *K* is a focus of *q* (Fig.6), the resulting bicircular quartic *q* (with cusps at the circular points *I* and *J*) is the well-known ***Pascal snail ("limaçon")***, a special trochoid and also a special circle-conchoid [3, p.88].

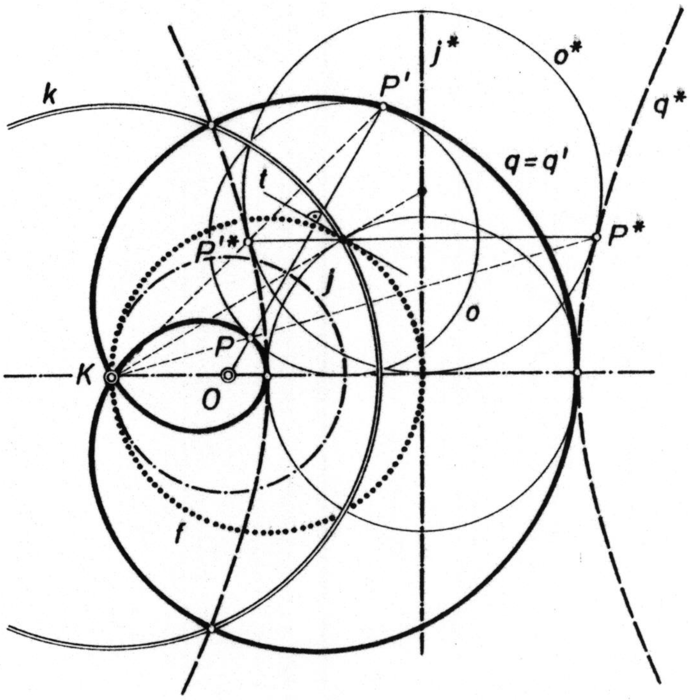


Fig.6: Pascal snail

Every axial curve q^* possesses a continuous set of *bitangent circles* o^*, centered on the mirror axis j^* and thus orthogonal to j^*. Consequently, any derived self-inverse curve $q = q'$ is also endowed with a set of bitangent circles $o = o'$, orthogonal to the circle j of self-inversion and touching q at sorresponding points P, P' (Fig.4). This leads to the classic

<u>*Theorem 2:*</u> *Each anallagmatic curve is the envelope of a one-parametric set of circles orthogonal to the circle of self-inversion.*

Such a set of circles may be determined by the locus f of the circle centers, the so-called *"deferent"*. This offers another mode of constructing anallagmatic curves. Having chosen a deferent f which is not completely contained in the inner region of the inversion circle j, any point M of f and outside of j determines a real orthogonal circle o of j. The contact points P, P' of this circle o with the envelope q are found on the perpendicular dropped from the inversion center O onto the tangent t of f at M (Fig.6). The contact points coincide (on the inversion circle j), if t is a common tangent of f and j; such a self-corresponding point $A = A'$ is a *vertex* of q, and the associated circle o represents a hyperosculating circle of curvature (Fig.4). Generally, each circle of curvature of a curve is mapped onto the corresponding circle of curvature of the inverse curve; this rule offers a valuable help for the designer.

Any intersection point F of the deferent f with the inversion circle j produces a bitangent point-circle and is a *focus* of the anallagmatic curve q. According to J.PLUECKER's definition, the ordinary foci of an analytic curve are the intersection points of isotropic tangents with proper contact points; they are to distinguish from extraordinary foci, which are the intersections of isotropic asymptotes. Since isotropic lines $x \pm iy = c$ are inverted into isotropic lines $x' \mp iy' = c'$ -- see equs. (2.2) --, the ordinary foci of inverse curves are corresponding points (Fig.4). Such a correspondence doesnot hold for extraordinary foci because of the exceptional character of the contact points I, J of isotropic asymptotes; thus, for instance, the centers of inverse circles in general are not associated by the inversion, although they lie on a ray through the inversion center.

An analytical solution of the problem of anallagmatic curves will be found in Section 12. One simple example may be mentioned already here: The curve q, defined by its polar equation

$$(3.1) \qquad r = a\cdot\cot(\phi/2),$$

is obviously self-inverse, as the points P, P' with the coordinates ϕ, r and $\phi' = \phi+\pi$, $r' = -a\cdot\tan(\phi/2)$ are corresponding in the inversion (2.1). The curve has the cartesian equation $(x^2+y^2)\times(y-2a)+a^2y=0$ and is the *symmetrical strophoid* [3, p.37]; this cubic could be obtained by inverting an equilateral hyperbola from a vertex (compare Fig.5).

4. Moebius Transformations

One of the historic ideas of C.F.GAUSS was the visualization of complex numbers $z = x + iy$ by the real points $P(x,y)$ of the Cartesian plane (published in 1831, but used already in the thesis of 1799). This concept, developed also by J.R.ARGAND (1806) and C.WESSEL (1797), became the base of complex function theory.

Conversely, every point $P(x,y)$ of the real plane may now be fixed by a single (but complex) number $z = x + iy$ which unites the (real) cartesian coordinates of P. If P is given by means of polar coordinates r, ϕ, the "Gauss coordinate" z is determined, with use of L.EULER's famous formula, by

$$(4.1) \qquad z = x + iy = r(\cos\phi + i\sin\phi) = r\cdot e^{i\phi}$$

Fundamental operations with complex numbers turn out to be represented by elementary mappings in the Gaussian plane [2]. Thus the addition of a constant b to a variable number z means a *translation* $z \to z' = z + b$, whose translation vector is determined by the arrow from the origin O ($z = 0$) to the point B ($z' = b$). Multiplication with a constant $c = a\cdot e^{i\alpha}$ means, due to (4.1), a *similarity* $z \to z' = c\cdot z$ which is composed of a rotation about O with the angle $\alpha = \arg c$, and a dilatation from O with the factor $a = |c|$. Passing from z (4.1) to the conjugate $\bar{z} = x - iy = r \cdot e^{-i\phi}$ means the *reflexion* in the x-axis (which is the locus of all real numbers $z = \bar{z}$).

As to the *inversion* $\iota: P \to P'$, introduced with equ.(2.1), it is described in complex notation by

$$(4.2) \qquad z' = a^2/\bar{z} \quad \text{or} \quad \bar{z}z' = a^2$$

Inversion with respect to an arbitrary circle (center B, radius $c = \bar{c} > 0$) is represented in the form

$$(4.3) \qquad z' - b = c^2/(\bar{z} - \bar{b}) \quad \text{or} \quad \bar{z}z' - b\bar{z} - \bar{b}z + b\bar{b} - c^2 = 0.$$

The composition of two or more inversions leads to a so-called *Moebius transformation* [2, p.207], named after F.A.MOEBIUS (1855). Such a mapping is still conformal and circle-preserving like all of the component inversions, but in general no longer involutory. It is directly conformal, if the number of the inversion components is even, and indirectly conformal, if this number is odd. A Moebius mapping $\mu: P \to P'$ of the first kind is always representable in the form

$$(4.4) \qquad z' = \frac{az + b}{cz + d} \quad \text{or} \quad czz' - az + dz' - b = 0,$$

with the natural restriction $D = ad - bc \neq 0$; otherwise z' would be independent of z. It is easy to confirm that the product of two direct Moebius mappings μ_1 and μ_2 is again a transformation $\mu_3 = \mu_2\mu_1$ of the same kind. Hence, all direct Moebius transformation constitue a *group* (with six real parameters).

On the other hand, the indirect Moebius transformations, described by

$$(4.5) \qquad z' = \frac{a\bar{z} + b}{c\bar{z} + d} \quad \text{or} \quad c\bar{z}z' - a\bar{z} + dz' - b = 0,$$

do not form a group, as the composition of two of them produces a direct Moebius mapping.

In the special case $c = 0$ equ. (4.4) reduces to $z' = (az+b)/d$, and μ is then a *direct similarity*, in particular a congruent transplacement, if $|a/d| = 1$ ($|a| = |d|$). Analogously, the mapping (4.5) reduces with $c = 0$ to an *opposite similarity*, in particular to an indirect isometry, if $|a| = |d|$.

For $c \neq 0$ the point $z_0 = -d/c$ is an exceptional point of the mapping μ (4.4), as its image $z'_0 = \infty$ is indefinite on the line at infinity; in the terminology of function theory z_0 is a "pole". Supposing this pole at the origin -- a situation always achievable by appropriate choice of the coordinate system -- we may simply assume $d = 0$ and $c = 1$, so that μ is described by $z' = a + (b/z)$. We learn from this reduced form (and so from the corresponding form in the indirect case) that a true Moebius mapping applied to a curve doesnot produce other shapes than a simple inversion would do.

II. INDIRECTLY CONGRUENT-INVERSE CURVE PAIRS

5. Analytical Preparation

Let q be a plane curve which by a certain inversion ι is mapped onto an indirectly congruent copy $q' \cong q$. When σ denotes the isometry which brings q' back to q, then $\mu = \sigma\iota$ is a direct Moebius transformation which maps q onto itself. Herewith the question for *indirectly congruent-inverse curve pairs* q, q' is lead back to the determination of *curves* q *which are invariant under a given direct Moebius mapping* μ.

Locating the center O of the inversion ι at the origin of a Gaussian plane, the *inversion* $\iota : P \to P'$ is described by

(5.1) $\quad z' = a^2/\bar{z} \quad$ with $a = \bar{a} > 0$.

An opposite isometry σ may always be composed of a reflexion in a certain line h and a translation along h [2, p.193]; the direction of the mirror axis h is indicated by the sum of two corresponding vectors. Choosing the x-axis parallel to h, the *opposite isometry* $\sigma : P' \to P''$ may be written in the form

(5.2) $\quad z'' = \bar{z}' + b \quad$ with $b = b' + ib''$.

Writing this "glide-reflexion" in cartesian coordinates, it reads

(5.3) $\quad x'' = x' + b', \quad y'' = -y' + b''$.

We see that the mirror axis h is determined by $y' = y'' = b''/2$ and that the translation vector is given with b'.

Combining equs. (5.1) and (5.2) we find that the *Moebius mapping* $\sigma\iota = \mu: P \to P''$ is described by

$$(5.4) \qquad z'' = \frac{a^2}{z} + b \quad \text{or} \quad zz'' - bz - a^2 = 0.$$

First of all let us ask for the *fixed points* $z = z''$ of μ. They are determined by the solutions of the quadratic equation

$$(5.5) \qquad z^2 - bz - a^2 = 0.$$

Hence there exist *two real fixed points* $M = M''$ and $N = N''$ (Fig.7) given by their Gauss coordinates

$$(.6) \qquad z_1 = \frac{1}{2}(b + R), \quad z_2 = \frac{1}{2}(b - R) \quad \text{with } R^2 = b^2 + 4a^2.$$

These values satisfy the relations

$$(5.7) \qquad z_1 + z_2 = b, \quad z_1 z_2 = -a^2 = \bar{z}_1 \bar{z}_2 .$$

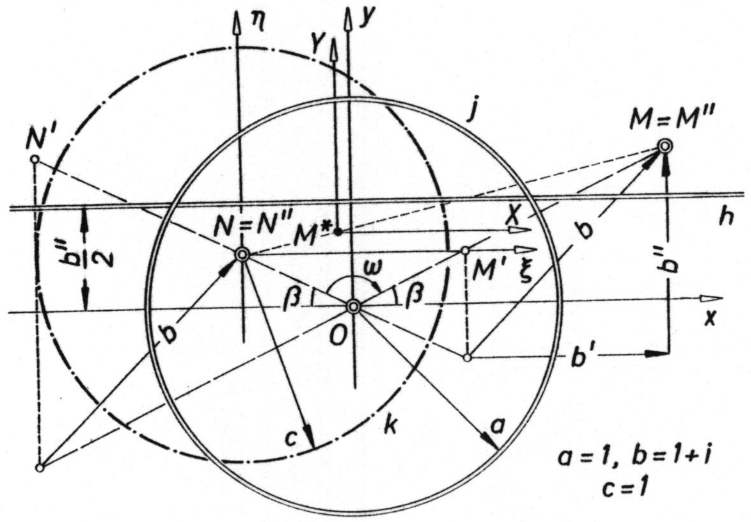


Fig.7: Direct Moebius mapping composed of an inversion and a glide-reflexion

Passing now by means of the translation $O \to N$ to a new system of Gaussian coordinates

(5.8) $\zeta = z - z_2 \quad (\zeta'' = z'' - z_2),$

the transformation formula (5.4) may be written as

(5.9) $\zeta\zeta'' - z_1\zeta + z_2\zeta'' = 0.$

The fixed points M and N are now given by $\zeta_1 = z_1 - z_2 = R$ and $\zeta_2 = 0$, respectively. They are distinct points for $R \neq 0$.

In a next step we apply an auxiliary *inversion* κ: $P \to P^*$ $(P'' \to P''^*)$ with arbitrary radius c, but having its center at N (Fig.7):

(5.10) $\zeta^* = c^2/\bar{\zeta} \quad (\zeta''^* = c^2/\bar{\zeta}'').$

Hereby the Moebius mapping μ (5.9) is changed into a *linear transformation* $\kappa\mu\kappa = \mu^*$: $P^* \to P''^*$, described by

(5.11) $\bar{z}_2\zeta^* - \bar{z}_1\zeta''^* + c^2 = 0.$

This mapping μ^* is a *direct similarity*, whose fixed point M^* is given with $\zeta_1^* = c^2/(\bar{z}_1 - \bar{z}_2) = c^2/\bar{R}$, provided $R \neq 0$.

In the *general case* $R \neq 0$ $(b \neq \pm 2ia)$ we introduce once more, again by translation, new coordinates

(5.12) $Z = \zeta^* - \zeta_1^* \quad (Z''^* = \zeta''^* - \zeta_1^*).$

Thus we finally obtain the "canonical" representation of μ^*:

(5.13) $\bar{z}_2 Z - \bar{z}_1 Z'' = 0$ or $Z'' = \Lambda \cdot Z$ with $\Lambda = \bar{z}_2/\bar{z}_1 = \rho \cdot e^{i\omega}.$

In general the similarity μ^* is composed of a rotation about the fixed point M^* $(Z_1 = 0)$ with the angle $\omega = \arg \Lambda$ and a dilatation from the same center with the factor $\rho = |\Lambda|$.

For $\omega \neq 0$ and $\rho \neq 1$ the similarity μ^* shows a certain spiral character [2, p.163] which becomes obvious by successive repetition of the mapping, as then any point is transported along a logarithmic spiral. The polar net about the fixed) origin M^* is, as a whole, invariant under μ^*. Returning to μ by means of the inversion κ (5.1o), the pencil of rays issuing from M^* is transformed into the elliptic pencil of circles passing through the fixed points M and N of μ (Fig.8), whereas the system of concentric circles about M^* is mapped onto the hyperbolic pencil of circles orthogonal to those of the elliptic pensil. This consideration offers a good insight into the effect of a direct Moebius mapping as illustrated in Fig.8, where the fixed points *M* and *N* have been placed at +i and -i.

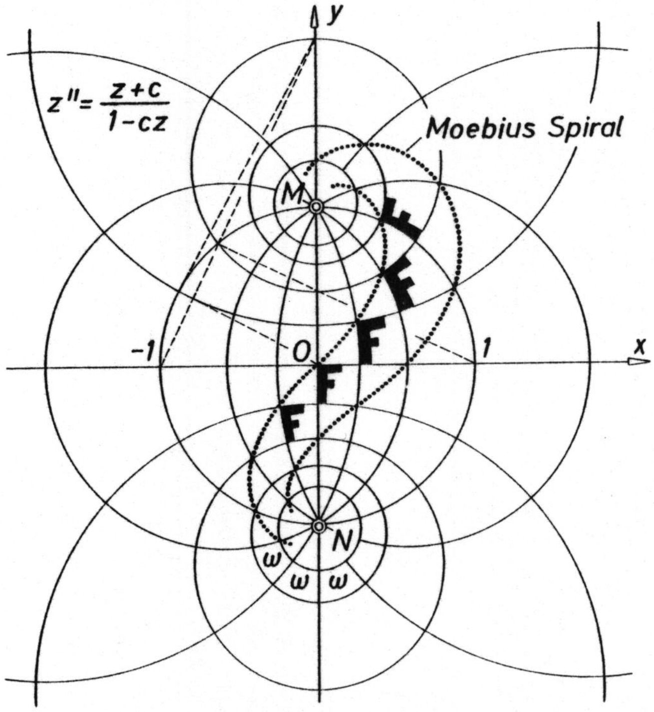


Fig.8: Invariant circle pencils of a direct Moebius mapping

By the chain of operations outlined above, the problem of curves q which are invariant under a given Moebius mapping μ has been lead back to the question for curves q^* which are invariant under a corresponding similarity μ^*. So we state

<u>Theorem 3:</u> A plane curve q which by a certain inversion is mapped onto an indirectly congruent copy q' is inverse to a curve q^* which admits a direct similarity or isometry.

This statemant proves to hold also in the postponed case $R = 0$ of coincident fixed points $M = N$ (Section 8). The next sections will show how to find the required curves q^*, invariant under the respective linear mapping μ^*.

6. Main Case

Before attacking the question for curves q^* which are invariant under the direct similarity μ^* (5.13), it is advisable to derive appropriate formulas for the characteristic constants ρ and ω of μ^*. With attention to (5.7) we find

(6.1) $\rho = |\Lambda| = |z_2/z_1| = a^2/z_1\bar{z}_1 = z_2\bar{z}_2/a^2.$

Denoting the argument of z_1 with β we have, due to (5.7), $\arg z_2 = \pi - \beta$, hence

(6.2) $\omega = \arg \Lambda = -\arg(z_2/z_1) = \arg z_1 - \arg z_2 = 2\beta - \pi \pmod{2\pi}.$

This means, in geometrical notation, $\omega = \sphericalangle NOM$ (Fig.7). Summarizing the meaning of $\Lambda = \bar{z}_2/\bar{z}_1$ (5.13), we note

<u>*Lemma 1:*</u> *The linear mapping μ^* (5.13) is indirectly congruent to the direct similarity which brings M to N and leaves O fixed.*

The present section is devoted to the *main case* $R \neq 0$ $(z_1 \neq z_2)$ with the additional assumption $b' = \mathrm{Re}\, b \neq 0$ which excludes the possibility $\rho = 1$: Due to the relations (5.7), $\rho = 1$ would induce $z_1 = a\cdot e^{i\beta}$, $z_2 = -a\cdot e^{-i\beta}$ and thus $b = z_1 + z_2 = 2ia \sin\beta$ without real part. The alternative $b' = 0$ which reduces the isometry σ (5.3) to a pure reflexion will be treated in Section 7.

The wanted curve q^*, invariant under the *"spiral similarity"* μ^*(5.13), may be described in parametric form $Z = Z(\phi)$. It will admit (as a whole) the "automorphism" μ^*, if

(6.3) $Z''(\phi) = \rho\cdot e^{i\omega} Z(\phi) = Z(\phi+\omega).$

In order to satisfy this relation, we write

(6.4) $Z(\phi) = e^{p\phi}e^{i\phi}\cdot w(\phi).$

The condition (6.3) will be fulfilled, if we put

(6.5) $\rho = e^{p\omega}$

and if the auxiliary function $w(\phi)$ satisfies the *functional equation*

$$(6.6) \qquad w(\phi + \omega) = w(\phi).$$

The substitution (6.5) is admissible, if $p = \omega^{-1} \ln \rho$ gives sense, hence if $\rho > 0$ and $\omega \neq 0$. Now, $\rho = z_2\bar{z}_2/a^2$ is surely positive, since $z_1 z_2 = -a^2 \neq 0$; on the other hand, $\omega = 2\beta - \pi = 0$ would suppose $\beta = \pi/2 \pmod \pi$, hence z_1, z_2 and $z_1 + z_2 = b$ were purely imaginary, which is excluded with $b' \neq 0$.

The remaining condition (6.6) means that $w(\phi)$ is a *periodic function* with the period $\omega \neq 0$. Such a function may be chosen arbitrarily by prescribing its values in the interval $0 \leq \phi \leq \omega$ with attention to the continuity restriction $w(0) = w(\omega)$. Finally the so conctructed auxiliary curve q^*, defined by equ. (6.4), has to be transformed back in recurring the substitutions (5.12), (5.10) and (5.8). The resulting curve q, obtained with

$$(6.7) \qquad z(\phi) = \frac{c^2}{\bar{Z}(\phi) + \bar{\zeta}_1^*} + z_2 \quad \text{with } \bar{\zeta}_1^* = c^2/R$$

- where R and z_2 are to be taken from equs. (5.6) --, represents a *solution* of the proposed problem: it will produce a congruent copy q', if transformed by the inversion ι (5.1). In general the auxiliary curve q^* shows, due to the automorphic similarity μ^*, a certain *periodic structure*: It consists of an infinite series of similar elementary arcs. This fact induces a correspondingly modified structure of the solution curve q (Figs.9a,b).

Searching for closed analytic functions $w(\phi)$, we have, apart from $w(\phi) = \text{const}$, trigonometric functions at our hands. These are, after introduction of the *"module"* $m = \pi/\omega$, in first line the functions $\cos 2m\phi$, $\sin 2m\phi$, and more generally

$$(6.8) \qquad w(\phi) = \sum_{j \text{ even}} (a_j \cos jm\phi + b_j \sin jm\phi)$$

with arbitrary complex coefficients a_j, b_j. Making use of Euler's formula (4.1), $w(\phi)$ might also be written as a sum of exponential functions $e^{ijm\phi}$ and $e^{-ijm\phi}$.

The simplest choice, $w(\phi) = a_0 \neq 0$, leads to a *logarithmic spiral* q^* with the equation

$$(6.9) \qquad Z = a_0 e^{p\phi} e^{i\phi}.$$

Such a spiral is an isogonal trajectory (or "loxodrome") of a ray pencil; in our case the spiral q^* intersects all rays issuing from the center M^* ($Z_1 = 0$) under the constant angle $\theta = \operatorname{arccot} p$ [3, p.221]. Consequently the inverse image q of q^*, whose equation would be obtained by means of formula (6.7), will be a θ-loxodrome of the elliptic pencil of circles with the base points M and N (Section 5). It may be called a *"Moebius spiral"*, as it is a point path of a continuous one-parametric group of Moebius transformations; some specimens of these double-spirals are to be seen in Figs. 8 and 9b.

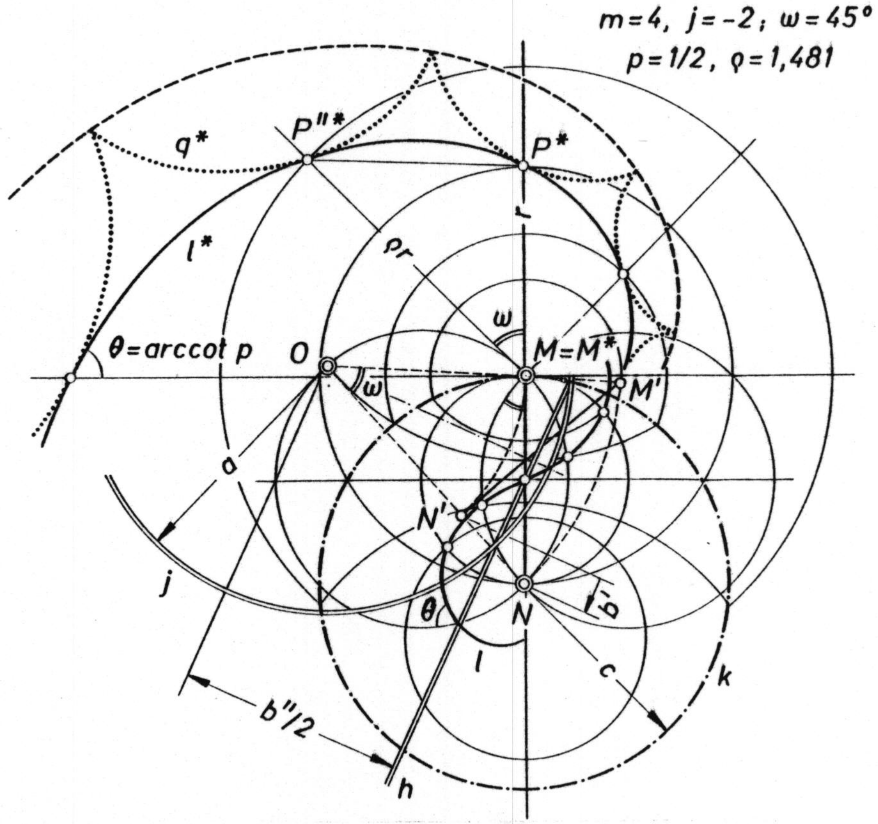


Fig.9a: Cusped spiraloid admitting a group of spiral similarities

Fig.9b shows a less trivial solution, generated by aid of the auxiliary function

(6.10) $$w(\phi) = a_0 + a_j e^{jmi\phi} \quad (j \text{ even}).$$

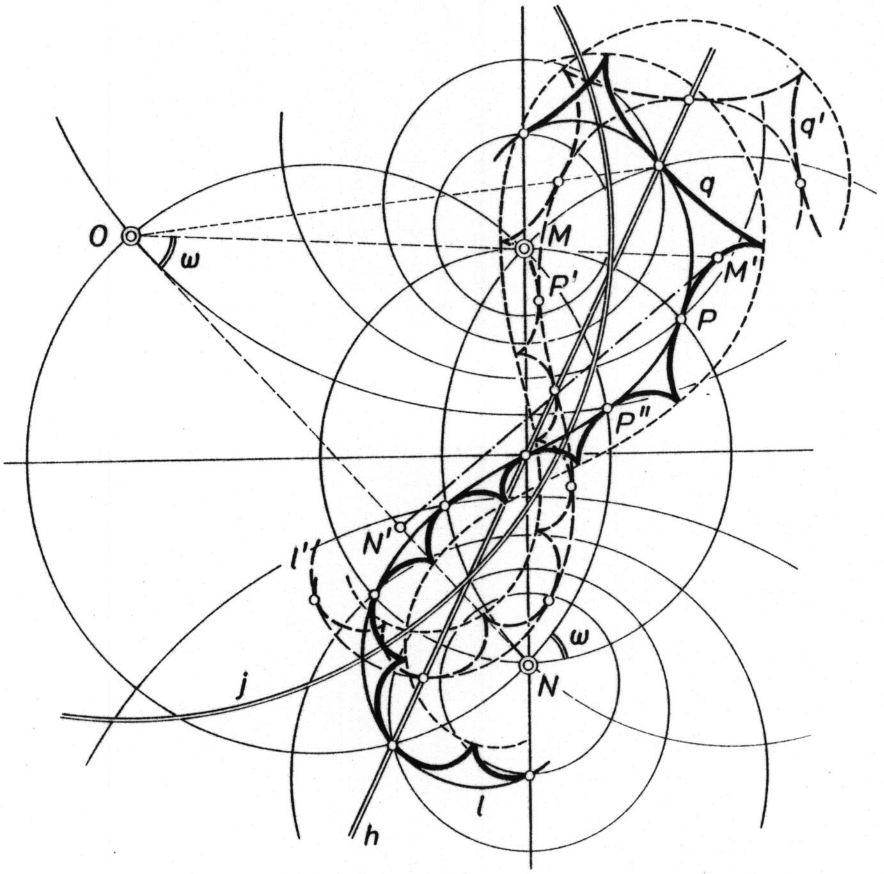


Fig.9b: Curve inverse to the spiraloid of Fig.9a and inverted into a glide-reflexion copy

The corresponding curve q^* (6.4) is a *"spiraloid"*, introduced by the author as a certain generalization of trochoids [5]. A spiraloid is cusped (as in Fig.9a), if

(6.11) $\quad |a_0|^2 : |a_j|^2 = [p^2 + (jm+1)^2] : (p^2+1).$

Such a cusped spiraloid may be generated as the envelope of a variable line which rotates with constant angular velocity $n = 1 + (jm/2)$ about one of its points, whereas this point moves, with the angular velocity 1 of its position vector, on a logarithmic spiral (6.9). The derived solution curve q is to be seen (in larger scale) in Fig.9b together with its congruent-inverse copy q'. It was constructed by means of the orthogonal system of circles which corresponds to the polar net about the origin M^* (compare Fig.8).

The most general solutions will be obtained analytically by a parametrization

(6.12) $\quad \phi = \psi(t) + \frac{t}{2m}, \quad w = \chi(t),$

where the arbitrary functions $\psi(t)$, real, and $\chi(t)$, complex, have the common period 2π. This modification has the effect that now the function $w(\phi)$ is not necessarily univalent as those functions which are given by a trigonometric polynomial (6.8). For a particular example see (7.9). -- Another way of enlarging the number of solutions consists in performing certain period-preserving operations with admissible functions $w(\phi)$, for instance by taking the reciprocal or any power or root of a chosen function $w(\phi)$, or by forming the product or the quotient of two functions $w_1(\phi)$ and $w_2(\phi)$. It may occur, however, that hereby the primitive period will reduce; so $w(\phi) = \tan m\phi$ has already the period ω and thus satisfies the condition (6.6).

Finally, let us consider the situation when we start with an arbitrarily chosen auxiliary curve q^* which is, as required, invariant under a spiral similarity μ^* (center M^*, rotation angle ω, dilatation factor ρ), e.g. the spiraloid of Fig.9a. Applying now an inversion κ with arbitrary center $N \neq M^*$ and arbitrary radius c, we obtain from q^* a curve q which will be a solution of our problem, as it is invariant under the Moebius transformation $\mu = \kappa\mu^*\kappa$. To reconstruct the inversion ι which maps q onto a congruent copy q', we choose an arbitrary pair of points P^*, P''^* corresponding each other in the similarity μ^*. Due to Lemma 1 and denoting with M the κ-image of M^*, we construct the triangle OMN indirectly similar to $M^*P^*P''^*$. Knowing now the inversion center O, we find the inversion radius a from $a^2 = OM \cdot ON$ and add the ι-images M', N' of M, N. The mirror axis h of the glide-reflexion σ is then parallel to MN' and NM' and passes through the midpoints of the segments MM' and NN' (Figs. 7 and 9a). The translation vector b' of σ is the component of $M'M$ (and $N'N$) parallel to h.

Now, with μ^* the initial curve q^* admits also the spiral similarities μ^{*n}, n integer. Hence there exist *infinitely many possible inversion centers* O_n, to be found in the exposed way after replacing ω by $n\omega$ and ρ by ρ^n. It can be shown that all these centers are situated on a *Moebius spiral*. For the proof let us locate the points M and N at the points +1 and -1 of a Gauss plane. For the Gauss coordinate z_n of O_n we have the relation

$$\frac{z_n + 1}{z_n - 1} = \rho^n \cdot e^{-in\omega} = e^{(p-i)n\omega}. \tag{6.13}$$

Putting $n\omega = \phi$, we see that all centers O_n lie on a Moebius transform of the logarithmic spiral $r = e^{-p\phi}$, that is a Moebius spiral. -- After all, Theorem 3 may be completed by

<u>Theorem 4:</u> *If an arbitrary curve q^* which admits a spiral similarity with the center M^*, is inverted from an arbitrary center $N \neq M^*$, the resulting curve q can be mapped by an infinity of suitable inversions onto an indirectly congruent copy. The centers of these inversions are situated on a Moebius spiral.*

Applying this theorem to a logarithmic spiral q^* (6.9), the angle ω is arbitrary; hence the resulting *Moebius spiral* q is accompanied by another Moebius spiral $\tilde{q}$ (of the same class) and can be inverted from any center O on $\tilde{q}$ into a congruent copy $q' \simeq q$. In particular, if q is a Moebius spiral with a point at infinity, $\tilde{q}$ is identical with q.

7. Reflexional Case

This section is devoted to the postponed case $b' = 0$. Here the constant $b = b' + ib''$, introduced in equ. (5.2), has a purely imaginary value $b = ib''$. Consequently (see Fig.7), the opposite isometry $\sigma: P' \to P''$ (5.3) reduces to a pure *reflexion* in the mirror axis h ($y' = y'' = b''/2$). Apart from the limit case $R = 0$ (which will be treated in Section 8) we have to distinguish between *two subcases* depending on the sign of the (real) discriminant $R^2 = 4a^2 + b^2 = 4a^2 - b''^2$ in equ. (5.6).

First Subcase: $|b''| < 2a$ ($R^2 > 0$, R real).

The Gauss coorciantes of the fixed points M and N of the Moebius mapping μ (5.4) read, according with formulas (5.6):

(7.1) $\quad z_1 = \frac{1}{2}(ib'' + R), \; z_2 = \frac{1}{2}(ib'' - R)$ with $R^2 = 4a^2 - b''^2 > 0.$

We state that the fixed points M and N are the intersection points of the mirror axis h with the inversion circle j. Their invariance under μ is evident.
Furthermore we see, in accordance with Lemma 1 (Section 6), that the κ-transform of μ, i.e. the linear mapping μ^* (5.13), is a *rotation* ($\rho = 1$), whose angle ω is determined by

(7.2) $\qquad \cos(\omega/2) = b''/2a.$

This circumstance induces a *cyclic symmetry* of the auxiliary curve q^*. Thus we obtain as a modification of Theorem 4

Theorem 5: If an arbitrary curve q^*, admitting a rotation about its center M^* with an angle ω, is inverted from an arbitrary point $N \neq M^*$, the resulting curve q can be mapped by suitable inversions onto a mirror copy q'. The base circles of these inversions all pass through M and N (M corresponding to M^* in the inversion $q^* \to q$), and the chord MN is seen from the circle center under the angle ω or an integer multiple of ω.

With $\rho = 1$ we have $p = 0$ in equ. (6.4). Hence the analytical solution of the problem begins with the equation

(7.3) $\quad Z(\phi) = e^{i\phi} \cdot w(\phi)$ with $w(\phi + \omega) = w(\phi),$

where the arbitrary periodic function $w(\phi)$ may be chosen in the same way as in Section 6. The so determined auxiliary curve q^* (7.3) has then to be transformed by means of formula (6.7) to obtain a solution curve q.

Due to its automorphic rotation μ, the auxiliary curve q consists of a sequence of congruent elementary arcs, each one corresponding to a period interval of the function $w(\phi)$. The number of these constituents is infinite in general, but *finite* for a *rational module* $m = \pi/\omega$: the curve q is then *closed*. Writing in such a case

$$(7.4) \qquad m = \lambda/\nu \quad (\lambda, \nu \text{ positive integers without common factor}),$$

we state: If the denominator ν is odd, then the curve q consists 2λ congruent arcs and ν-times encircles the center before closing; if ν is even, the respective numbers are λ and $\nu/2$. Taking for $w(\phi)$ a finite trigonometric polynomial of the pattern (6.8), it can be shown -- by means of the substitution $e^{i\phi/\nu} = \tau$ and use of Euler's formula (4.1) -- that the corresponding curves q^* and q are *algebraic* and *rational*, i.e. of genus 1.

The simplest examples are obtained with

$$(7.5) \qquad w(\phi) = a_0 + a_j e^{jmi\phi} \quad (j \text{ even}).$$

The auxiliary curve q^* is then a *trochoid*, in particular a cusped one, if $|a_0| = |jm+1|$. A solution belonging to the class $m = 3/2$ ($\omega = 120°$) and $j = -2$ is illustrated in Fig.10. Choosing for q^* Steiner's three-cusped hypocycloid |3, p.142|, a monocirculas quartic touching the line at infinity at the circular points I and J, the derived solution curve q is a tricirular sextic with three real cusps and an isolated double point at N. There exist two possible inversions (bas circles j_1 and j_2, see Theorem 5) which map q onto a mirror copy q'.

The special case $m = 1$ ($\omega = 180°$) deserves special attention, as it was mentioned already by J.L.KAVANAU |2, p.379|. Here the automorphic rotation μ is a half-turn, with other words q has a *center of symmetry*, M. Due to formula (7.2) we have $b'' = 0$, hence the mirror axis $\bar{h} = MN$ passes through the inversion center O (which is the midpoint of the segment MN).

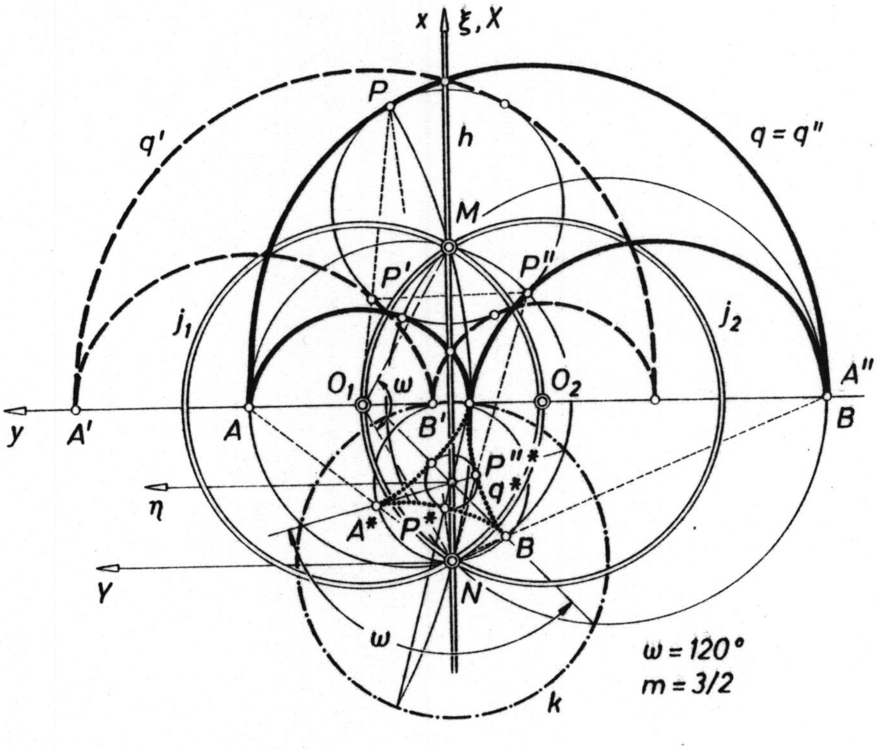


Fig.10: Tricircular sextic, inverse to Steiner's hypocycloid and inverted into a mirror copy

KAVANAU's statement that any inverse image q of an arbitrary centric curve q' admits an inversion which maps it onto a mirror copy q' appears now as a corollary of Theorem 5. The Moebius mapping μ (5.4), described by $zz'' = a^2$, is involutory (a "Moebius involution"). -- A simple example is to be seen in Fig.11 (and also in |2, p.379|). It shows a bicircular quartic q, derived from an ellipse q'; this example also belongs to the class (7.5), where it is to be found with $m = 1$, $j = -2$. Another example ($m = 4$, $\omega = 45°$) is contained in Fig.26.

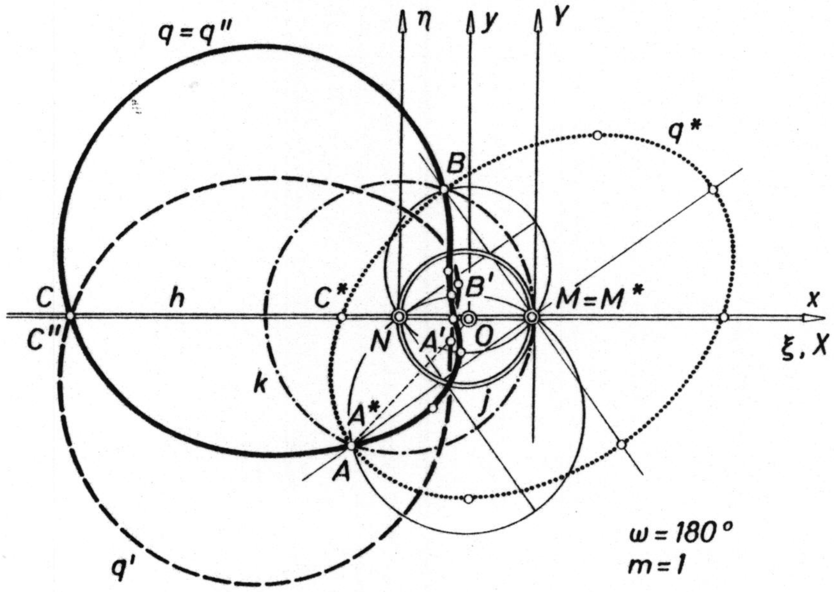


Fig.11: Bicircular quartic, inverse to an ellipse and inverted into a mirror copy

Starting with a centric Moebius spiral q' (like the curve l in Fig.9b) and inverting it from one of its two asymptotic points, we obtain a ***logarithmic spiral*** q |3, p.221|. This leads to the well-knon property of the logarithmic spiral that it produces, if inverted from its center, concentric mirror copy.

Second Subcase: $|b''| > 2a$ ($R^2 < 0$, R purely imaginary)

The Gauss coordinates of the fixed points M and N of the Moebius mapping μ (5.4) read, according with formulas (5.6):

(7.6) $z_1 = \frac{i}{2}(b'' + S)$, $z_2 = \frac{i}{2}(b'' - S)$ with $S^2 = b''^2 - 4a^2 > 0$.

Hence M and N are on the y-axis and symmetrical to the mirror axis h ($y = b''/2$); moreover they correspond each other in the inversion ι (5.1), as $z_1\bar{z}_2 = \bar{z}_1 z_2 = a^2$. These fixed points $M = N'$

and $N = M'$ may be constructed by intersecting the y-axis with an anallagmatic circle $o = o'$ (orthogonal to j) which is centered on h.

As the factor $\Lambda = \bar{z}_2/\bar{z}_1$ in equ. (5.13) is real, the direct similarity μ' reduces to a pure *dilatation* ($\omega = 0$) from the center M'; it is characterized by the factor

$$(7.7) \qquad \rho = (b'' - S)^2/4a^2 > 0.$$

To avoid the failure of formula (6.5), we replace $\omega = 0$ by $\omega = 2\pi$. Thus we arrive, in analogy to (6.4) and with suppression of the superfluous factor $e^{i\phi}$, to the equation

$$(7.8) \qquad Z = e^{p\phi} \cdot w(\phi) \text{ with } 2\pi p = \ln \rho \text{ and } w(\phi + 2\pi) = w(\phi)$$

which describes the auxiliary curve q'. The corresponding curve q is then obtained by means of the transformation (6.7), where $R = iS$. Suitable periodic functions $w(\phi)$ are again to be found in Section 6 for $m = 1/2$.

The simplest choice, $w(\phi) = r_0 e^{i\phi}$, determines a logarithmic spiral q' and leads again to a *Moebius spiral* q.

To have an example for a periodic function $w(\phi)$ given in parametric form (6.12), let us take

$$(7.9) \qquad \phi = t - \frac{1}{2}\sin 2t, \quad w = 1 + \frac{i}{3}\cos t.$$

The so determined auxiliary curve q' is distinguished (due to a special choice of the coefficients) by two series of cusps distributed over two rays l_1', l_2' (Fig.12a). Correspondingly, the derived solution curve q shows two series of cusps on two circles l_1, l_2 passing through the fixed points M and N of μ (Fig.12b).

Finally we state

<u>*Theorem 6:*</u> *If an arbitrary curve q', admitting a dilatation with the factor ρ from a center M', is inverted from an arbitrary center $N \neq M'$, the resulting curve q can be mapped onto a mirror copy q' by suitable inversions, whose centers are all collinear with M' and N. These centers O_n divide the segment MN corresponding to the ratio $MO_n : NO_n = 1 : \rho^n$ (n integer).*

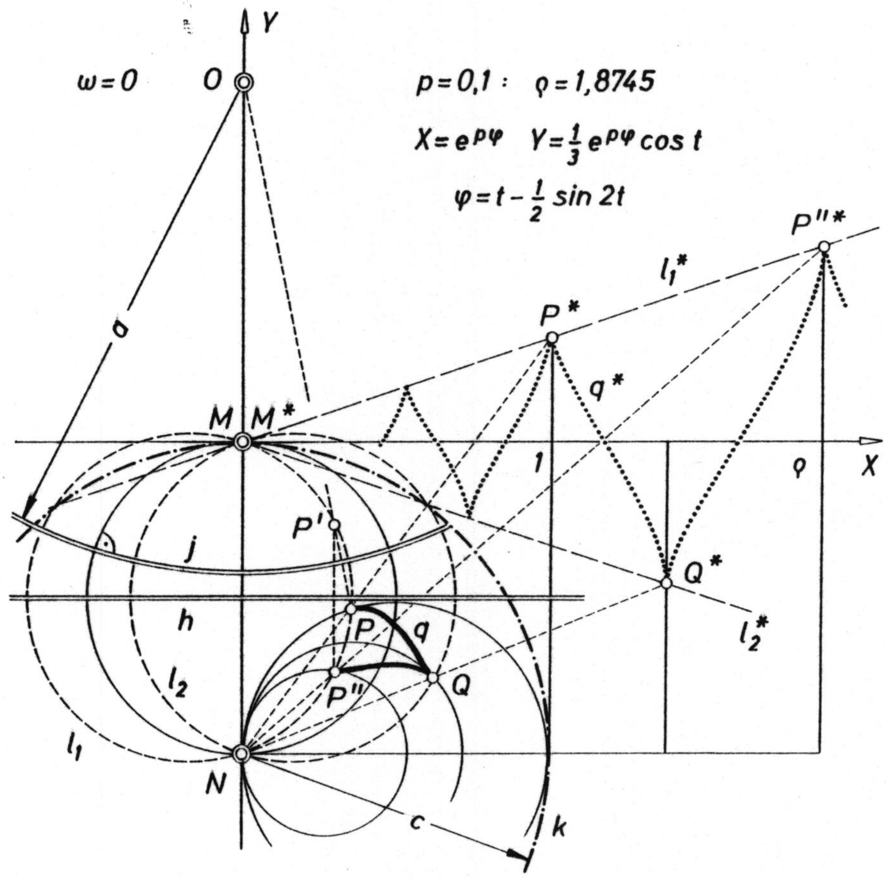


Fig.12a: Auxiliary curve admitting a group of dilatations

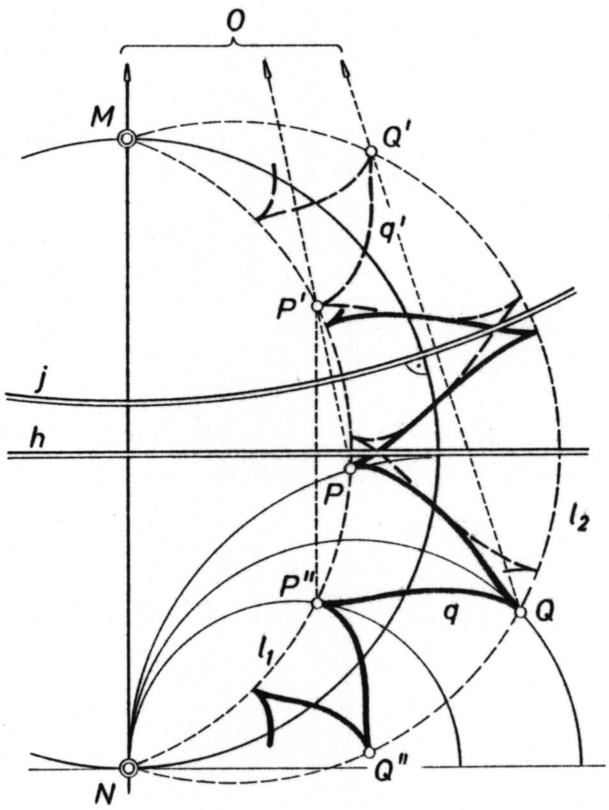


Fig.12b: Solution curve inverse to the auxiliary curve of Fig.12a and inverted into a mirror copy

8. Limit Case

A case still open concerns the possibility of a vanishing discriminant $R^2 = 0$ in equs. (5.6). This possibility arises with $b = \pm 2ia$ and thus belongs to the reflexional case $b' = 0$ (Section 7). It is sufficient to consider only the supposition $b = 2ia$. Now we have *coinciding fixed points* $M = N$ $(z_1 = z_2 = ia)$ at the contact point of the mirror axis h $(y = a)$ with the inversion circle j. The equation (5.9) of the Moebius mapping μ has the form

(8.1) $$\zeta\zeta'' + ia(\zeta'' - \zeta) = 0.$$

The application of the inversion κ (5.10), having its center at $N = M$ and (for simplicity) the radius $c = a$, leads to the linear mapping $\mu^* = \kappa\mu\kappa$:

(8.2) $$\zeta''^* - \zeta^* = ia.$$

We see that now μ^* is a *translation* along the y-axis. As the coordinate transformation (5.12) fails (the fixed point M^* of μ^* being at infinity), we simply put $Z = \zeta^*$ and write

(8.3) $$Z'' = Z + ia.$$

The question for auxiliary curves q^* which admit the translation μ^* (8.3) is answered to by

(8.4) $$Z(\phi) = \frac{ia}{2\pi}\phi + w(\phi) \quad \text{with } w(\phi + 2\pi) = w(\phi).$$

The arbitrary periodic function $w(\phi)$ is just the same as before and may be taken from Section 6 with $m = 1/2$. The curve q^* determined in this way finally has to be transformed back by means of

(8.5) $$z = \frac{a^2}{\bar{Z}} + ia,$$

in order to obtain the corresponding solution curve q.

Theorem 7: *If an arbitrary periodic curve q^*, admitting a translation of length a, is inverted from an arbitrary center M, the resulting curve q is mapped onto a mirror copy q' by the inversion in a circle of radius a, passing through M and touching there the mirror axis which is perpendicular to the translation lines. The radius a may be the primitive period of q^* or any integer multiple of it.*

Fig.13 shows as an example a solution curve q derived from a *cycloid* q . The corresponding auxiliary function is of the form $w(\phi) = f + g\,e^{i\phi}$.

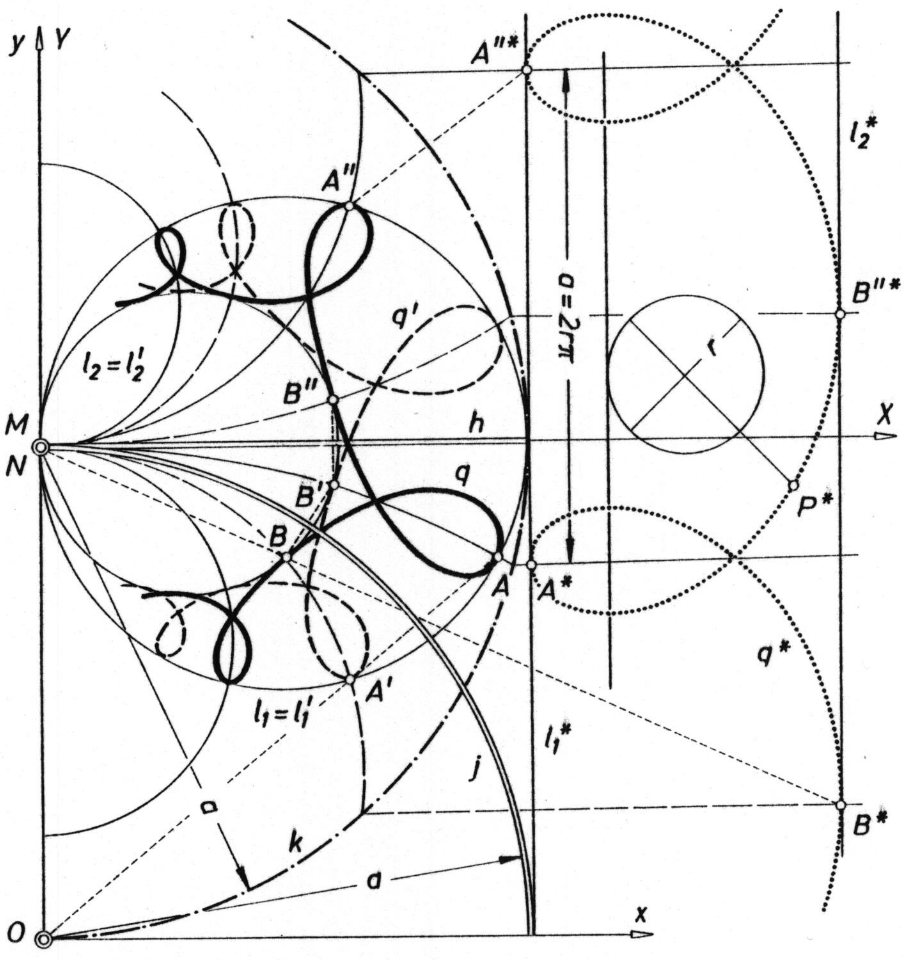


Fig.13: Curve inverse to a cycloid and inverted into a mirror copy

III. DIRECTLY CONGRUENT-INVERSE CURVE PAIRS

9. Preparation

Let q be a plane curve which by a certain inversion ι is mapped onto a directly congruent copy q'. When σ denotes the direct isometry which brings q' back to q, then $\mu = \sigma\iota$ is an indirectly conformal Moebius transformation (Section 4) which maps q onto itself. Herewith the question for ***directly congruent-inverse curve pairs*** q, q' is led back to the determination of *curves* q *which are invariant under a given indirect Moebius mapping* μ. The difference to the similar situation in Chapter II consists in the circumstance that now μ is indirectly conformal and therefore does not necessarily possess real fixed points.

Placing again the inversion center O at the origin of a Gaussian plane, the *inversion* $\iota: P \to P'$ is described by

$$(9.1) \qquad z' = a^2/\bar{z} \quad \text{with } a = \bar{a} > 0$$

The *transplacement* $\sigma: P' \to P''$ may be composed of a rotation about the center O (angle α) and a translation. Under the admissible assumption that the x-axis is parallel to the translation vector, the isometry σ will be described by

$$(9.2) \qquad z'' = e^{i\alpha} z' + b \quad \text{with } b = \bar{b} \geq 0.$$

It should be noticed that the transplacement σ can be performed by means of a simple rotation about the fixed point H_1 of σ which is given with

$$(9.3) \qquad z'_0 = z''_0 = \frac{b}{1 - e^{i\alpha}} = \frac{b}{2}\left(1 + i \cot\frac{\alpha}{2}\right).$$

The special case $\alpha = 0$ shows certain peculiarities, but must not be rejected. For $b \neq 0$ the isometry σ is a pure translation along the x-axis (rotation pole H_1 at infinity); for $b = 0$ σ reduces to identity, so that this case ($\mu = \iota$) leads to anallagmatic curves $q = q'$ (Sections 3 and 12).

Combining equs. (9.1) and (9.2), we obtain for the *Moebius mapping* $\sigma\iota = \mu: P \to P''$ the representation

$$(9.4) \qquad z'' = \frac{a^2 e^{i\alpha}}{\bar{z}} + b \quad \text{or} \quad \bar{z}z'' - b\bar{z} - a^2 e^{i\alpha} = 0.$$

If there exists a real *fixed point* of the Moebius transformation μ (9.4), its Gauss coordinate $z = z''$ will satisfy the conjugate pair of equations

$$z\bar{\bar{z}} - b\bar{z} = a^2 e^{i\alpha}, \quad z\bar{\bar{z}} - bz = a^2 e^{-i\alpha}. \tag{9.5}$$

Considering, for a moment, z and $\bar{z}$ as independent quantities, we find, with attention to

$$b(z - \bar{z}) = a^2(e^{i\alpha} - e^{-i\alpha}) = 2ia^2 \sin\alpha, \tag{9.6}$$

two possible value pairs:

$$\begin{aligned} z_{12} &= (b^2 + 2ia^2\sin\alpha \pm R)/2b \\ \bar{z}_{12} &= (b^2 - 2ia^2\sin\alpha \pm R)/2b \end{aligned} \quad \text{with } R^2 = (b^2 + 2a^2\cos\alpha)^2 - 4a^4. \tag{9.7}$$

These values satisfy the relations

$$z_1 + \bar{z}_2 = \bar{z}_1 + z_2 = b, \; z_1 z_2 = -a^2 e^{-i\alpha}, \; \bar{z}_1\bar{z}_2 = -a^2 e^{i\alpha}. \tag{9.8}$$

Apart from the exceptional case $b = 0$ which will be treated in Section 12 ("cyclic case"), further developments will essentially depend on the sign of the (real) discriminant R^2. For $R^2 > 0$ (R real), the value pairs $z_1, \bar{z}_1$ and $z_2, \bar{z}_2$ are pairs of conjugate complex numbers, hence determine two real fixed points $M = M''$ and $N = N''$ of the Moebius mapping μ ("hyperbolic case", Section 10). In the limit case $R = 0$ the fixed points coincide in a real point $M = N$ ("parabolic case", Section 11). For $R^2 < 0$ (R purely imaginary), there exist no real fixed points of μ; this "elliptic case" (Section 13) can be led back to the cyclic case ($b = 0$) which also belongs to the category $R^2 < 0$.

10. Hyperbolic Case

We suppose $R^2 > 0$, hence $b > 2a|\sin(\alpha/2)|$. The values z_1, z_2 (9.7), determining the *real fixed points* M and N of the Moebius mapping μ (9.4), have equal imaginary part. Hence M and N lie on the line $by = a^2\sin\alpha$, parallel to the x-axis (Fig.14) or coinciding with it if $\alpha = 0$. The fixed points may be constructed by means of two auxiliary circles invariant under μ. The first one, h_1, is centered at the pole H_1 (9.3) and orthogonal to the inversion circle j, hence self-inverse (Property iv in Section 2); its invariance is evident, as σ is a rotation about H_1. The second circle, h_2, has its center H_2 at the point with the coordinates $x = b/2$, $y = -x\tan(\alpha/2)$ and meets j at opposite points; due to Property v its invariance is easily checked. The circles h_1 and h_2 intersect each other in the required fixed points M and N (Fig.14).

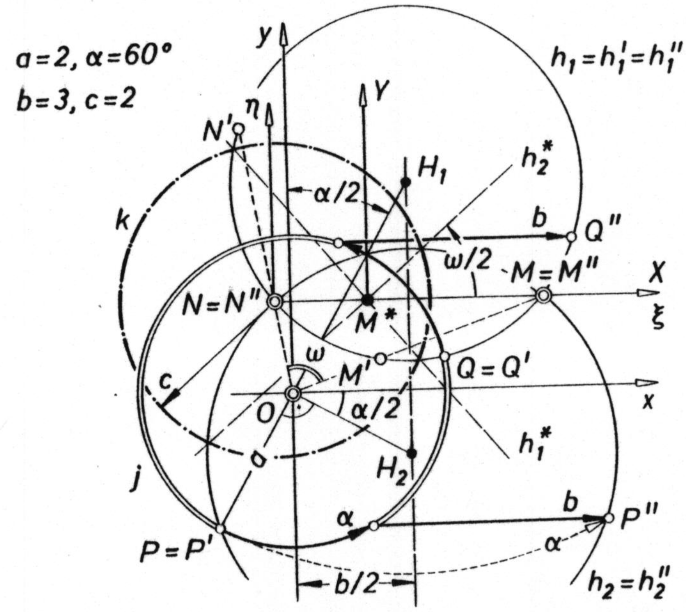


Fig.14: Indirect Moebius mapping, composed of an inversion and a transplacement (hyperbolic case)

Introducing -- like in Section 5, equ. (5.8) -- new coordinates with

(10.1) $\zeta = z - z_2 \quad (\zeta'' = z'' - z_2)$,

the equation (9.4) of the Moebius mapping μ takes, due to (9.8), the form

(10.2) $\bar{\zeta}\zeta'' - \bar{z}_1\bar{\zeta} + \bar{z}_2\zeta'' = 0.$

The new coordinates of the fixed points M and N are now $\zeta_1 = z_1 - z_2 = R/b$ and $\zeta_2 = 0$, respectively.

In a next step we apply an *inversion* $\kappa: P \to P'$ $(P'' \to P''')$ from the center N and with an arbitrary radius c:

(1o.3) $\zeta' = c^2/\bar{\zeta}$ $(\zeta''' = c^2/\bar{\zeta}'')$.

Herewith μ is transformed into a linear mapping $\kappa\mu\kappa = \mu' : P' \to P'''$, described by

(10.4) $z_2\bar{\zeta}' - z_1\zeta''' + c^2 = 0$.

This is an *indirect similarity*, whose finite fixed point M' is given by $\zeta_1' = c^2/(\bar{z}_1 - \bar{z}_2) = bc^2/R$; N' is at infinity $(\zeta_2' = \infty)$. Anew translating the coordinate system by means of

(10.5) $Z = \zeta' - \zeta_1'$ $(Z'' = \zeta''' - \zeta_1')$, ,

we attain the canonical form

(1o.6) $z_2\bar{Z} - z_1 Z'' = 0$ or $Z'' = \Lambda \cdot \bar{Z}$ with $\Lambda = z_2/z_1 = \rho \cdot e^{i\omega}$.

Corresponding to this expression the similarity μ' is composed of a reflexion in the X-axis, a rotation about the fixed point M' (angle ω) and a dilatation (factor ρ) from the center M'. The rotational component may be eliminated by replacing the mirror axis $Y = 0$ by another one which makes with it the angle $\omega/2$ (Fig.14). Hence μ' may be denoted as a "dilative reflexion" |2, p.198|. With respect to $\Lambda = z_2/z_1$ we have

<u>*Lemma 2:*</u> *The linear mapping μ' in the hyperbolic case is an indirect similarity with the fixed point M'. It is composed of a dilatation from M' with the factor $\rho = ON:OM$, and a reflexion in an axis passing through M' and forming with the line MN the angle $\omega/2$, where $\omega = \sphericalangle MON$.*

Consulting Fig.14 we see that the mentioned mirror axis is the κ-image h_2' of the second auxiliary circle h_2. The first circle, h_1, is mapped by the inversion κ onto another line h_1, perpendicular to h_2^* and also invariant under the dilative reflexion μ'. Consequently, the orthogonal circle pair h_1, h_2 is invariant under μ, a property already used in constructing the fixed points M and N. -- The effect of an indirect Moebius mapping μ is illustrated in Fig.15; for simplicity, the fixed points have been placed at +i and -i.

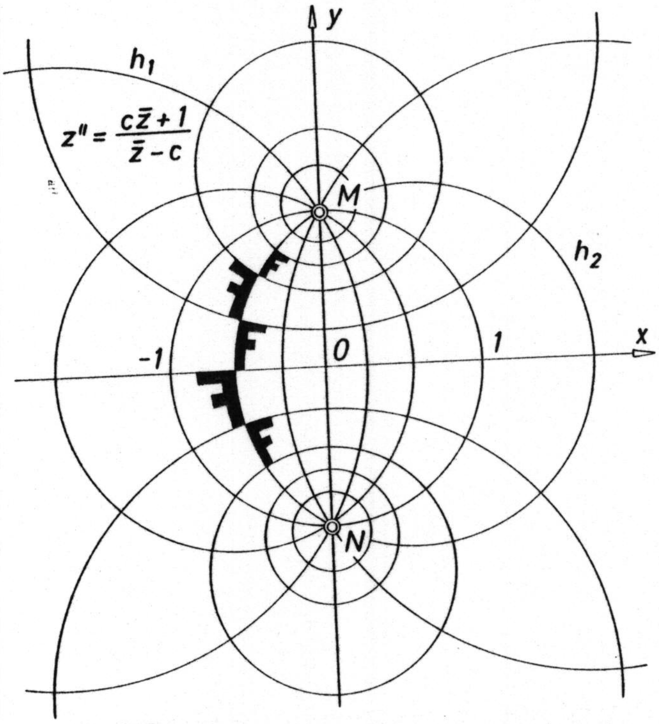


Fig.15: Invariant circle pencils of an indirect Moebius mapping with two real fixed points

For analytical purposes it will be convenient to have formulas for the characteristic constants ρ and ω. From $\Lambda = z_2/z_1$ we find with attention to relations (9.8):

(10.7) $$\rho = |\Lambda| = a^2/z_1\bar{z}_1 = z_2\bar{z}_2/a^2.$$

Putting $\arg z_1 = \beta$, we have $\arg z_2 = \pi - \alpha - \beta$, hence

(10.8) $$\omega = \arg \Lambda = \arg z_2 - \arg z_1 = \pi - \alpha - 2\beta \pmod{2\pi}$$
$$\text{with } \sin\beta = a^2 \sin\alpha / b|z_1| = (a/b)\sqrt{\rho}\,\sin\alpha.$$

The possibility $\rho = 1$ can be excluded, as then μ' would be a pure reflexion, hence μ a pure inversion (Property vii in Section 2). This case, characterized by $\alpha = b = 0$, leads only to anallagmatic curves $q = q'$ (Section 3).

In order to find curves q^* which are *invariant under the similarity* μ^* (10.6). we look for functions $Z(\phi)$ with the property

(10.9) $$Z^*(\phi) = \rho\, e^{i\omega}\, \bar{Z}(\phi) = Z(\phi+\omega).$$

Expressing such a function in the form

(10.10) $$Z(\phi) = e^{p\phi} e^{i\omega/2} \cdot w(\phi) \text{ with } e^{p\omega} = \rho,$$

the relation (10.9) holds, if the auxiliary function $w(\phi)$ satisfies the condition

(10.11) $$w(\phi + \omega) = \bar{w}(\phi).$$

This yields for the real and imaginary parts of $w = u + iv$ the *functional equations*

(10.12) $$u(\phi + \omega) = u(\phi), \quad v(\phi + \omega) = -v(\phi).$$

Both of these *arbitrary functions* are *periodic*: $u(\phi)$ has the period ω, $v(\phi)$ the period 2ω. In principle the values of both of these (real) functions might be arbitrarily prescribed in the interval $0 \leqq \phi \leqq \omega$, with attention to the continuity restrictions $u(\omega) = u(0)$ and $v(\omega) = -v(0)$. It may be remarked that $\omega = 0$ cannot occur, as this would suppose the excluded value $\rho = 1$.

Having now chosen a suitable function $w(\phi) = u(\phi) + iv(\phi)$, the corresponding curve q^* (10.10) has to be transformed back in recurring equs. (10.5), (10.3) and (10.1). The resulting curve q is represented by

(10.13) $$z(\phi) = \frac{c^2}{\bar{Z}(\phi) + \bar{\zeta}_1^*} + z_2 \text{ with } \zeta_1^* = bc^2/R,$$

where R and z_2 are to be taken from equ. (9.7). The curve q represents a *solution* of our problem: Transformed by means of the inversion ι (9.1), q will produce a congruent copy q'.

The auxiliary curve q^* (10.10) shows, due to its automorphic similarity μ^*, a certain periodic structure: It consists of an infinite series of similar elememtary arcs. This circumstance induces a corresponding structure of the solution curve q (Fig.16).

Asking for closed analytical function pairs $u(\phi)$, $v(\phi)$ we may, after introduction of the module $m = \pi/\omega$, use the following trigonometrical polynomials or convergent series:

$$u(\phi) = \sum (a_j \cos jm\phi + b_j \sin jm\phi),$$
(10.14)
$$v(\phi) = \sum (a_j \cos jm\phi + b_j \sin jm\phi)$$

with arbitrary real coefficients a_j, b_j. Fig.16 shows a simple example, derived from a function of the kind

(10.15) $$w(\phi) = a_0 + ib_1 \sin m\phi.$$

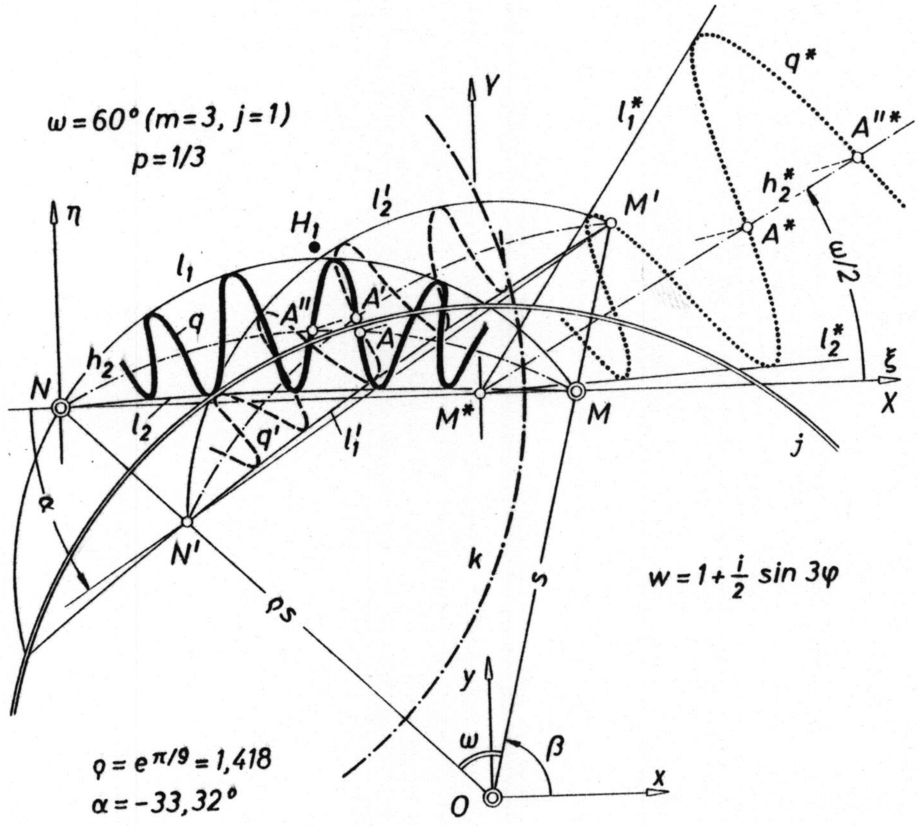


Fig.16: Curve inverted into a directly congruent copy (hyperbolic type)

The corresponding auxiliary curve $q^{\cdot}$ is known as orthogonal projection of a cylindro-conical loxodrome.

The most general solutions would be obtained by simultaneous parametrization of the functions $u(\phi)$ and $v(\phi)$ in the manner of equs. (6.12).

Starting with an arbitrary suitable curve $q^{\cdot}$ and going back through the preceding developmemts, we state

Theorem 8: *If an arbitrary periodic curve $q^{\cdot}$, admitting a dilative reflexion (center $M^{\cdot}$, factor ρ, axis $h_2^{\cdot}$), is inverted from an arbitrary center $N \neq M^{\cdot}$, the resulting curve q can be mapped onto a directly congruent copy q' by means of a suitable inversion. The center O of such an inversion is determined by the relations $OM:ON = 1:\rho$ and $\sphericalangle MON = \omega$, where M denotes the image of $M^{\cdot}$ and $\omega/2$ is the angle which the axis $h_2^{\cdot}$ makes with the line MN. Since the factor ρ may be replaced by any odd power ρ^n, all of the possible inversion centers lie on a circle passing through M and N.*

11. Parabolic Case

In the limit case of a vanishing discriminant we have $R = 0$ and, still postponing the "cyclic case" $b = 0$ (Section 12), $b > 0$. From equs. (9.7) we take

(11.1) $$b = 2a\sin(\alpha/2) \quad \text{with } \alpha \neq 0,$$

and

(11.2) $$z_1 = z_2 = a(\sin\frac{\alpha}{2} + i\cos\frac{\alpha}{2}) = ia\,e^{-i\alpha/2}.$$

Hence the fixed points of the Moebius mapping μ (9.4) coincide in a single (real) point $M = N$, situated on the inversion circle. Both of the invariant circle pencils are now parabolic.

After performing the steps (10.1) and (10.3) we arrive at the *linear mapping* $\mu^{\cdot} = \kappa\mu\kappa$:

(11.3) $$z_1(\bar{\zeta}^{\cdot} - \zeta''^{\cdot}) + c^2 = 0.$$

Putting, for simplicity, $c = a$ and

(11.4) $$\zeta^{\cdot} = Z - \frac{ia}{2}\cos\frac{\alpha}{2},$$

we obtain the canonical form

(11.5) $$Z'' = \bar{Z} + s \text{ with } s = a \sin\frac{\alpha}{2} = \frac{b}{2},$$

which shows that μ is an opposite isometry. From its cartesian representation

(11.6) $$X'' = X + s, \quad Y'' = -Y$$

we see that this *glide-reflexion* |2, p.193| is composed of a reflexion in the X-axis and a translation of length $s = b/2$ along that axis. In other words we have

<u>Lemma 3:</u> *The linear mapping μ in the parabolic case is a glide-reflexion parallel to the x-axis which brings the inversion center O to the fixed point M = N.*

In order to find appropriate auxiliary curves q which admit the glide-reflexion μ, we suppose for their representing function $Z(\phi)$ the relation

(11.7) $$Z''(\phi) = \bar{Z}(\phi) + s = Z(\phi + \pi).$$

Putting

(11.8) $$Z(\phi) = \frac{s\phi}{\pi} + w(\phi),$$

the relation (11.7) requires the condition

(11.9) $$w(\phi + \pi) = \bar{w}(\phi).$$

With respect to the similar condition (10.11), suitable functions $w(\phi) = u(\phi) + iv(\phi)$ are given by formulas (10.14) with $m = 1$, or corresponding parametric representations. Any periodic curve q determined in this way has then to be transformed back by means of

(11.10) $$z(\phi) = \frac{2a^2}{2\,\bar{Z}(\phi) + ia\cos(\alpha/2)} + ia\,e^{-i\alpha/2}.$$

Starting with an arbitrarily given suitable curve q we have

<u>Theorem 9:</u> *Let q be an arbitrary periodic curve admitting a glide-reflexion μ with the translation vector s, and O,M an arbitrary pair of corresponding points of μ. Then the inversion in the circle with the center M and passing through O transforms q into a curve q with the following property: Its inverse image q′ with respect to the circle centered at O and passing through M is directly congruent to q; q and q′ are*

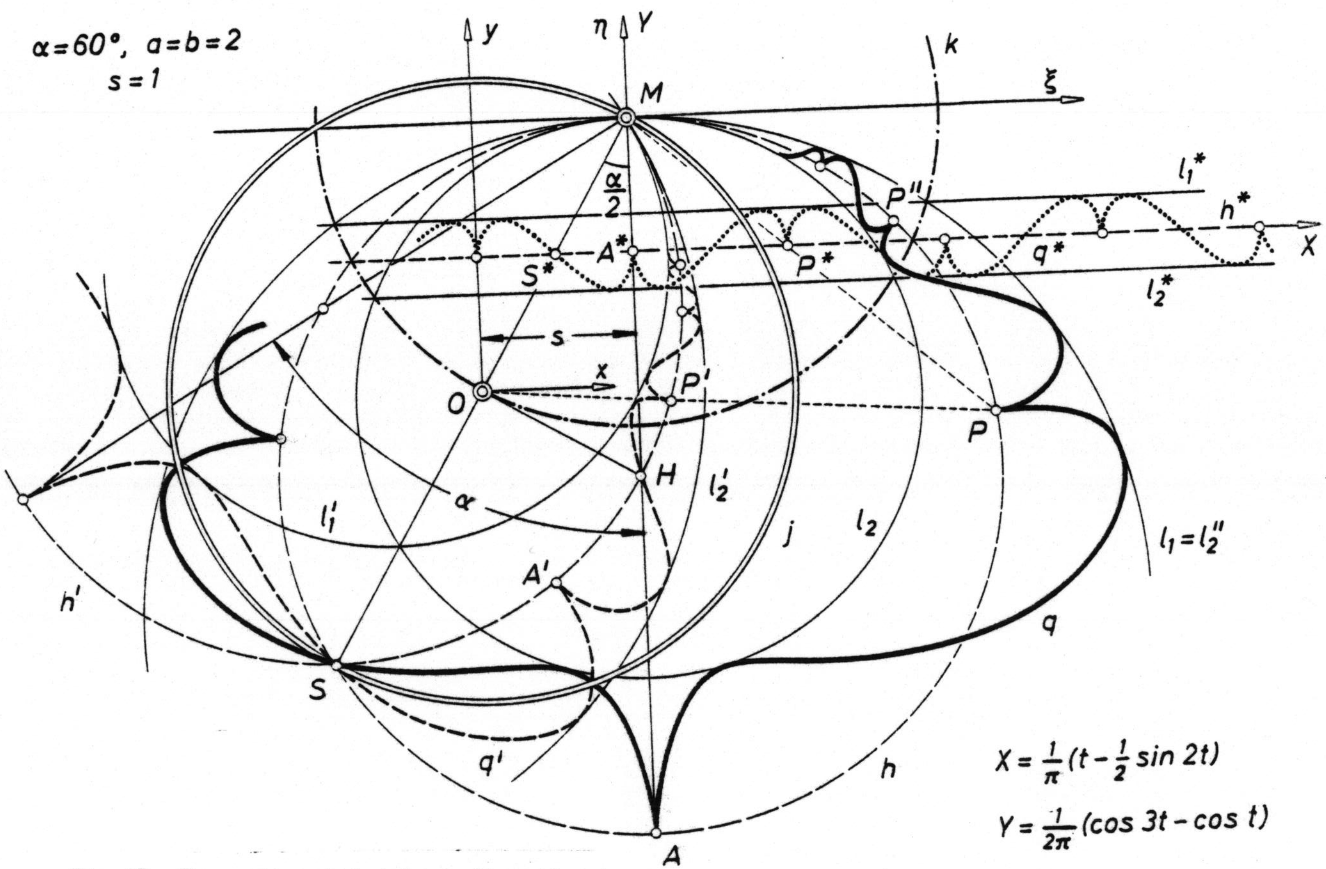


Fig.17: Curve inverted into a directly congruent copy (parabolic type)

related to each other by a rotation about M. Since the primitive vector s may be replaced by any odd multiple ns, there exist an infinity of possible inversion centers O for the same curve q; they all lie on a circle passing through M.

Fig.17 shows an example based on the auxiliary curve $q^{`}$ with the parametric representation

$$(11.11) \qquad X = \frac{s}{2\pi}(2t - \sin 2t), \quad Y = \frac{s}{2\pi}(\cos 3t - \cos t).$$

Since (more or less by chance) this curve possesses a center of symmetry at the midpoint S' of the segment MO, the derived solution curve q also offers an illustration for the case $\omega = \pi$ in Theorem 5: Hence the inverse image q' of q is not only directly congruent to q, but also a mirror copy of q. Moreover, due to the infinite set of mirror axes of $q^{`}$, the solution curve q is anallagmatic in infinitely many ways (Theorem 1).

12. Cyclic Case

Before entering into the investigation of the general elliptic case $R^2 < 0$, it is recommendable to clear the situation in the postponed special case $b = 0$, denoted as "cyclic case" (Section 9). Here we have $R^2 = -4a^4\sin^2\alpha$, hence also $R^2 < 0$, if we exclude $\alpha = 0 \pmod{\pi}$. Keeping in mind that the rotation angle α of the isometry σ (9.2) -- which now is a pure *rotation about the inversion center O* -- is determined only up to even multiples of π, the provisorily made restriction will prove to be of no importance.

Thus we have to do with *Moebius mappings* $\mu = \sigma\iota = \iota\sigma$, described by

$$(12.1) \qquad \bar{z}z'' = a^2 e^{i\alpha}.$$

Obviously such a transformation has no real fixed points, if $\alpha \neq 0$. Consequently there is no occasion to perform the operations used until now. Nevertheless we will find by individual reasoning solution curves invariant under μ. Such a curve $q = q''$, admitting with μ also the rotation $\mu^2 = \sigma^2$ (angle 2α) will show a cyclic periodicity. Therefore we may suppose for its representation $z = z(\phi)$ either

(12.2) $$z''(\phi) = z(\phi + \alpha + 2k\pi), \quad k \text{ integer},$$

or

(12.3) $$z''(\phi) = z(\phi + \beta) \text{ with } \beta = \alpha + (2k+1)\pi$$

Since α may be augmented or diminished by any integer multple of 2π, it is sufficient to consider only the case $k = 0$. Notwithstanding we have to distinguish between *two types* of solutions.

Solutions of Type A

Inserting the relation (12.2) into equ. (12.1), we obtain

(12.4) $$\bar{z}(\phi) \cdot z(\phi + \alpha) = a^2 e^{i\alpha}.$$

This multiplicative *functional equation* can be transformed into an additive one by passing to logarithms or, better, by means of the substitution

(12.5) $$z(\phi) = a\, e^{i\phi} \cdot \frac{1 + w(\phi)}{1 - w(\phi)}.$$

For the so introduced auxiliary function $w(\phi)$ we find the condition

(12.6) $$w(\phi + \alpha) + \bar{w}(\phi) = 0.$$

It requires for the real and imaginary parts of $w(\phi) = u(\phi) + iv(\phi)$ the relations

(12.7) $$u(\phi + \alpha) = -u(\phi), \quad v(\phi + \alpha) = v(\phi),$$

similar to equs. (10.12). Both of the real functions are *periodic*, $u(\phi)$ with period 2α, $v(\phi)$ with period α. Using the module $m = \pi/\alpha$, simple analytic solutions are offered with

(12.8) $$u(\phi) = \sum (a_j \cos jm\phi + b_j \sin jm\phi),$$
$$v(\phi) = \sum (a_j \cos jm\phi + b_j \sin jm\phi)$$

with arbitrary real coefficients a_j, b_j. It should be noted that we cannot do without a real part $u(\phi)$, since with $u(\phi) = 0$ the fraction in formula (12.5) would have the absolute value 1, hence the solution curve q would reduce to the inversion circle j or a part of it. On the other hand, with vanishing imaginary part $v(\phi) = 0$ the solution curve q will be simply described by its polar equation

(12.9) $$r = a\frac{1 + u(\phi)}{1 - u(\phi)}$$

The most general solutions of Type A can be obtained by parametrizing the variables ϕ, u and v in a manner which was indicated with equs. (6.12).

Solutions of Type B

The alternative (12.3) leads over (12.1) to the *functional equation*

(12.10) $$\bar{z}(\phi)\cdot z(\phi + \beta) = -a^2 e^{i\beta} \text{ with } \beta = \alpha + \pi.$$

It seems impossible to transform this multiplicative equation into an additive one, but it may be normed by putting

(12.11) $$z(\phi) = a\, e^{i\phi} W(\phi)$$

which leads to the simpler condition

(12.12) $$\bar{W}(\phi)\cdot W(\phi + \beta) = -1.$$

Since the auxiliary function $W(\phi)$ is *periodic* (period 2β), we write it as quotient of two periodic functions,

(12.13) $$W(\phi) = P(\phi)/Q(\phi),$$

and try to determine these functions in such a way that

(12.14) $$P(\phi + \beta) = -\bar{Q}(\phi), \quad Q(\phi + \beta) = \bar{P}(\phi).$$

These relations yield

(12.15) $$P(\phi + 2\beta) = -P(\phi), \quad Q(\phi + 2\beta) = -Q(\phi).$$

Hence both of the functions have the period 4β, and simple analytic solutions are given, with use of the module $n = \pi/\beta$, by

(12.16)
$$\begin{aligned} P(\phi) &= a_1\cos\frac{n\phi}{2} + a_3\cos\frac{3n\phi}{2} + a_5\cos\frac{5n\phi}{2} + \dots \\ &\quad + b_1\sin\frac{n\phi}{2} + b_3\sin\frac{3n\phi}{2} + b_5\sin\frac{5n\phi}{2} + \dots, \\ Q(\phi) &= \bar{a}_1\sin\frac{n\phi}{2} - \bar{a}_3\sin\frac{3n\phi}{2} + \bar{a}_5\sin\frac{5n\phi}{2} - \dots \\ &\quad - \bar{b}_1\cos\frac{n\phi}{2} + \bar{b}_3\cos\frac{3n\phi}{2} - \bar{b}_5\cos\frac{5n\phi}{2} + \dots, \end{aligned}$$

where a_j and b_j are arbitrary complex coefficients. If all coefficients are real, $W(\phi)$ is a real function, and the solution curve q is simply described by its polar equation

$$(12.17) \qquad r = a\,W(\phi).$$

The two types of solutions are not convertible into each other, since the attempt of a substitution $W = (1+w)/(1-w)$ does not yield condition (12.6). Thus we state

<u>Theorem 10:</u> *Plane curves q which coincide with their inverse image q' after a rotation through an angle α about the inversion center are representable by equs. (12.5) or (12.11), where the respective arbitrary periodic functions $w(\phi)$ or $W(\phi)$ satisfy the corresponding conditions (12.6) or (12.12).*

<u>*Discussion and Examples*</u>

Since each solution curve $q = q''$ admits the *rotation* $\mu^2 = \sigma^2$ (angle 2α), it consists of a sequence of *congruent arcs*. In case of a *rational module* $m = \pi/\alpha$ the curve q (supposed as continuous) is *closed*. Writing in such a case $|m| = \lambda/\nu$ (λ, ν positive integers without common factor), we state that q consists of λ congruent arcs and ν-times surrounds the center before closing, if λ is *even*. Then ν is odd, and the automorphism $\mu^\lambda = (\sigma\iota)^\lambda = \sigma^\lambda$ is a rotation through the angle $\lambda\alpha = \nu\pi$, thus equivalent to a half-turn: such a curve q has a *center of symmetry* (Figs.19,20). -- Otherwise, if λ is *odd*, the automorphism $\mu^\lambda = \sigma^\lambda\iota$ is a mapping composed of the inversion ι and a rotation through the angle $\nu\pi$. For ν even, the rotation is equivalent to identity, hence the curve q is *anallagmatic* (Section 3, Fig.21). For ν odd, the rotation is equivalent to a half-turn and the curve q is *anti-anallagmatic*, i.e. related with its inverse image q' by the centric symmetry with respect to the inversion center (Fig,22).

Those solution curves q which are obtained by means of trigonometric polynomials (12.8) or (12.16) with rational module $m = \lambda/\nu$ or $n = \lambda/(\lambda+\nu)$ are *algebraic* and *rational* (of genus 0). For Type A this is to show by introducing the new parameter $t = e^{i\phi/\nu}$ (occasionally $t = e^{2i\phi/\nu}$) which also allows to determine the order of q. Analogous substitutions are suitable for Type B.

With $m = 1/2$ ($\alpha = 2\pi$) we obtain only *anallagmatic curves*. The choice $w = a_1 \cos(\phi/2)$, for instance, leads to a self-inverse sextic of Type A. The symmetric strophoid (3.1) belongs to Type B and would be found (after change of sign at m) with $W = \cot(\phi/2)$.

The category $m = 1$ ($\alpha = \pi$) throughout consists of *anti-anallagmatic curves*. They might be considered as self-inverse with respect to the imaginary circle $x^2 + y^2 = -a^2$. With $w = \varepsilon \cos \phi$, for instance, we obtain a curve q of Type A with the polar equation

$$(12.18) \qquad r = a\frac{1 + \varepsilon \cos \phi}{1 - \varepsilon \cos \phi} = \frac{2a}{1 - \varepsilon \cos \phi} - a.$$

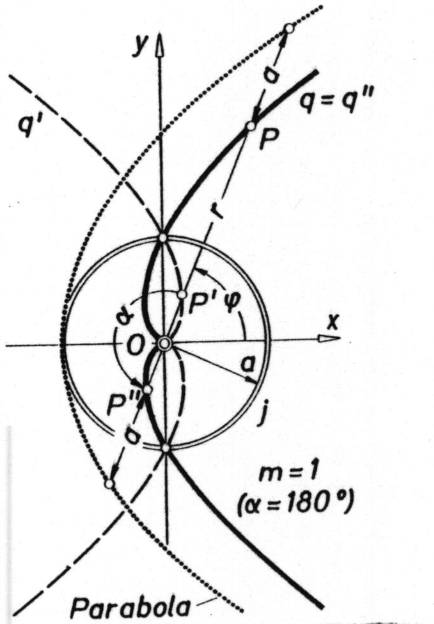


Fig.18: Anti-anallagmatic quartic (focal conchoid of a parabola, cyclic type A)

Since the first term in the last expression represents a conic with the numerical eccentricity ε and with one focus at the origin, we see that q (a monocircular quartic) is a *focal conchoid of a conic* [3,p.106]. Fig.18 shows the particular specimen derived from a parabola ($\varepsilon = 1$); its equation may be written in the forms

$$(12.19) \qquad r = a \cdot \cot^2(\phi/2)$$

or

$$(x^2+y^2)(y^2 - 4ax) = a^2y^2.$$

With $m = 2$ ($\alpha = \pi/2$) the resulting (centric) curves q coincide with their inverse image q' after a quarter-turn. It seems that most of the few known examples of non-trivial congruent-inverse curve pairs belong to this class [1]. A simple representant of Type B is generated by $W = \cot\phi$; its equations read

(12.20) $\quad r = a\cdot\cot\phi$ or $(x^2+y^2)y^2 = a^2x^2$.

This monocircular quartic with a tacnode at the center (Fig.19) is known under the name of "*kappa-curve*" |3, p.74; , p.123|. Due to its centric symmetry, this curve q is also a mirror copy of its inverse image q'.

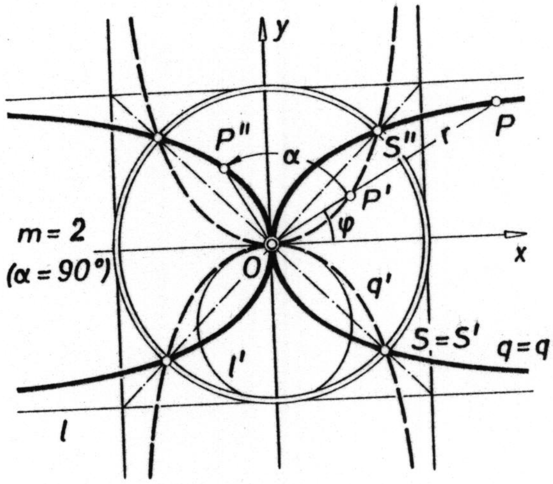
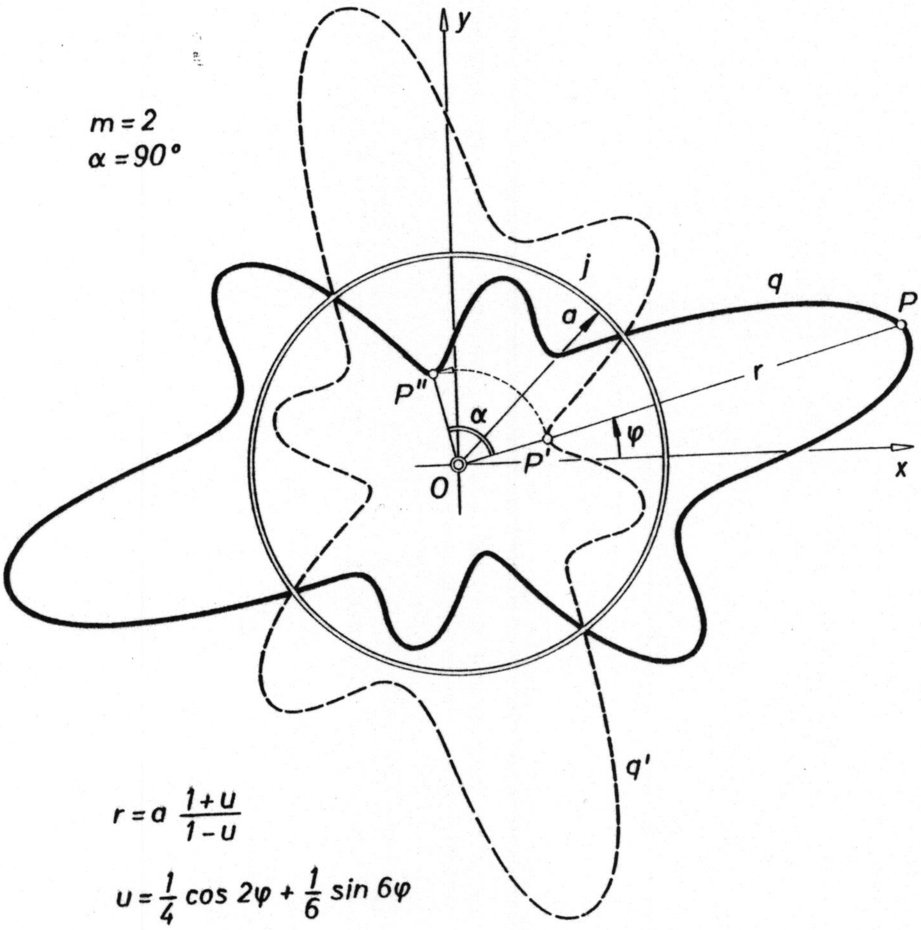


Fig.19: Kappa-curve inverted into a congruent copy (cyclic type B)

A less simple example of Type A is shown in Fig.20. It is derived from a function

(12.21) $\quad w = a_1\cos 2\phi + b_2\sin 6\phi$

and is centric, but without axial symmetry; its order is 14.



$$r = a\,\frac{1+u}{1-u}$$

$$u = \frac{1}{4}\cos 2\varphi + \frac{1}{6}\sin 6\varphi$$

Fig.20: Directly congruent-inverse curve pair (cyclic type A)

The solution curve of Fig.21 belongs to Type A with the module $m = 5/2$ ($\alpha = 2\pi/5 = 72°$). It is determined by $w = a_1 \cos(5\phi/2)$, anallagmatic and of order 14.

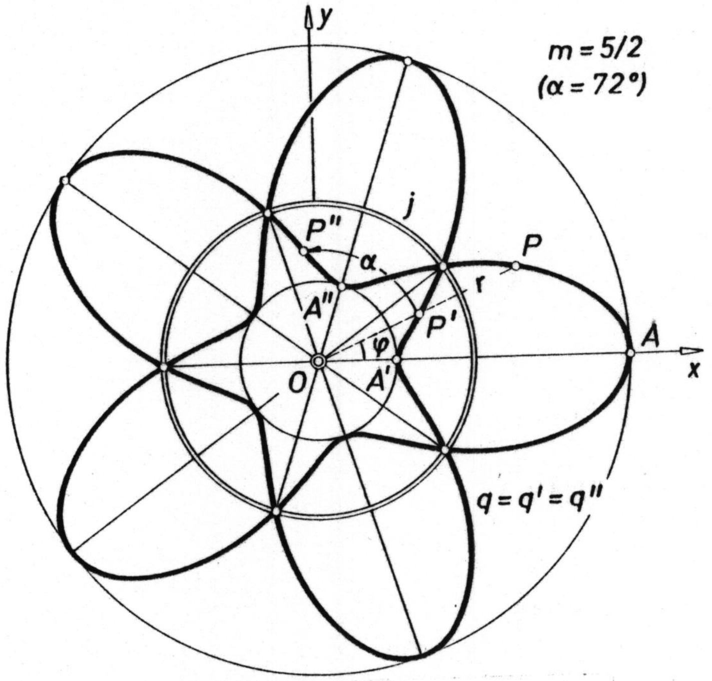


Fig.21: Anallagmatic curve of order 14 (cyclic type A)

The last example, shown in Fig.22, belongs to Type A with the module $m = 3$ ($\alpha = \pi/3 = 60°$) and is obtained with a function

(12.22) $$w = ia_0 + a_1 \cos 3\phi .$$

The curve q is anti-anallagmatic and of order 14. -- A curve q of Type A (12.5) has *cusps* (like that in Fig.22), if for a certain real value of the parameter ϕ the derivative $\dot{z} = dz/d\phi$ vanishes. This requires for the auxiliary function $w(\phi)$ the condition

(12.23) $$w^2 + 2i\dot{w} = 1,$$

hence for its components u, v the relations

(12.24) $\quad u^2 - v^2 - 2\dot{v} = 1, \quad uv + \dot{u} = 0.$

In the present case (12.22) these conditions read

(12.25) $\quad a_1^2 \cos^2 3\phi - a_0^2 = 1, \quad a_0 = 3\tan 3\phi.$

They have been satisfied for $\phi = 15°$, say, with $a_0 = 3$, $a_1 = 2\sqrt{5}$.

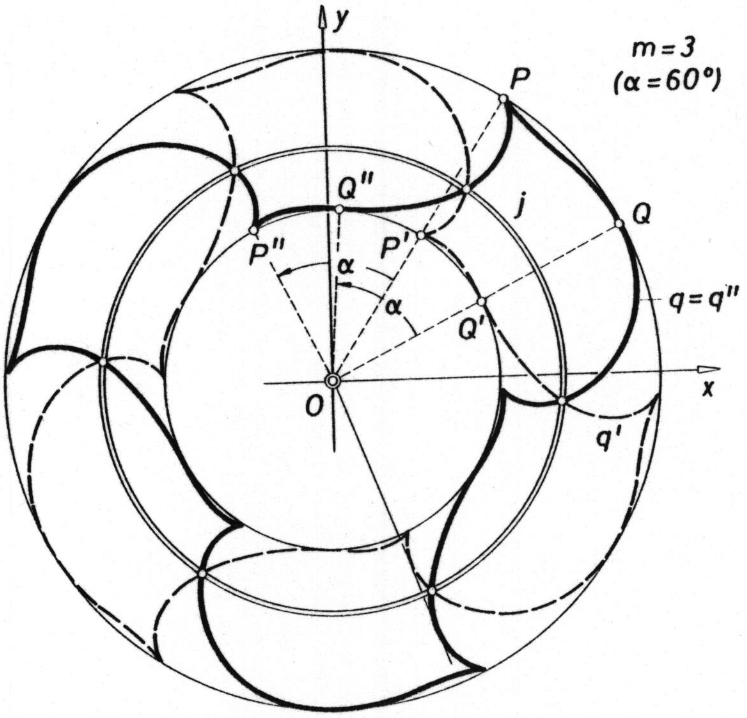


Fig.22: Anti-anallagmatic curve of order 14 (cyclic type A)

13. Elliptic Case

For the discussion of the general elliptic case $R^2 < 0$ we put $R = iS$ ($S \neq 0$, real). With exclusion of the cyclic case $b = 0$ (Section 12) we obtain from equs. (9.7):

$$
\begin{aligned}
z_{12} &= |b^2 + i(2a^2 \sin\alpha \pm S)|/2b, \\
(13.1) \qquad \bar{z}_{12} &= |b^2 - i(2a^2 \sin\alpha \pm S)|/2b \\
&\text{with } S^2 = 4a^4 - (b^2 + 2a^2 \cos\alpha)^2.
\end{aligned}
$$

As now neither $z_1, \bar{z}_1$ nor $z_2, \bar{z}_2$ are conjugate value pairs, the (indirect) Moebius mapping μ (9.4) has *no real fixed points*. |By the way, the fixed points are imaginary and have the (complex) cartesian coordinates $x_{12} = \frac{1}{2}(z_{12} + \bar{z}_{12})$, $y_{12} = \frac{i}{2}(\bar{z}_{12} - z_{12})$|. Denoting now in this section with M and N those real points which are determined by the Gauss coordinates z_1 and z_2, respectively, let us forget the second line of equs. (13.1). From now on the symbols $\bar{z}_1, \bar{z}_2$ shall mean (as usually) the conjugates of z_1, z_2, i.e.

$$(13.2) \qquad \bar{z}_{12} = |b^2 - i(2a^2 \sin\alpha \pm S)|/2b.$$

After this agreement the following relations hold:

$$(13.3) \qquad z_1 + \bar{z}_1 = z_2 + \bar{z}_2 = b, \quad z_1 z_2 = -a^2 e^{-i\alpha}, \quad \bar{z}_1 \bar{z}_2 = -a^2 e^{i\alpha},$$

furthermore

$$(13.4) \qquad \bar{z}_1 z_2 - b\bar{z}_1 - a^2 e^{i\alpha} = 0. \quad \bar{z}_2 z_1 - b\bar{z}_2 - a^2 e^{i\alpha}.$$

With respect to the equation (9.4) of the Moebius mapping $\mu: P \to P''$ we have

<u>Lemma 4:</u> *The points $M(z_1)$ and $N(z_2)$ are corresponding to each other in the indirect Moebius mapping μ ($M'' = N$, $N'' = M$). They are the fixed points of the direct Moebius mapping μ^2.*

This *commutative point pair* M, N of μ may be constructed in the following way (Fig.23). Due to the common real part $b/2$ of z_1 and z_2 we know the connecting line $l = MN$ ($x = b/2$); the μ-image l'' of l, a circle easy to find, intersects l at M and N.

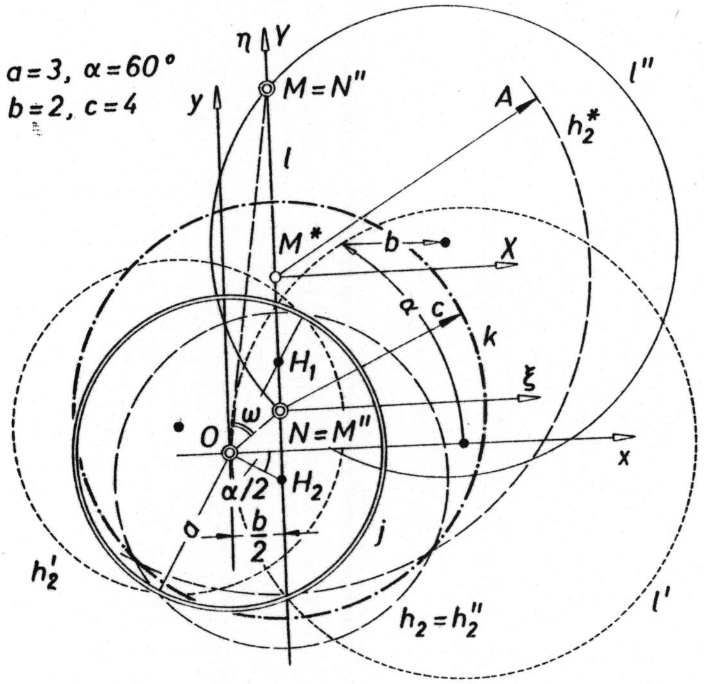


Fig.23: Indirect Moebius mapping, composed of an inversion and a transplacement (elliptic case)

Using relations (13.3) the equation (9.4) of the *Moebius mapping* μ may be written as

(13.5) $\quad \bar{z}z'' - (z_2+\bar{z}_2)\bar{z} + \bar{z}_1\bar{z}_2 = 0.$

Although M and N are not the fixed points of μ, we will perform the same operations as before. At first we pass to new coordinates

(13.6) $\quad \zeta = z - z_2 \quad (\zeta'' = z'' - z_2),$

thus translating the origin to N (Fig.23). hereby the equation of μ obtains the form

(13.7) $\quad \bar{\zeta}\zeta'' - \bar{z}_2(\bar{\zeta} - \zeta'') + \bar{\zeta}_1\bar{z}_2 = 0$ with $\zeta_1 = z_1 - z_2 = iS/b.$

Now we apply an auxiliary *inversion* κ (center N, radius c), defined by

(13.8) $$\zeta' = c^2/\bar{\zeta} \quad (\zeta''' = c^2/\bar{\zeta}'').$$

Replacing the result by its conjugate (which means the same), we obtain

(13.9) $$\zeta_1 z_2 \bar{\zeta}' \zeta''' + c^2 z_2 (\bar{\zeta}' - \zeta''') + c^4 = 0.$$

In contrast to the preceding cases, this transformed mapping $\mu' = \kappa\mu\kappa$ is not linear, but still a Moebius transformation. It interchanges the κ-image M' of M with the κ-image N' of N which is at infinity, hence M' is the *pole* of μ' (Section 4). Using this pole as origin for new coordinates

(13.10) $$Z = \zeta' - \zeta_1' \quad (Z'' = \zeta''' - \zeta_1')$$
$$\text{with } \zeta_1' = c^2/\bar{\zeta}_1 = ibc^2/S,$$

the linear terms in equ. (13.9) will vanish. With attention to $\mathrm{Re}\,\zeta_1' = 0$ and $\zeta_1\zeta_1' = -c^2$ we obtain

(13.11) $$\zeta_1 z_2 \bar{Z} Z'' + c^4 z_1/\zeta_1 = 0,$$

and finally the canonical form of μ':

(13.12) $$\bar{Z} Z'' = \frac{b^2 c^4}{S^2} \cdot \frac{z_1}{z_2} = A^2 e^{i\omega}.$$

Herewith the problem of the general elliptic case is led back to the *cyclic case* (Section 12), since equs. (13.12) and (12.1) are of the same type: μ' is a "cyclic" Moebius mapping, composed of an inversion from the pole M' (radius A) and a rotation about M' (angle ω). The polar net about the pole M', as a whole invariant under μ', corresponds (by means of the inversion κ) to an orthogonal system of circle pencils invariant under μ: the elliptic pencil of the circles passing through M and N (without a fixed element), and the orthogonal hyperbolic pencil (containing one invariant circle, the κ-image of the inversion circle $Z\bar{Z} = A^2$ of μ'). These facts provide a good insight into the effect of an indirect Moebius transformation without real fixed points; see Fig.24, where the commutative point pair M,N has been located at the points $\pm i$ of a Gauss plane.

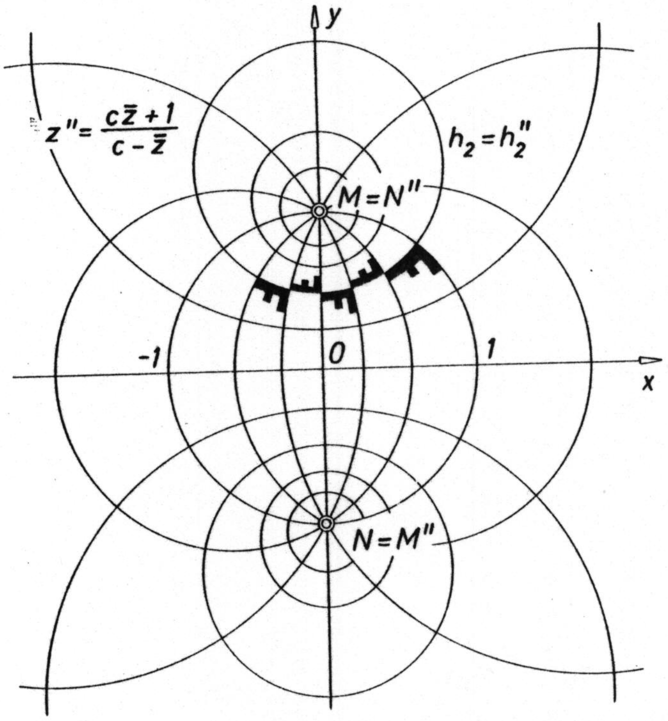


Fig.24: Invariant circle pencils of an indirect Moebius mapping without real fixed points

Before entering into details we roughly state

<u>Theorem 11:</u> Each one of the components of a directly congruent-inverse curve pair q,q′ of elliptic type is inverse to a curve q˙ which belongs to a directly congruentinverse pair of the cyclic type.

To obtain a solution of the general elliptic problem, i.e. a curve q which is invariant under the Moebius mapping μ (9.4), we have to start with a solution of the cyclic problem, i.e. with a curve q˙ invariant under μ˙ (13.12). The equation $Z = Z(\phi)$

of q^{*} is then to be transformed back by means of

$$(13.13) \qquad z(\phi) = \frac{c^2}{\bar{Z}(\phi) + \bar{\zeta}_1^{*}} + z_2 \text{ with } \zeta_1^{*} = ibc^2/S,$$

where S and z_2 are to be taken from equ. (13.1). Corresponding to the types A and B of the auxiliary curves q^{*} (Section 12) we may distinguish between the same types at the solution curves q.

It is worth to recall that the isometry σ (9.2) which relates the solution curve q with its inverse image q' may be performed by means of a pure rotation through the angle α about the center H_1 (9.3), which now is at the inside of the inversion circle j (Fig.23).

To add some more details, in particular formulas for the essential constants A and ω of the cyclic Moebius mapping μ^{*}, we deduce from equs. (13.12) with attention to relations (13.3):

$$(13.14) \qquad A^2 = \frac{b^2c^4}{S^2} \cdot \left|\frac{z_1}{z_2}\right| = \frac{b^2c^4}{a^2S^2} \cdot z_1\bar{z}_1 .$$

The so determined circle with the center M' and the radius A is invariant under μ^{*}; hence it is the κ-image of a circle invariant under μ. The latter was already mentioned as a fixed element of the invariant hyperbolic circle pencil and may be found as follows (Fig.23): Its center H_2 is on the line $l = MN$ and has the coordinates $x = b/2$, $y = -x\tan(\alpha/2)$; the circle itself, h_2, meets the inversion circle j at opposite points (compare the same construction in the hyperbolic case, Fig.14).

As to the rotation angle ω of μ^{*}, we learn from equ. (13.12):

$$(13.15) \qquad \omega = \arg(z_1/z_2) = \arg z_1 - \arg z_2 = \measuredangle NOM .$$

In words (Fig.23) we have

<u>*Lemma 5:*</u> *The rotation angle of the cyclic Moebius mapping μ^{*} (13.12) is equal with the angle under which the commutative point pair M,N of the elliptic Moebius mapping μ (9.4) is seen from the inversion center O.*

Instead of the relation (13.15) we may write

$$(13.16) \qquad \omega = \arg z_1 + \arg \bar{z}_2 = \arg(z_1\bar{z}_2),$$

and as with respect to equ. (13.3) $|z_1\bar{z}_2| = |z_1 z_2| = a^2$, we have

$$(13.17) \qquad z_1\bar{z}_2 = a^2 e^{i\omega} = a^2(\cos\omega + i\sin\omega).$$

Taking from relation (13.4)

$$(13.18) \qquad z_1\bar{z}_2 = bz_1 + a^2 e^{-i\alpha} = \frac{1}{2}(b^2 + 2a^2\cos\alpha + iS)$$

and comparing this expression with (13.17), we obtain the formulas

$$(13.19) \qquad \cos\omega = \frac{b^2 + 2a^2\cos\alpha}{2a^2}, \quad \sin\omega = \frac{S}{2a^2}.$$

Looking for *algebraic solutions* we have only to prescribe a suitable value of ω, i.e. a *rational module* $m = \pi/\omega$. Such a choice of $\omega = \pi/m$ implies, due to the first formula (13.19), a single condition for the quantities a, b and α, namely

$$(13.20) \qquad 2a^2(\cos\omega - \cos\alpha) = b^2 \text{ with } \cos\alpha < \cos\omega.$$

In the example of Fig.25, based on $m = 2$ ($\omega = \pi/2$), this condition is satisfied with $\alpha = 2\pi/3$ and $a = b$, for instance. Choosing as auxiliary curve q the kappa-curve defined by its polar equation (12.20) with A instead of a, we obtain a tricircular sextic q with two tacnodes ("bretzel curve").

Another algebraic example, belonging to the module $m = 4$ ($\omega = \pi/4 = 45^\circ$) and shown in Fig.26, is constructed with $\alpha = 3\pi/4 = 135^\circ$ and $b^2 = 2\sqrt{2}\,a^2$. The auxiliary curve q, defined with

$$(13.21) \qquad w = \frac{1}{5}\sin(4\phi + 60^\circ)$$

in equ. (12.5) (where A, Z replace a, z), is a four-lobed cyclic curve of order 10. The corresponding elliptic solution curve q has the order 18.

Summarizing the main facts we state

<u>*Theorem 12:*</u> *Let q' be an arbitrary solution curve of the cyclic problem, i.e. admitting an inversion of radius A from its center M' followed by a rotation about M with an angle ω. If q is transformed by an inversion κ with arbitrary center $N \neq M'$ and radius c, the resulting curve q admits certain inversions which map it onto a directly congruent copy q'. The center O of such an inversion is determined by the relations $OM:ON = MN^2A^2/c^4$ and $NOM = \omega$ or an integer multiple $n\omega$, where*

M denotes the κ-image of M˘ . All possible inversion centers O are situated on a well-defined Apollonian circle.

The last part of the theorem follows from equ. (13.14) with respect to $|z_1| = OM$, $|z_2| = ON$ and $S/b = |z_1 - z_2| = MN$.

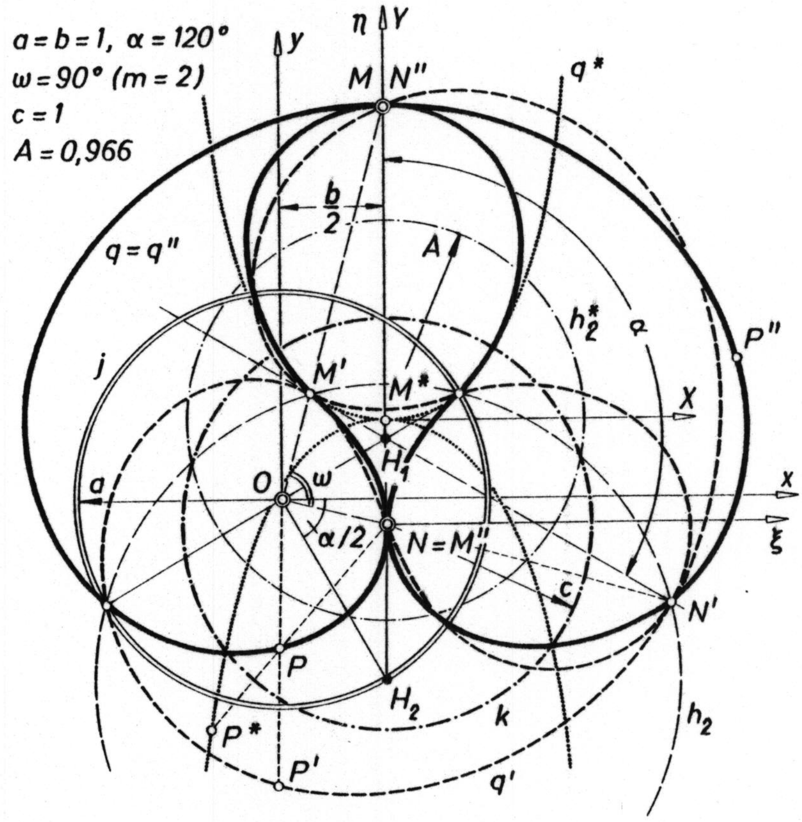


Fig.25: Circular sextic (bretzel-curve) inverted into a directly congruent copy (elliptic type B)

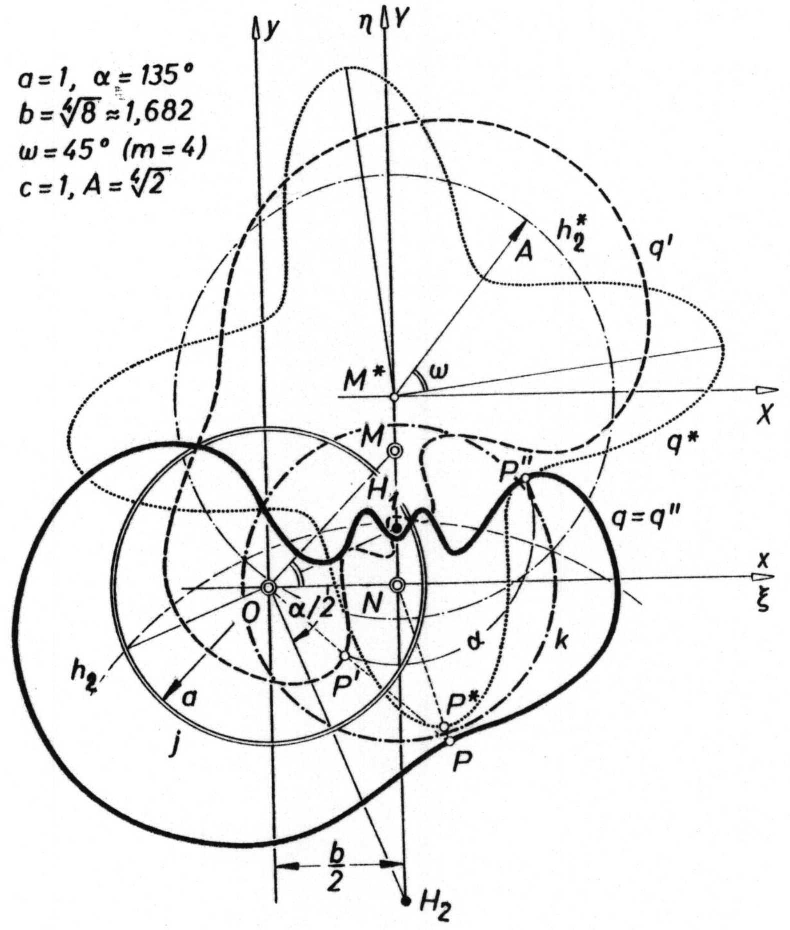


Fig.26: Curve of order 18 inverted into a directly congruent copy (elliptic type A)

IV. COMPLEMENTS

14. Stereographic Projection

Stereographic projection is a classic mapping of the surface of a *sphere* Ω onto a *plane* Π. It was known already to the ancient Greeks (HIPPARCHOS, 150 B.C.) and consists in a central projection from a surface point U_0 of Ω onto a plane Π which is perpendicular to the radius ending at U_0. Since this mapping is *conformal* and *circle-preserving* |2, p.390|, it is important in geography and astronomy, but also in crystallography, complex function theory (Riemann's sphere as an alternative to the Gauss plane), and of course in circle geometry. Here we intend to show its use for deriving directly congruent-inverse curve pairs in the cyclic case (Section 12).

In cylindrical coordinates r_0, ϕ_0, z_0 the sphere Ω may be given by

(14.1) $$r_0^2 + z_0^2 = a^2 \quad (\phi_0 \text{ arbitrary}).$$

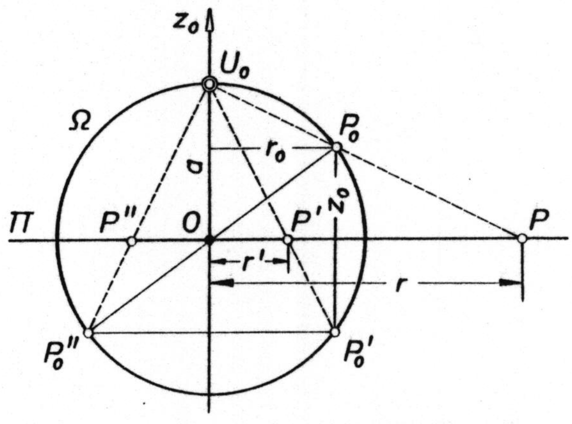


Fig.27: Stereographic projection of a sphere

Taking as center of projection the "north pole" $U_0(r_0 = 0, z_0 = a > 0)$ and as picture plane the equator plane Π $(z = 0)$, a surface point $P_0(r_0, \phi_0, z_0) \neq U_0$ is mapped onto that point P of Π (Fig.27), which is determined by the polar coordinates

$$(14.2) \qquad r = \frac{ar_0}{a - z_0} = \sqrt{\frac{a + z_0}{a - z_0}}, \quad \phi = \phi_0.$$

Inversion of the sign of z_0 induces the reflexion $P_0 \to P_0'$ of the sphere in the equator plane, and in the latter a transformation $P \to P'$ described by $r' = a^2/r$, hence

Lemma 6: The reflexion of the sphere Ω in the equator plane Π appears in stereographic projection as the inversion with respect to the equator circle j.

Transplacing the points P_0' and P' by means of a half-turn about the z_0-axis to P_0'' and P'' (Fig.27) we obtain

Lemma 7: The reflexion of the sphere Ω in its center O appears in stereographic projection as the anti-inversion with respect to the equator circle j.

These facts allow a simple construction of anallagmatic or anti-anallagmatic curves. In the first case we take an arbitrary *cylinder* Γ perpendicular to Π and determine its intersection curve q_0 with the sphere Ω: The stereographic projection of q_0 is an *anallagmatic curve* q, provided that the spherical curve q_0 does not split into two separate parts (which may occur). If, for instance, the base of the cylinder Γ is a conic, then q_0 is a spherical quartic, and its stereographic projection q is either a bicircular quartic or a circular cubic; the latter possibility arises, if Γ passes through the projection center U_0. In the second case we take an arbitrary *cone* Δ issuing from the center O of Ω, and determine its intersection q_0 with Ω: The stereographic projection of q_0 is an *anti-anallagmatic curve* q, provided q_0 does not split. If, for instance, Δ is a quadratic cone, then q_0 is a spherical quartic, and its stereographic projection q is again a bicircular quartic or a circular cubic (in general of genus 1).

Now let us proceed to directly congruent-inverse curve pairs q, q' of the *cyclic type* (Section 12). The basic *indirect Moebius mapping* $\mu = \sigma\iota$ (12.1), composed of the inversion ι and a rotation σ with angle α about the inversion center O, is the stereographic image of an *opposite isometry* $\mu_0 = \sigma_0 \iota_0$ on the sphere Ω; this automorphism of Ω is composed of the reflexion ι_0 in the equator plane Π and the rotation $\sigma_0 = \sigma$ (about the z_0-axis). A curve q in Π invariant under μ is the stereographical image of a spherical curve q_0 which is invariant under μ_0. Such a curve consists of a sequence of congruent arcs; continuity supposed, it will intersect the equator circle j in a series of points with angular distance α. We might arbitrarily prescribe one of the mentioned elementary arcs beginning and ending on j, and then repeat it by applying the isometry μ_0.

Analytically, the spherical curve q_0 may be defined by a (real) function

$$(14.3) \qquad z_0 = z_0(\phi) \quad \text{with} \quad z_0(\phi + \alpha) = -z_0(\phi).$$

Using the module $m = \pi/\alpha$ suitable functions are offered with trigonometrical polynomials of the kind indicated for $u(\phi)$ in the first equation (12.8), or by appropriate parametric representations $\phi(t)$ and $z_0(t)$. The polar equation $r = r(\phi)$ of q is then determined by

$$(14.4) \qquad r^2 = a^2 . \frac{a + z_0(\phi)}{a - z_0(\phi)} .$$

The spherical curve q_0 lies on a ruled surface Φ defined by equ. (14.3); Φ consists throughout of normals of the z_0-axis and therefore is, in the language of line geometry, a *"conoid"*.

The simplest example is obtained with $m = 2$ ($\alpha = \pi/2$) and

$$(14.5) \qquad z_0 = \varepsilon a \cos 2\phi \quad (\varepsilon > 0).$$

In this case Φ is Pluecker's cubic conoid, known as "cylindroid". Its intersection with the sphere Ω is a sextic q_0 with double points at the north pole U_0 and the south pole V_0. Consequently the stereographic image q of q_0 is a *monocircular quartic* with a double point at the center $O = V$. Its equation in polar coordinates is given by

(14.6) $$r^2 = a^2 \frac{1 + \varepsilon \cos 2\phi}{1 - \ \cos 2\phi}\ .$$

The cartesian equation, obtained by elimination of r and ϕ from $x = r \cos\phi$, $y = r \sin\phi$ and (14.6), reads

(14.7) $$(x^2+y^2)(Bx^2 + Ay^2) = a^2(Ax^2 + By^2) \text{ with } A = 1+\varepsilon,\ B = 1-\varepsilon.$$

The double point O of q is a real node for $\varepsilon \ \ 1$ and isolated for $\varepsilon \ \ 1$ (Fig.28); the curve is an oval, if $\varepsilon \leqq \sqrt{5} - 2 = 0{,}236$. As for $\varepsilon \neq 1$ there are no other singularities than the double point O, the quartic q is of genus 2. In the limit case $\varepsilon = 1$ we meet again the (rational) kappa-curve $r = a \cot\phi$ (with a tac-node at O and a node at the point at infinity of the x-axis, Fig.19).

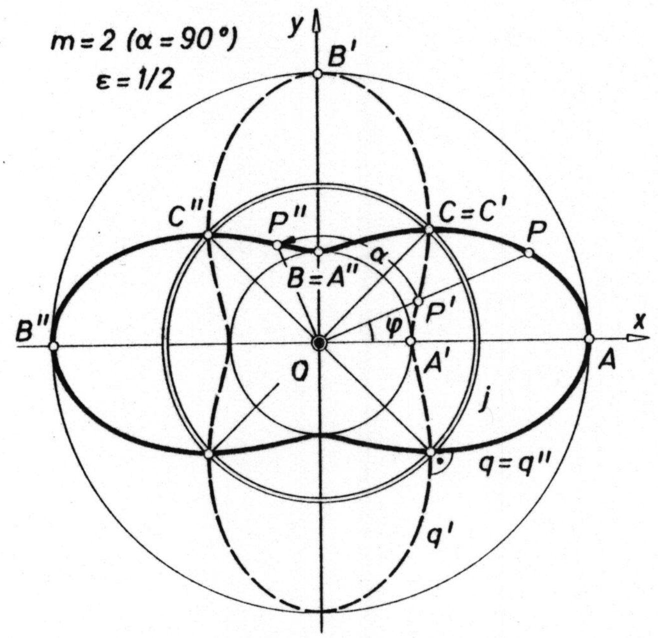


Fig.28: Circular quartic inverted into a congruent copy

The quartic q in Fig.28 reminds of a Cassinian curve (as mentioned in the invitation for the competition), but is not at all identical with it, as it is not bicircular. In order to obtain a Cassini curve we replace the conoid Φ (14.5) by a *hyperbolic paraboloid*, equally invariant under μ_0:

(14.8) $$cz_0 = x_0^2 - y_0^2 = r_0^2 \cos 2\phi \quad (c > 0).$$

It intersects the sphere Ω in a quartic q_0, sometimes denoted as "tennis-ball curve" (Fig.29). The stereographic projection q of q_0 is abicircular quartic. To deduce its polar equation we need the inverted formulas (14.2), i.e.

(14.9) $$r_0 = \frac{a^2 r}{r^2 + a^2}, \quad z_0 = a\frac{r^2 - a^2}{r^2 + a^2}.$$

Inserting these expressions in equ. (14.8)., we find

(14.10) $$r^4 - 2b^2 r^2 \cos 2\phi - a^4 = 0 \quad \text{with } b^2 = 2a^3/c.$$

The cartesian equation of q reads

(14.11) $$(x^2+x^2)^2 - 2b^2(x^2-y^2) = a^4$$

and testifies that q is indeed a *Cassini curve*. This quartic, having only two double points at the absolute circular points I and J (Section 2), is of genus 1. It is an oval, if $a^4 \geq 3b^4$ hence $c^2 \geq 12a^2$.

Similar procedures may be performed for any even module m. In cases of *odd* modules it is preferable to use, instead of a conoid Φ, a *cone* Δ having its vertex at the center O of the sphere Ω. It may be given in the form

(14.12) $$z_0/r_0 = u(\phi) \text{ with } u(\phi + \alpha) = -u(\phi), \ \alpha = \pi/m.$$

This cone Δ intersects the sphere Ω in a curve q_0, whose stereographic image q is described by the polar equation

(14.13) $$r^2 - 2ar \cdot u(\phi) - a^2 = 0.$$

The simplest example, obtained with $m = 3$ ($\alpha = \pi/3 = 60^0$) and

(14.14) $$u(\phi) = \varepsilon \cos 3\phi \quad (\varepsilon > 0),$$

operates with a cubic cone Δ containing the z_0-axis as (isolated) double line. Its spherical intersection q_0 is a sextic

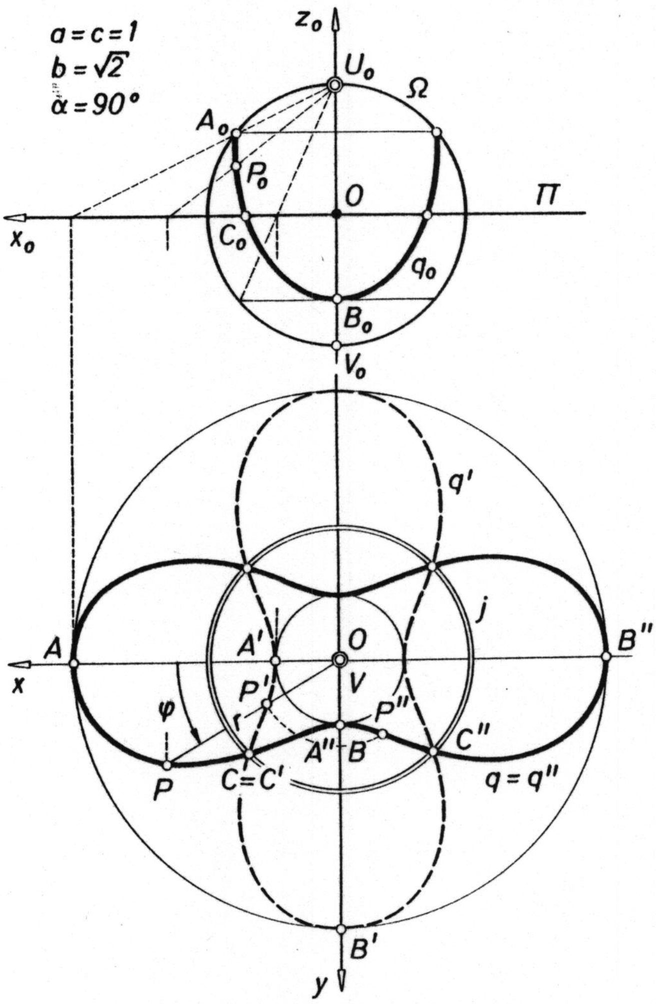


Fig.29: Cassinian quartic as stereographic image of a spherical quartic (tennis-ball curve), and inverted into a congruent copy

with double points at the poles U_0 and V_0 of Ω. Hence the stereographic image q of q_0, described by

$$(14.15) \qquad r^2 - 2\varepsilon a \cos 3\phi - a^2 = 0$$

and shown in Fig.30, is a *monocircular quartic* with an isolated double point at the center O. Its cartesian equation reads

$$(14.16) \qquad (x^2+y^2)(x^2+y^2-a^2) = 2\varepsilon a x(x^2 - 3y^2).$$

As the quartic q has no other singularities than the double point O (the circular points I,J being only contact points with the line at infinity), the curve is of genus 2.

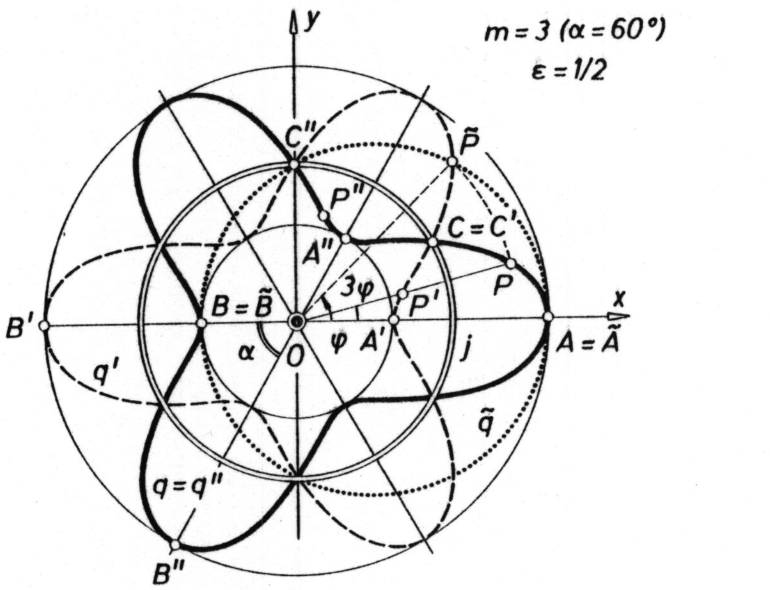


Fig.30: Monocircular quartic inverted into a congruent copy

Additional remarkable examples are derivable from *spherical helices* q_0, i.e. curves of constant slope $\tan\beta$ on the sphere Ω. Such a curve may be kinematically generated as the path of a point of a plane which is rolling on a fixed cone of revolution with the aperture angle 2β. Due to the invariance of the distance $a > 0$ between the cone vertex O and the generating point $P_0 \neq O$, the path q_0 of P_0 is traced on a sphere Ω with the center O and the radius a, and since at each instant the path tangent is perpendicular to the rolling plane, all tangents of q_0 have indeed the same inclination β against a base plane Π orthogonal to the axis of the cone. Considering merely the elements on the surface of the sphere Ω we have a great circle g_0 contained in the moving plane and rolling on a pair of congruent small circles $p_0, \bar{p}_0$ belonging to the fixed cone. Hence the spherical helix q_0, being an orthogonal trajectory of the position system of the great circle g_0, may be interpreted as a *spherical involute of the circle pair* $p_0 \cdot \bar{p}_0$.

Evidently the spherical helix q_0 is periodic and admits a rotation through the angle $2\pi/\sin\beta - 2\pi = 2\alpha$ about the axis of the fixed cone. It consists of a sequence of congruent elementary arcs, joined with each other in cusps which are distributed alternatively over the circles $p_0, \bar{p}_0$ (Fig.31). Led by the kinamatical generation of the helix q_0 and identifying the axis of the fixed cone with the z_0-axis, it is not difficult (see [4]) to deduce the following parametrical representation of q_0:

$$(14.17) \qquad \begin{aligned} x_0 + iy_0 &= \frac{a}{2(m+1)} \left| (2m+1)e^{it} - e^{(2m+1)it} \right|, \\ z_0 &= a\cos\beta \cdot \cos mt \quad \text{with} \quad \sin\beta = m/(m+1). \end{aligned}$$

The first line confirms the well-known fact that the orthogonal projection of the spherical helix onto the base plane Π ($z = 0$) is an *epicycloid* (A.ENNEPER, 1882).

Since the helix q_0 admits the decisive isometry μ_0 (angle $\alpha = \pi/m$), its stereographic image q is a curve which coincides with its inverse q' after a rotation through the angle α about the center O (Fig.31). From the fact that q_0 is an orthogonal trajectory of the system of great circles g_0 mentioned above,

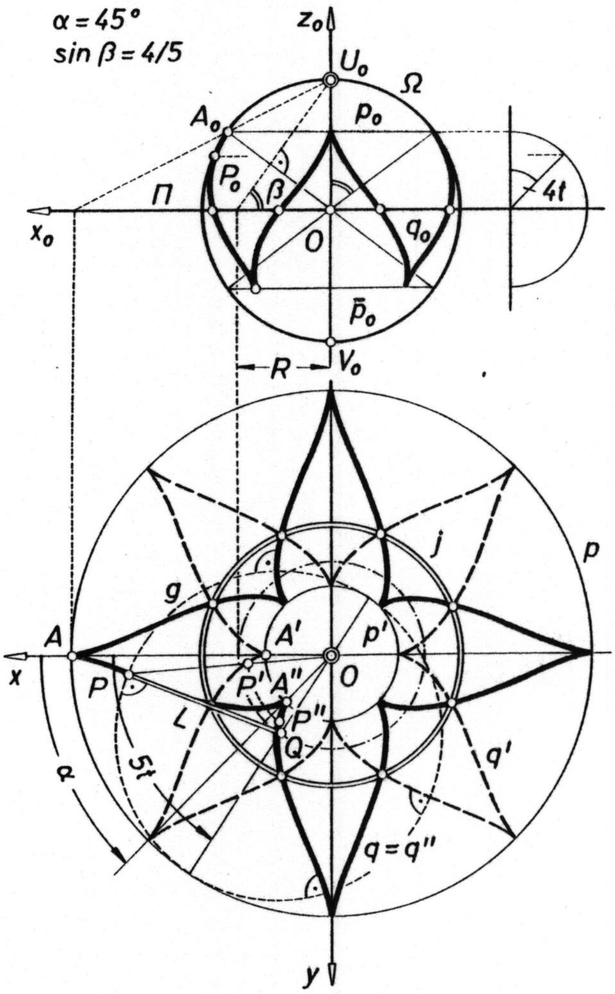


Fig.31: Tractrix of a circle as stereographic image of a spherical helix and inverted into a congruent copy

it follows, due to the properties of stereographic projection, that the image curve q is an orthogonal trajectory of a rotational set of circles g. This means that q is a *tractrix of a circle* |4|: Let PQ be a bar of constant length $L = a/\sin\beta$, whose endpoint Q is led along a circle of radius $R = a\cot\beta = L\cos\beta$ $(< L)$; provided the other endpoint P can move only in the instantaneous direction of the bar (for instance because of a wheel or a sharp edge), it will describe our tractrix q. -- The example of Fig.31 illustrates the situation for the module $m = 4$ $(\alpha = 45^0,\ \beta = \arcsin(4/5) = 53{,}13^0)$. The drag bar PQ is of length $L = 5a/4$, and the director circle of Q has the radius $R = 3a/4$. Since the module is rational, the tractrix q is algebraic (of order 18). A parametrical representation of q would be obtained by inserting the expressions (14.17) into the formula

$$(14.18) \qquad x+iy = \frac{a(x_0+iy_0)}{a - z_0}$$

which is equivalent with equ. (14.2).

The presented examples show that, in spite of the restriction to the cyclic case, stereographic projection is not quite useless for our purpose. However it would be difficult to attac the general problem of congruent-inverse curve pairs by means of this tool, as then the involved automorphisms μ_0 of the sphere Ω were collineations unhandy to operate with.

15. Fan Transformations

Let us denote with "*fan transformation*" that planar point-to-point mapping $\tau: P \to \tilde{P}$ which in polar coordinates is defined by

$$(15.1) \qquad \tilde{r} = r,\ \tilde{\phi} = n\phi \quad (n > 0).$$

Such a mapping, introduced under the name of "angle stretching" by E.MUELLER (1917), changes only the polar angles and thus reminds of the opening $(n > 1)$ or the closing $(n < 1)$ of a fan; with $n = 1$ it reduces to identity. For rational values of the constant factor n the mapping τ is algebraic, but of rather high degree (already 4 for $n = 2$ or $1/2$).

Now let q be a plane curve which is related with its inverse q' by a rotation through an angle $\alpha = \pi/m$ about the inversion center O (Section 12) and represented in polar coordinates by

(15.2) $\qquad r = f(\phi)$ with $f(\phi)\cdot f(\phi + \alpha) = a^2$.

Applying to q the fan transformation τ (15.1), we obtain a curve $\tilde{q}$ described by

(15.3) $\qquad \tilde{r} = f(\tilde{\phi}/n) = g(\tilde{\phi})$ with $g(\tilde{\phi})\cdot g(\tilde{\phi} + n\alpha) = a^2$.

Hence $\tilde{q}$ is a curve of the same cyclic kind, but with the module $\tilde{m} = m/n$. In particular with $n = m$ we have $\tilde{m} = 1$ und thus

<u>*Theorem 13:*</u> *Any solution curve q of the cyclic case with the module m is by a fan transformation with the factor $n = m$ mapped onto an anti-anallagmatic curve $\tilde{q}$.*

Conversely we may derive solutions q of the cyclic case by applying a fan transformation τ onto an arbitrary anti-anallagmatic curve $\tilde{q}$. Starting, for instance, from a *circle* $\tilde{q}$ which meets the inversion circle j at opposite points (Fig.3), i.e.

(15.4) $\qquad \tilde{x}^2 + \tilde{y}^2 - a^2 = 2b\tilde{x}$ or $\tilde{r}^2 - 2b\tilde{r}\cos\tilde{\phi} - a^2 = 0$,

we find solution curves q determined by

(15.5) $\qquad r^2 - 2br\cos m\phi - a^2 = 0$.

All these curves with different modules are fan transforms of each other, if they are derived from the same circle $\tilde{q}$.

With $m = 2$ ($\alpha = 90^0$) we obtain the centric curve q of Fig.32. Its cartesian equation reads

(15.6) $\qquad (x^2+y^2)(x^2+y^2-a^2)^2 = 4b^2(x^2-y^2)^2$

and says that q is a *bicircular sextic* with an isolated double point at the center. The curve is an oval, if $15b^2 \leq a^2$.

With $m = 3$ ($\alpha = 60^0$) we meet again the anti-anallagmatic quartic (14.15) of Fig.30.

With $m = 3/2$ ($\alpha = 120^0$ we obtain the anallagmatic curve of Fig.33; it is of order 10.

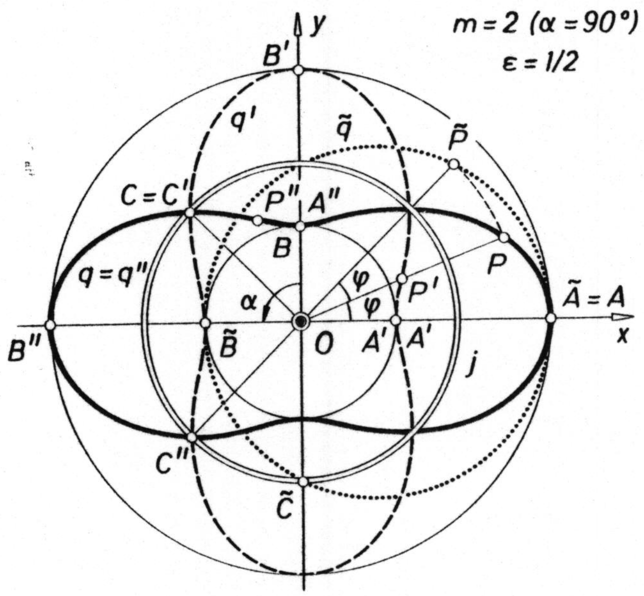


Fig.32: Bicircular sextic as a fan transform of a circle

A possibility to modify the fan transformation (15.1) without derogating the purpose in mind consists in replacing r by any power r^p. Especially with $p = n$ we obtain *conformal mappings*. For reasons of dimension we may then write

$$(15.7) \qquad \tilde{r}/a = (r/a)^n, \quad \tilde{\phi} = n\phi$$

or, in complex notation,

$$(15.8) \qquad \tilde{z}/a = (z/a)^n.$$

For $n = 2$ or $n = 1/2$ this mapping is quadratic. Applying the (inverted) mapping (15.7) to the *circle* $\tilde{q}$ (15.4), we find curves q determined by

$$(15.9) \qquad r^{2n} - 2a^{n-1}br^n \cos n\phi - a^{2n} = 0.$$

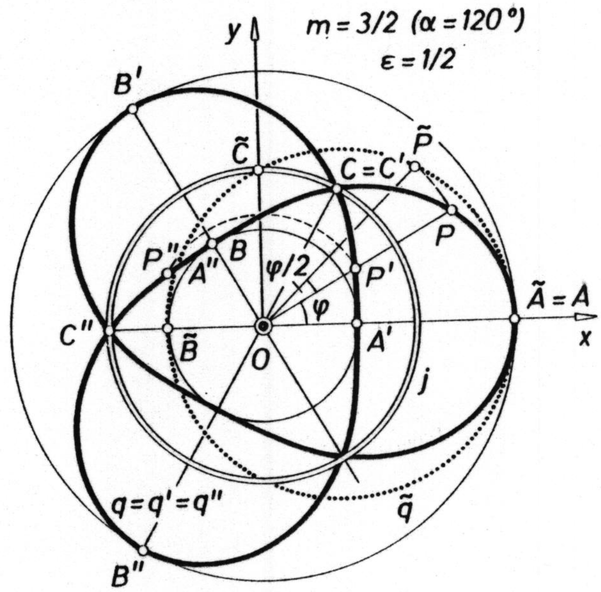


Fig.33: Curve of order 10 as a fan transform of a circle

With $n = 2$ ($\alpha = 90°$) we meet again the *Cassinian quartic* (14.10); with $n = 1/2$ ($\alpha = 360°$) we obtain the anallagmatic *Pascal snail* of Fig.6.

The last example concerns the curve with the equations

$$(15.10) \qquad r = a \cdot \cot 2\phi \quad \text{or} \quad 4x^2y^2(x^2+y^2) = a^2(x^2-y^2)^2.$$

This *circular sextic*, depicted in Fig.34, is known as the "windmill" |3, p.22|. It is a fan transform of the anallagmatic strophoid (3.1) as well as of the kappa-curve (12.20), the latter being a conformal mapping of the anti-anallagmatic quartic (12.19); compare Figs. 18 and 19.

Further examples might be added ad libitum.

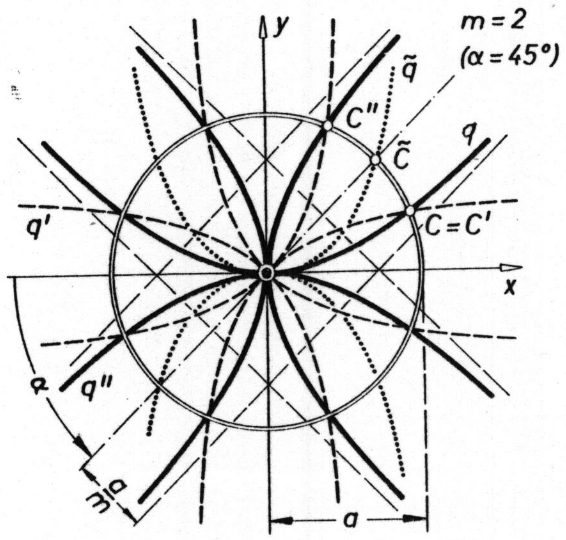


Fig.34: Circular sextic ("windmill") as a fan transform of the kappa-curve (Fig.19)

References

|1| J.L.KAVANAU: *Curves and Symmetry, I.* Science Software Systems, Inc. (Los Angeles, 1982).

|2| D.PEDOE: *A Course of Geometry.* University Press (Cambridge, 1970).

|3| H.WIELEITNER: *Spezielle ebene Kurven (Sammlung Schubert, Bd.56).* Göschen (Leipzig, 1908).

|4| W.WUNDERLICH: *Über die Schleppkurven des Kreises.* Sitzungsber.Akad.Wiss.Wien *156*, 155-173 (1948).

|5| W.WUNDERLICH: *Sur une certaine généralisation des cycloides.* Simon Stevin *36*, 57-71 (1962).