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SINGLE-DISK CAM MECHANISMS WITH OSCILLATING DOUBLE ROLLER FOLLOWER

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Abstract—Planar cam mechanisms of the kind considered here are subject to two conditions [Duca and Simionescu, *Mechanism and Machine Theory* **15,** 213–220 (1980)]. By weakening one of the conditions, which is restrictive, more practicable forms are achievable. The design is based on the choice of a certain part of either the cam profile or of the transmission law. Mathematical analysis leads to a functional equation, which is the same as that in the similar problem with flat-faced follower pairs, solved in a previous paper [Wunderlich, *J. Mechanisms* **6,** 1–20 (1971)]. All formulas for synthesis are developed. Analytical solutions, among them algebraic ones, provide particular examples of plane curves with a movable closed chord polygon.

1. INTRODUCTION

Cam mechanisms serve to transform a rotation into an oscillating motion. Such a mechanism consists of three elements: The fixed *frame* Σ_0, the rotating *cam* Σ_1, and the oscillating *follower* Σ_2. In general the follower is combined with a spring which presses it against the cam to secure steady contact. To avoid vibrations which may occur in high-speed mechanisms, the spring may be replaced by a second follower, rigidly connected with the first and operated by a second cam integral with the first cam. The idea of combining the two cams into one restricts the variety of possible transmission laws and leads to rather difficult geometrical problems[2].

In the commonly used *planar type* of cam mechanisms the motions Σ_1/Σ_0 and Σ_2/Σ_0 are rotations about parallel axes. Reducing everything to a design plane orthogonal to the axes, we have a disk profile c turning about a center O, and a follower profile p rotating about a pivot P and always touching c at a varying point T (Fig. 1). Denoting the rotation angles of Σ_1 and Σ_2 by ϕ and ψ, respectively, the behaviour of the mechanism is characterized by the *transmission law* $\psi(\phi)$, a function with period 2π.

To construct the unknown cam profile c for a given follower profile p, and for prescribed transmission law $\psi(\phi)$, we consider the mechanism from the point of view of an observer in the cam system Σ_1: c appears then as the envelope of all the positions of p occupied in the course of the relative motion Σ_2/Σ_1. The common normal of c and p at the contact point T passes through the instantaneous pole I of this motion. According to the well-known Aronhold–Kennedy Theorem, this relative pole I is situated on the line OP and divides the segment OP in the ratio $OI : PI = d\psi : d\phi$ (Fig. 1).

For a *flat-faced follower*—p being a straight line (passing through P)—the analytical determination of the cam profile c has been given in a previous paper[2]. In the same paper the problem of single-disk cams operating a flat-faced follower pair has been completely solved: The respective cam profiles c are characterizable as "curves with an isoptic‡ circle"[3]. The simplest example, indicated already by Goldberg[4], uses an ellipse c to operate the arms of a right-angle follower pair; see [2], p. 7.

Circular follower profiles are of special interest for roller followers; here the follower arm has a circular roller p of radius e at its extremity A. For this type of cam mechanism it is convenient to consider the relative path $\bar{c}$ of the roller center A: The working cam profile c is equidistant (at the distance e) to that more important "pitch profile" $\bar{c}$ (Figs. 5 and 9). In [2] the author doubted the existence of single-disk cams with an oscillating pair of rigidly connected roller followers that function precisely. This wrong conjecture, caused by omitting a very special arrangement, was corrected in a recent note of Duca and Simionescu[1]. To delete an unnecessary restriction and to clarify the character of admissible transmission laws, the subject will be treated anew. Moreover, a close connection will be revealed between the actual problem and the settled problem of flat-faced follower pairs.

2. EXISTENCE CONDITIONS

Let us consider a planar single-disk cam mechanism with pitch profile $\bar{c}$ rotating about the center

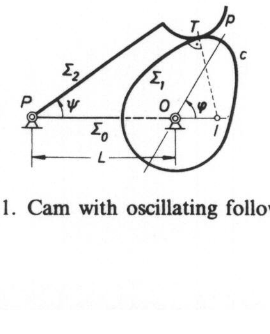

Fig. 1. Cam with oscillating follower.

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‡An isoptic curve is the locus of points from which two tangents to a given curve make a constant angle.

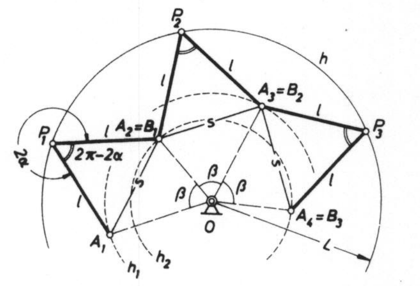

Fig. 2. Adjacent positions of a double follower.

O. The arms PA and PB of the double follower are supposed to have equal length $PA = PB = l$; the case of unequal follower arms seems to be rather hopeless. The invariant angle $\sphericalangle APB = 2\alpha$ will be measured in that sector which does not contain the cam center O. The distance from O to the follower pivot P is denoted by L (Fig. 2).

Starting from an arbitrary position $A_1P_1B_1$ of the follower, and observing all from the cam system Σ_1, we identify after a certain time an adjacent position $A_2P_2B_2$ with $A_2 = B_1$ Proceeding in this way, we obtain an *equilateral polygon* $A_1P_1A_2P_2\ldots$ of side length l, whose vertices P_i are situated on a circle h with center O and radius L, whereas the vertices A_i are alternatively distributed over two concentric circles h_1 and h_2 (Fig. 2). These points form a *chord polygon* $A_1A_2A_3\ldots$ of $\bar{c}$ with constant side length $s = 2l \cdot \sin\alpha$. Each of the chords $A_iA_{i+1} = A_iB_i$ is seen from the center O under the same angle β. In general this angle is variable, and it depends on the value π/β whether the polygon is closed or not: For π/β *rational*, the polygon *closes* after a finite number of steps; otherwise the polygon never closes. Consequently, in general (i.e. for $\beta \neq \text{const}$) the pitch profile $\bar{c}$ will contain, at any instant when π/β takes an irrational value, two infinite and everywhere dense point sets $\{A_1A_3A_5\ldots\}$ and $\{A_2A_4A_6\ldots\}$ on certain concentric circles h_1, h_2, and this circumstance excludes practicable cams[2].

The situation, however, is quite different in the case $\beta = \text{const}$, overlooked in [2], but investigated in [1]: Observing the mechanism from the follower system Σ_2, we can say that all points O from wherat the (fixed) segment AB subtends a constant angle β must lie on a circle passing through A and B. Due to $OP = L = \text{const}$, this circle has its center at P, and we obtain the *necessary condition*

$$l = L, \tag{2.1}$$

stated as condition (1) in [1]. As an immediate consequence we have $\beta = \alpha$. Moreover, to exclude infinite chord polygons we have to assume a *rational value* of $\pi/\beta = \pi/\alpha$, explicitly

$$\pi/\alpha = n = \lambda/\mu > 1, \tag{2.2}$$

where λ and $\mu < \lambda$ are natural numbers without common factor. The corresponding condition (2) in [1], demanding an integer value of $n(\mu = 1)$, imposes a superfluous restriction, which is mischievous for the actual purpose (compare Figs. 5 and 9). However, there is another restriction to be mentioned: it will turn out in Section 3 that the characteristic number $n = \lambda/\mu$ must have an *odd denominator* μ; otherwise the pitch profile $\bar{c}$ would cross itself.

We shall see that under circumstances that satisfy the three stated conditions the construction leads to suitable results, although only small numbers λ and μ are useful for practical purposes.

3. CAM DESIGN

Let us measure the rotation of the follower APB by the angle ψ which the bisector of the angle $\sphericalangle APB = 2\alpha$ makes with the base line OP. Considering the mechanism from the cam system Σ_1, we see that two adjacent positions APB and $A'P'B'$ with $A' = B$ are mirror images with respect to the axis OB, provided condition (2.1) is satisfied (Fig. 3). Hence the passage from APB to $A'P'B'$ induces the change of ψ to $-\psi$.

Due to $\sphericalangle AOB = \sphericalangle A'OB' = \alpha$ the next position $A''P''B''$ with $A'' = B'$ is related with APB by rotation about O through the angle 2α. Continuing the procedure of adding adjacent positions, we arrive at a remarkable *linkage* which is closed, if—according with condition (2.2)—the ratio π/α is a rational number $n = \lambda/\mu$. This overconstrained linkage consists of a chain of 2λ isosceles triangles joined by 2λ spokes with a fixed central hinge at O. If all joints are movable, the linkage admits a one-parametric deformation. Since all odd-numbered spokes form invariant angles 2α, they constitute a rigid system, and the same holds for all even-numbered spokes. Hence the linkage is composed of 2λ triangles and two λ-gons; the number of (binary) joints is $2\lambda + 1$. Figure 4 shows the linkage for the case $n = 3$ ($\alpha = 60°$).

From the fact that the deviation angle ψ of the follower changes its sign during the phase $APB \rightarrow A'P'B'$, it follows (for reasons of supposed continuity) that there exists at least one mean position $A_1P_1B_1$ with $\psi_1 = 0$. Starting from such a zero position, the chain of adjacent positions $A_iP_iB_i$ with $A_i = B_{i-1}$ generates a *regular polygon* $A_1A_2\ldots A_{2\lambda}$, provided μ is odd (condition 3); for μ even, the

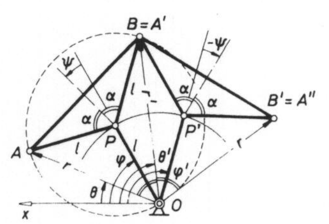

Fig. 3. Chebyshev dyads forming a plagiograph of Sylvester.

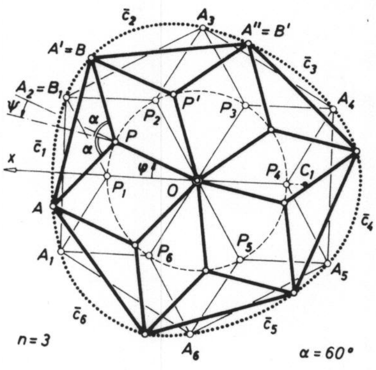

Fig. 4. Overconstrained linkage.

polygon would close already after λ steps. All vertices A_i belong to the required pitch profile $\bar{c}$ of the cam; for μ even, they would be double points.

In the case of an *integer number* $n(\lambda = n, \mu = 1)$ the regular chord polygon $A_1A_2 \ldots A_{2n}$ is *convex*. To construct a suitable profile curve $\bar{c}$, we may prescribe *arbitrarily* (but within reasonable limits) the arc $\bar{c}_1$ from A_1 to $A_2 = B_1$. The rest of the profile $\bar{c}$ is then *completely determined*: Using an auxiliary linkage of the kind represented in Fig. 4, and leading its joint A along the given arc $\bar{c}_1$, the other joints A', A'', ... will describe the missing portions $\bar{c}_2, \bar{c}_3, \ldots, \bar{c}_{2n}$ of the required profile $\bar{c}$. In the case of a *fraction* $n = \lambda/\mu$ ($\mu > 1$ and odd), the regular chord polygon $A_1A_2 \ldots A_{2\lambda}$ is *star-shaped* (Fig. 9). Now we may arbitrarily choose only a partial arc $\bar{c}_1$ from A_1 to one of the two neighbouring vertices, but the way to find the rest of the profile $\bar{c}$ is the same as before. The angular distance from A_1 to its neighbours is $\delta = \pm\pi/\lambda$.

The decisive *point-to-point mapping* χ: $A \rightarrow A'$ which transforms the initial are $\bar{c}_1$ into a new part $\bar{c}_1' = \bar{c}_2$ of the profile $\bar{c}$, and generally any known arc $\bar{c}_i$ into another part $\bar{c}_i' = \bar{c}_{i+1}$, is mechanically affected by means of a simple device consisting of the triangle APA' and the crank OP turning about O (Fig. 3). With respect to $OP = PA = PA' = l$ this mechanism of freedom 2 may be called a *generalized Chebyshev dyad*; its original form appears with $\sphericalangle\, APA' = 2\alpha = \pi (n = 2$, Fig. 8). The mapping χ, which is also simple to execute graphically, is an algegbraic two-to-two transformation of order 4: When the point A is led along a straight line g (not passing through O), then the image point A' describes a (symmetric) coupler quartic g' of the slider-crank mechanism g-$AA'P$-O. In Fig. 4 the initial arc $\bar{c}_1$ was chosen as a part of a circle (center C_1); consequently, the derived arc $\bar{c}_1' = \bar{c}_2$ is part of a (symmetric) ~~computer~~ sextic generated by the four-bar C_1-$AA'P$-O. coupler

Repeating the transformation χ, we obtain the mapping χ^2: $A \rightarrow A''$ which is, due to $OA = OA''$ and $\sphericalangle\, AOA'' = 2\alpha$, a rotation about the center O through the angle 2α. It is effected by means of a particular *plagiograph* of Sylvester, composed of two Chebyshev dyads (Fig. 3). Consequently the next portion $\bar{c}_3 = \bar{c}_2' = \bar{c}_1''$ of the profile $\bar{c}$ is directly *congruent* with $\bar{c}_1$ (Fig. 4). Analogously we have $\bar{c}_4 \simeq \bar{c}_2$, $\bar{c}_5 \simeq \bar{c}_3 \simeq \bar{c}_1$, etc. Thus, we state that the working profile c of the cam also admits the rotation with angle 2α about O, and likewise consists of alternating arcs of two kinds.

Figure 5 shows a possible form of a single-disk cam mechanism with an oscillating double roller follower, based on the scheme of Fig. 4 ($n = 3, \alpha = 60°$). The arrangement is somewhat unconventional, as the follower pivot P lies within the cam disk.

4. ANALYTICAL SOLUTION

Introducing polar coordinates r, θ in the cam system Σ_1, it may be convenient for the observer in Σ_1 to count the polar angles in opposite sense (Fig. 3). The important *mapping* χ: $A(r, \theta) \rightarrow A'(r', \theta')$ is then determined—applying the cosine theorem to triangle AOA'—by

$$r^2 + r'^2 - 2rr' \cos\alpha = 4l^2 \sin^2\alpha, \quad \theta' = \theta + \alpha. \quad (4.1)$$

The rotation χ^2: $A(r, \theta) \rightarrow A''(r'', \theta'')$ is described by

$$r'' = r, \quad \theta'' = \theta + 2\alpha. \quad (4.2)$$

Provided we know the polar equation $r = r(\theta)$ of the chosen arc $\bar{c}_1$ of the pitch profile $\bar{c}$ of the cam (Section 3), we find the equation $r' = r'(\theta')$ of the corresponding arc $\bar{c}_1' = \bar{c}_2$ by solving the quadratic eqn (4.1), thus, obtaining

$$r' = r\cos\alpha + \sqrt{(4l^2 - r^2)} \cdot \sin\alpha. \quad (4.3)$$

With respect to condition (2.2) and the consequences revealed in Section 3, the function $r(\theta)$ is supposed to be defined in an interval of length $\delta = \pi/\lambda = \alpha/\mu$, for instance in $0 \leqq \theta \leqq \delta$. The associated function $r'(\theta')$ is then obtained in the interval $\alpha \leqq \theta' \leqq \alpha + \delta = (\mu + 1)\alpha/\mu$. For reasons of continuity eqn (4.1) has to be satisfied with $r = r(0)$ and $r' = r(\delta)$. All of the additional portions $\bar{c}_3, \bar{c}_4, \ldots$ of the profile $\bar{c}$ might be found by successively con-

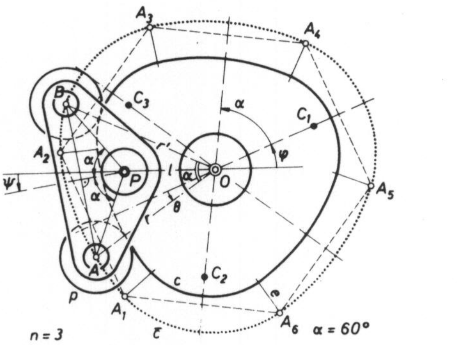

Fig. 5. Single-disk cam mechanism with oscillating double roller follower.

tinuing the procedure, but we know already that they are congruent either with $\bar{c}_1$ or with $\bar{c}_2$. This fact is in accordance with eqns (4.2), although the latter are not the only possible consequences of eqn (4.1); see [3], eqn (3.5). Unifying now all the partial functions defined in different intervals of length δ, we obtain a periodic function describing the profile $\bar{c}$ as a whole.

In order to find the *transmission law* $\psi(\phi)$ associated with the given profile function $r(\theta)$, we take from Fig. 3:

$$\cos(\phi - \theta) = r(\theta)/2l, \quad \psi + \alpha = 2(\phi - \theta). \quad (4.4)$$

By solving these equations we obtain ϕ and ψ as functions of θ, hence a parametric representation of the required law.

Moreover, we see from Fig. 3 that the rotation angles ϕ and ϕ', belonging to adjacent follower positions APB and $A'P'B'$, are related by

$$\phi + \phi' = 2\theta' = 2\theta + 2\alpha = 2\phi - \psi + \alpha. \quad (4.5)$$

With attention to the corresponding oscillation angles ψ and $\psi' = -\psi$ we state a *linear transformation*

$$\phi' = \phi - \psi + \alpha, \quad \psi' = -\psi. \quad (4.6)$$

Repeating this step, we attain

$$\phi'' = \phi' - \psi' + \alpha = \phi + 2\alpha, \quad \psi'' = \psi. \quad (4.7)$$

This means that $\psi(\phi)$ is a periodic function with period $2\alpha = 2\pi/n = 2\mu\pi/\lambda$. Taking into account the natural period 2π, it follows that any linear combination $u \cdot 2\alpha + v \cdot 2\pi$ with integers u and v is also a period of $\psi(\phi)$. Choosing for u, v a solution pair of the Diophantine equation $\mu u + \lambda v = 1$, we find the *fundamental period* $2\delta = 2\pi/\lambda$. Hence it holds

$$\psi(\phi) = \psi(\phi + 2\delta) \quad \text{with} \quad \delta = \pi/\lambda = \alpha/\mu. \quad (4.8)$$

Introducing now $\alpha = \mu\delta$ in eqn (4.6) and remembering that μ is odd (Section 2), we find the following pair of values also corresponding in the transmission law:

$$\tilde{\phi} = \phi - \psi + \delta, \quad \tilde{\psi} = -\psi. \quad (4.9)$$

The geometrical interpretation of this linear mapping $(\phi, \psi) \rightarrow (\tilde{\phi}, \tilde{\psi})$ means that the Cartesian graph of the law $\psi(\phi)$ admits a certain *affinity*. Its effect is illustrated in Fig. 6 by applying it to the rectangle with the vertices $(0, 0)$, $(\delta, 0)$, (δ, ε), $(0, \varepsilon)$; the image under (4.9) is the parallelogram with the vertices $(\delta, 0)$, $(2\delta, 0)$, $(2\delta - \varepsilon, -\varepsilon)$, $(\delta - \varepsilon, -\varepsilon)$.

Consequently, the transmission law $\psi(\phi)$ can be prescribed only in a relatively small interval of length $\delta = \pi/\lambda$; the rest is completed by application of eqn (4.9) and repetition with the period 2δ. In choosing the initial part it is necessary to take care to avoid discontinuities and corners of the graph.

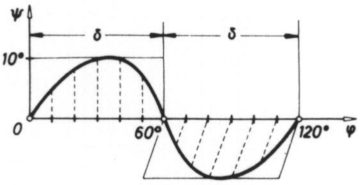

Fig. 6. Transmission law of the cam mechanism of Fig. 5.

To construct a cam for a given transmission law $\psi(\phi)$ which satisfies the condition (4.9), the polar function $r(\theta)$ of the pitch profile $\bar{c}$ may be derived from eqn (4.4). We obtain

$$\theta = \phi - \frac{1}{2}(\psi + \alpha), \quad r = 2l \cdot \cos\frac{\psi + \alpha}{2}, \quad (4.10)$$

where ϕ serves as the independent variable.

5. CONNECTION WITH THE PROBLEM OF FLAT-FACED FOLLOWER PAIRS

The solution expounded in the preceding sections depends on a suitable choice of a part of the pitch profile $\bar{c}$ or of the corresponding part of the transmission law $\psi(\phi)$. Hence the complete cam profile is "patched up" of components of different kind: Compare the example of Figs. 4 and 5, where the profile $\bar{c}$ is composed of circular arcs $\bar{c}_1$, $\bar{c}_3$, $\bar{c}_5$ and of parts $\bar{c}_2$, $\bar{c}_4$, $\bar{c}_6$ of coupler sextics. From the mathematical point of view an *analytically closed solution* would be more welcome. This means that the function $r'(\theta')$ should be the analytic continuation of $r(\theta)$. Hence $r(\theta)$ should satisfy the *functional equation*

$$r^2 + r'^2 - 2rr'\cos\alpha = 4l^2\sin^2\alpha \quad \text{with} \quad r' = r(\theta + \alpha). \quad (5.1)$$

This equation is of just the same type as eqn (4.1) in [2] which is the key equation for the similar problem of single-disk cam mechanisms with a flat-faced follower pair. Indeed there exists a close connection between the two problems: Erecting in Fig. 3 the perpendiculars $p \perp OA$ at A, $p' \perp OA'$ at A' etc., we obtain an *equiangular polygon* inscribed in a fixed circle with radius $2l$. Since its angles have the common constant value $\pi - \alpha$, the sides $p, p', \ldots$ represent adjacent positions of a rigidly connected pair of *flat-faced followers* operated by a single cam disk, whose profile c^* is the envelope of the varying positions of the polygon sides. Hence c^* is a *curve with the isoptic circle* $(O, 2l)$; this characteristic property of c^* was mentioned in Section 1. Conversely we state: *Our pitch profile* $\bar{c}$ *is a pedal curve of* c^*, i.e., the locus of the bases $(A, A', \ldots)$ of the perpendiculars dropped from the center O onto the tangents $(p, p', \ldots)$ of c^*.

Instead of adopting the analytical solution (4.9) from [2] and duly adapting it for the present purpose, we prefer to develop the solution of eqn (5.1) directly. First we introduce an auxiliary Cartesian frame $(O; X, Y)$ which rotates about the origin O in such a manner that the symmetral $X = Y$ always coincides with the bisector of the angle AOB (Fig. 7). Hence the coordinate axes form equal angles $\measuredangle XOA = \measuredangle BOY = \sigma$ with the rays OA and OB; their constant value is

$$\sigma = (\pi/4) - (\alpha/2). \tag{5.2}$$

The radial distances $r = OA$ and $r' = OB$ are linear combinations of the coordinates X and Y of the pivot P, namely

$$r = 2(aX + bY), \quad r' = 2(bX + aY)$$
with
$$a = \cos\sigma, \quad b = \sin\sigma. \tag{5.3}$$

Evidently the quantities X and Y are periodic functions of θ, with the same period 2α as $r(\theta)$ and $r' = r(\theta + \alpha)$. Moreover they satsify the relations

$$X(\theta + \alpha) = Y(\theta), \quad Y(\theta + \alpha) = X(\theta). \tag{5.4}$$

These relations correspond to the passage from APB to $A'P'B'$, and may be formally checked by means of the inversions

$$X = \frac{ar - br'}{2\sin\alpha}, \quad Y = \frac{ar' - br}{2\sin\alpha}. \tag{5.5}$$

Inserting now the expressions (5.3) in eqn (5.1), this functional equation reduces to

$$X^2 + Y^2 = l^2 \quad \text{with} \quad Y(\theta) = X(\theta + \alpha), \tag{5.4}$$

a simpler form of the condition $OP = l$. Due to eqns (5.4) the variable *difference*

$$X^2 - Y^2 = \Delta \tag{5.7}$$

is a periodic function of θ satisfying the relation

$$\Delta(\theta + \alpha) = -\Delta(\theta). \tag{5.8}$$

For an *arbitrary function* $\Delta(\theta)$ we find

$$X^2 = \frac{1}{2}(l^2 + \Delta), \quad Y^2 = \frac{1}{2}(l^2 - \Delta). \tag{5.9}$$

Making use of eqns (5.3), we finally attain the required *solution* of the functional eqn (5.1):

$$r = \sqrt{2}[a\sqrt{l^2 + \Delta(\theta)} + b\sqrt{l^2 - \Delta(\theta)}]$$
with
$$a = \cos\left(\frac{\pi}{4} - \frac{\alpha}{2}\right), \quad b = \sin\left(\frac{\pi}{4} - \frac{\alpha}{2}\right). \tag{5.10}$$

Arguing as in Section 4, we state that the auxiliary function $\Delta(\theta)$, possessing the commensurable periods 2α and

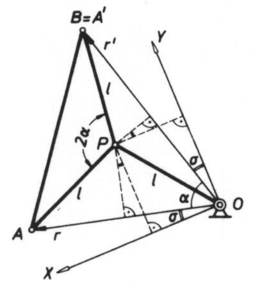

Fig. 7. Auxiliary Cartesian frame.

2π, has the fundamental period 2δ and satisfies the condition

$$\Delta(\theta - \delta) = -\Delta(\theta) \quad \text{with} \quad \delta = \pi/\lambda = \alpha/\mu. \tag{5.11}$$

Consequently, $\Delta(\theta)$ can be chosen only in an interval of length δ.

To secure real values of the square roots in formula (5.10), the restriction $|\Delta(\theta)| < l^2$ is necessary. The associated transmission law $\psi(\phi)$ is to calculate as indicated in eqns (4.4).

Another way to solve the problem has been used in [5]: Fastening to the follower APB a third arm PQ of length l and which is positioned along the bisector of the angle APB, its endpoint Q will describe a certain path q in the cam system Σ_1. Since Q repeatedly passes through the center O, this path q has the shape of a *rosette*. Choosing one of the congruent leaves of q (within rather narrow limits), and repeating it, the motion Σ_2/Σ_1 is completely determined by leading P along the circle (O, l) and simultaneously Q along the rosette q. Hence the pitch profile $\bar{c}$ of the cam is kinematically generated as the common path of the points A and B. The corresponding eqn (6.2) in [5], describing the profile $\bar{c}$, is easy to derive, but not so advantageous as eqn (5.10) above.

6. ALGEBRAIC EXAMPLES

The simplest choice of a suitable auxiliary function $\Delta(\theta)$, satisfying condition (5.11), is

$$\Delta = m^2 \cos\lambda\theta \quad \text{with} \quad 0 < m < l.$$

The resulting profile $\bar{c}$, represented by its polar eqn (5.10) or in Cartesian coordinates by

$$x = r \cdot \cos\theta, \quad y = r \cdot \sin\theta, \tag{6.2}$$

is *algebraic*, but in general not rational.

In the case $n = 2$, for instance, we have $\lambda = 2$, $\alpha = \delta = \pi/2$, $a = 1$, $b = 0$. The pitch profile $\bar{c}$, represented by

$$r^2 = 2(l^2 + m^2\cos 2\theta)$$

or

$$(x^2 + y^2)^2 = 2(l^2 + m^2)x^2 + 2(l^2 - m^2)y^2, \tag{6.3}$$

is a *bicircular quartic* known as "Booth's lemniscate" (Fig. 8).

Corresponding to a remark in Section 5, it is the central pedal of Goldberg's ellipse[4]. This curve $\bar{c}$ and also the equidistant cam profile c are ovals, if $m^2 < l^2/3$. The follower now has stretched arms, since $\sphericalangle APB = 2\alpha = \pi$. Considering the closed chain of adjacent follower positions around the cam, we detect an apparently unknown property of Booth's lemniscate: It contains a *movable chord rhombus* $AA'A''A'''$ with constant side length $s = 2l$ (Fig. 8).

The *order* N of a curve $\bar{c}$ belonging to the family (6.1) is the number of intersection points (real and imaginary) with any straight line, and may be found by considering the line equation $Ax + By = C$, after inserting there the quantities (6.2), (5.10) and (6.1), and then passing, by means of Euler's formula, to the algebraic parameter $t = e^{i\theta}$. The degree N of the resulting equation in t has (with exception of the case $n = 2$) the value

$$N = \begin{cases} 4(\lambda + 2) & \text{for } \lambda \text{ odd}, \\ 2(\lambda + 2) & \text{for } \lambda \text{ even}. \end{cases} \qquad (6.4)$$

Thus the pitch profile $\bar{c}$ in Fig. 9 which illustrates the case $n = 4/3$ is of order 12; here we have $\lambda = 4$, $\alpha = 135°$, $\sigma = -22.5°$, $a = 0.924$, $b = -0.383$. Since the double follower is right-angled and the pivot P is outside of the cam disk, the arrangement looks more familiar than in those devices which are obtained under the restrictive condition of integer values n in [1].

A vast variety of algebraic cam profiles is obtainable by combining several terms of the kind (6.1) in a trigonometric polynomial

$$\Delta = \sum_j (a_j \cos j\lambda\theta + b_j \sin j\lambda\theta), \quad j \text{ odd}. \qquad (6.5)$$

Suitable choice of coefficients a_j, b_j will allow approximation to any wanted transmission law, and that without discontinuities of derivatives.

Similar but less simple algebraic solutions are obtained by means of the other method, indicated at the end of Section 5, by choosing an algebraic rosette q (e.g. a suitable hypotrochoid, see Fig. 6 in [5]).

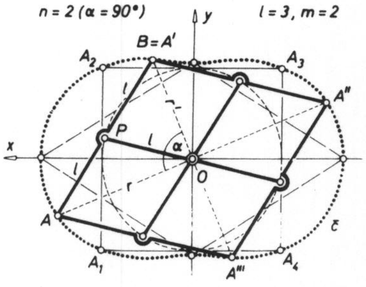

Fig. 8. Booth's lemniscate with movable chord rhombus.

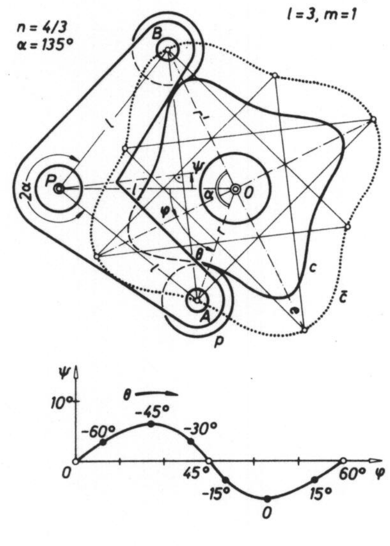

Fig. 9. Cam mechanism and transmission law.

In 1939 Fischer[6] raised the problem concerning the existence of plane curves (apart from the circle) in which an equilateral chord polygon with a certain side length s always closes, wherever it begins. Obviously the solution of our cam problem offers a family of special solutions of the geometric problem: The chain of adjacent follower segments AB of length $s = 2l \sin\alpha$ is a movable (and deforming) chord polygon of the pitch profile $\bar{c}$; see the rhombus in Fig. 8, the hexagon in Figs. 4 and 5, and the star-shaped octogon in Fig. 9. Although special, as distinguished by the fact that all chords subtend the same constant angle α from the center O, our solutions are insofar remarkable, as they contain an infinity of algebraic examples, whereas Fischer's solutions, if analytical at all, are transcendent. By the way, for the purely geometric question the restrictions μ odd and $|\Delta| < l^2$ may be dropped.

7. CONCLUSION

Planar single-disk cam mechanisms with a oscillating double roller follower—whose possibility was doubted in [2], but proved in [1]—exist under three conditions:

1. The common length l of the follower arms PA and PB must be equal to the distance between the rotation center O of the cam and the pivot P of the follower.

2. The angle $\sphericalangle APB = 2\alpha$ of the follower arms must be commensurable with the full angle 2π.

3. The rational fraction $\pi/\alpha = n = \lambda/\mu > 1$ must have an odd denominator μ. (For practical purposes only small integers λ and μ are useful.)

The cam disk shows a certain periodicity (cyclic symmetry): it admits the rotation about the center O through the angle $2\delta = 2\pi/\lambda = 2\alpha/\mu$. The polar equation $r = r(\theta)$ of the pitch profile $\bar{c}$ (equidistant to the working cam profile c) satisfies the functional equation

$$r^2(\theta) + r^2(\theta + \alpha) - 2r(\theta)r(\theta + \alpha)\cos\alpha = 4l^2\sin^2\alpha. \tag{7.1}$$

Analytical solutions of this equation are represented by formula (5.10). They depend on the arbitrary choice of an auxiliary periodic function $\Delta(\theta)$ obeying the law $\Delta(\theta + \delta) = -\Delta(\theta)$. Appropriate functions $\Delta(\theta)$, e.g. (6.1) or (6.5), provide algebraic solutions; these curves are also of purely geometric interest, as they possess movable closed chord polygon (Section 6). There exists a close connection between the present problem and the similar problem of flat-faced follower pairs operated by a single-disk cam (Section 5).

The transmission law $\psi(\phi)$, relating the oscillatory motion of the follower with the rotation of the cam, has the period 2δ and satisfies the conditions

$$\tilde{\phi} = \phi - \psi + \delta, \quad \tilde{\psi} = -\psi. \tag{7.2}$$

This means that the Cartesian graph of $\psi(\phi)$ admits a certain affine mapping onto itself.

For the synthesis of the cam mechanism with a given follower angle 2α either of two possibilities is indicated:

(a) A section of the pitch profile $\bar{c}$, subtending the angle ϕ from the center O, may be suitably chosen. The rest of the profile is then completely determined and simply derivable (Section 3). The corresponding transmission law is obtained by means of eqns (4.4).

(b) Prescribing the transmission law $\psi(\phi)$ in an interval of length δ, the rest is completely determined by the relations (7.2). The pitch profile $\bar{c}$ of the associated cam is found by means of eqns (4.10).

A remarkable overconstrained linkage, connected with the cam mechanism, is discussed in Section 3.

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EINSCHEIBEN-NOCKENTRIEBE MIT DOPPELTEM ROLLEN-SCHWINGHEBEL

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Kurzfassung - Ebene Nockentriebe der genannten Art - deren Möglichkeit in [2] bezweifelt, jedoch in [1] nachgewiesen wurde - existieren unter drei Bedingungen:

1. Die gemeinsame Länge l der beiden Schwinghebelarme PA und PB muß mit dem Abstand des Nockenzentrums O vom Hebellager P übereinstimmen.
2. Der Winkel $\sphericalangle APB = 2\alpha$ muß mit dem vollen Winkel 2π kommensurabel sein.
3. Der rationale Bruch $\pi/\alpha = n = \lambda/\mu > 1$ muß einen ungeraden Nenner μ haben.

Die zugehörige Nockenscheibe gestattet die automorphe Drehung um O durch den Winkel $2\delta = 2\pi/\lambda = 2\alpha/\mu$. Die Polargleichung $r = r(\theta)$ des Leitprofils $\bar{c}$ befriedigt die Funktionalgleichung (7.1). Analytische Lösungen, darstellbar durch (5.10), hängen von einer willkürlichen Funktion $\Delta(\theta)$ ab, welche die Bedingung $\Delta(\theta+\delta) = -\Delta(\theta)$ erfüllt. Passende Annahmen von $\Delta(\theta)$, z.B. (6.1) und (6.5), liefern sogar algebraische Lösungen, die auch von rein geometrischem Interesse sind.

Das Übertragungsgesetz $\psi(\phi)$ des Getriebes weist ebenfalls die Periode 2δ auf und unterliegt der Bedingung (7.2), welche besagt, daß das kartesische Diagramm eine gewisse Affinität verträgt.

Für die Synthese des Nockentriebs bieten sich zwei entgegengesetzte Wege an, die auf der Wahl eines (relativ kleinen) Abschnitts entweder des Leitprofils oder aber des Übertragungsgesetzes beruhen.

Bestehende Zusammenhänge mit dem verwandten Problem eines geradflankigen Schwinghebelpaares [2], mit einem übergeschlossenen Gelenkwerk (Fig. 4 und 8) sowie mit einer alten geometrischen Frage [6] werden herausgestellt.