SINGLE-DISK CAM MECHANISMS WITH
OSCILLATING DOUBLE ROLLER FOLLOWER

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Abstract—Planar cam mechanisms of the kind considered here are subject to two conditions [Duca and Simionescu, Mechanism and Machine Theory 15, 213–220 (1980)]. By weakening one of the conditions, which is restrictive, more practicable forms are achievable. The design is based on the choice of a certain part of either the cam profile or of the transmission law. Mathematical analysis leads to a functional equation, which is the same as that in the similar problem with flat-faced follower pairs, solved in a previous paper [Wunderlich, J. Mechanisms 6, 1–20 (1971)]. All formulas for synthesis are developed. Analytical solutions, among them algebraic ones, provide particular examples of plane curves with a movable closed chord polygon.

1. INTRODUCTION

Cam mechanisms serve to transform a rotation into an oscillating motion. Such a mechanism consists of three elements: The fixed frame $\Sigma_w$, the rotating cam $\Sigma_c$, and the oscillating follower $\Sigma_f$. In general the follower is combined with a spring which presses it against the cam to secure steady contact. To avoid vibrations which may occur in high-speed mechanisms, the spring may be replaced by a second follower, rigidly connected with the first and operated by a second cam integral with the first cam. The idea of combining the two cams into one restricts the variety of possible transmission laws and leads to rather difficult geometrical problems[2].

In the commonly used planar type of cam mechanisms the motions $\Sigma_1/\Sigma_2$ and $\Sigma_2/\Sigma_3$ are rotations about parallel axes. Reducing everything to a design plane orthogonal to the axes, we have a disk profile $c$ turning about a center $O$, and a follower profile $p$ rotating about a pivot $P$ and always touching $c$ at a varying point $T$ (Fig. 1). Denoting the rotation angles of $\Sigma_1$ and $\Sigma_2$ by $\phi$ and $\psi$, respectively, the behaviour of the mechanism is characterized by the transmission law $\psi(\phi)$, a function with period $2\pi$.

To construct the unknown cam profile $c$ for a given follower profile $p$, and for prescribed transmission law $\psi(\phi)$, we consider the mechanism from the point of view of an observer in the cam system $\Sigma_1$: $c$ appears then as the envelope of all the positions of $p$ occupied in the course of the relative motion $\Sigma_2/\Sigma_1$. The common normal of $c$ and $p$ at the contact point $T$ passes through the instantaneous pole $I$ of this motion. According to the well-known Aronhold–Kennedy Theorem, this relative pole $I$ is situated on the line $OP$ and divides the segment $OP$ in the ratio $O'I = d\psi : d\phi$ (Fig. 1).

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1An isoptic curve is the locus of points from which two tangents to a given curve make a constant angle.

For a flat-faced follower — $p$ being a straight line (passing through $P$) — the analytical determination of the cam profile $c$ has been given in a previous paper[2]. In the same paper the problem of single-disk cams operating a flat-faced follower pair has been completely solved: The respective cam profiles $c$ are characterizable as “curves with an isoptic circle”[3]. The simplest example, indicated already by Goldberg[4], uses an ellipse $c$ to operate the arms of a right-angle follower pair; see [2], p. 7.

Circular follower profiles are of special interest for roller followers; here the follower arm has a circular roller $p$ of radius $e$ at its extremity $A$. For this type of cam mechanism it is convenient to consider the relative path $\tilde{c}$ of the roller center $A$: The working cam profile $c$ is equidistant (at the distance $e$) to that more important “pitch profile” $\tilde{c}$ (Figs. 5 and 9). In [2] the author doubted the existence of single-disk cams with an oscillating pair of rigidly connected roller followers that function precisely. This wrong conjecture, caused by omitting a very special arrangement, was corrected in a recent note of Duca and Simionescu[1]. To delete an unnecessary restriction and to clarify the character of admissible transmission laws, the subject will be treated anew. Moreover, a close connection will be revealed between the actual problem and the settled problem of flat-faced follower pairs.

2. EXISTENCE CONDITIONS

Let us consider a planar single-disk cam mechanism with pitch profile $\tilde{c}$ rotating about the center

![Fig. 1. Cam with oscillating follower.](image-url)
where \( \lambda \) and \( \mu < \lambda \) are natural numbers without common factor. The corresponding condition (2) in [1], demanding an integer value of \( n(\mu = 1) \), imposes a superfluous restriction, which is mischievous for the actual purpose (compare Figs. 5 and 9). However, there is another restriction to be mentioned: it will turn out in Section 3 that the characteristic number \( n = \lambda / \mu \) must have an odd denominator \( \mu \); otherwise the pitch profile \( \tilde{c} \) would cross itself.

We shall see that under circumstances that satisfy the three stated conditions the construction leads to suitable results, although only small numbers \( \lambda \) and \( \mu \) are useful for practical purposes.

### 3. Cam Design

Let us measure the rotation of the follower \( A'PB' \) by the angle \( \psi \) which is the bisector of the angle \( \angle APB = 2\alpha \) makes with the base line \( OP \). Consider the mechanism from the cam system \( \Sigma \), we see that two adjacent positions \( APB \) and \( A'B'B' \) with \( A' = B \) are mirror images with respect to the axis \( OB \), provided condition (2.1) is satisfied (Fig. 3). Hence the passage from \( APB \) to \( A'B'B' \) induces the change of the angle \( \psi \) to \(-\psi\).

Due to \( \angle AOB = \angle A'OB' = \alpha \) the next position \( A''P''B'' \) with \( A'' = B'' \) is related with \( APB \) by rotation about \( O \) through the angle 2\( \alpha \). Continuing the procedure of adding adjacent positions, we arrive at a remarkable linkage which is closed, if—according with condition (2.2)—the ratio \( n/\alpha \) is a rational number \( n = \lambda / \mu \). This overconstrained linkage consists of a chain of \( 2\lambda \) isosceles triangles joined by \( 2\lambda \) spokes with a fixed central hinge at \( O \). If all joints are movable, the linkage admits a one-parametric deformation. Since all odd-numbered spokes form invariant angles 2\( \alpha \), they constitute a rigid system, and the same holds for all even-numbered spokes. Hence the linkage is composed of \( 2\lambda \) triangles and two \( \lambda \)-gons; the number of (binary) joints is \( 2\lambda + 1 \). Figure 4 shows the linkage for the case \( n = 3 \) (\( \alpha = 60^\circ \)).

From the fact that the deviation angle \( \psi \) of the follower changes its sign during the phase \( APB \rightarrow A'PB' \), it follows (for reasons of supposed continuity) that there exists at least one mean position \( A,P,B \), with \( \psi = 0 \). Starting from such a zero position, the chain of adjacent positions \( APB \), with \( A_i = B_{i-1} \) generates a regular polygon \( A_1A_2 \ldots A_{2\lambda} \), provided \( \mu \) is odd (condition 3); for \( \mu \) even, the

\[
\frac{\pi}{\alpha} = n = \frac{\lambda}{\mu} > 1,
\]

stated as condition (1) in [1]. As an immediate consequence we have \( \beta = \alpha \). Moreover, to exclude infinite chord polygons we have to assume a rational value of \( \pi/\beta = n/\alpha \), explicitly

\[
\frac{\pi}{\alpha} = n = \frac{\lambda}{\mu} > 1,
\]
polygon would close already after $\lambda$ steps. All vertices $A_i$ belong to the required pitch profile $\tilde{c}$ of the cam; for $\mu$ even, they would be double points.

In the case of an integer number $n(\lambda = n, \mu = 1)$ the regular chord polygon $A_1 A_2 \ldots A_{2\lambda}$ is convex. To construct a suitable profile curve $\tilde{c}$, we may prescribe arbitrarily (but within reasonable limits) the arc $\tilde{c}_1$ from $A_1$ to $A_2 = B$. The rest of the profile $\tilde{c}$ is then completely determined: Using an auxiliary linkage of the kind represented in Fig. 4, and leading its joint $A$ along the given arc $\tilde{c}_1$, the other joints $A', A''$, $\ldots$ will describe the missing portions $\tilde{c}_2, \tilde{c}_3, \ldots, \tilde{c}_n$ of the required profile $\tilde{c}$. In the case of a fraction $n = \lambda/\mu$ ($\mu > 1$ and odd), the regular chord polygon $A_1 A_2 \ldots A_{2\lambda}$ is star-shaped (Fig. 9). Now we may arbitrarily choose only a partial arc $\tilde{c}_1$ from $A_1$ to one of the two neighbouring vertices, but the way to find the rest of the profile $\tilde{c}$ is the same as before. The angular distance from $A_1$ to its neighbours is $\delta = \pm \pi/\lambda$.

The decisive point-to-point mapping $\chi^0: A \rightarrow A'$ which transforms the initial arc $\tilde{c}_1$ into a new part $\tilde{c}'_1 = \tilde{c}_1$ of the profile $\tilde{c}$, and generally any known arc $\tilde{c}_i$ into another part $\tilde{c}'_i = \tilde{c}_i, i$, is mechanically affected by means of a simple device consisting of the triangle $APA'$ and the crank $OP$ turning about $O$ (Fig. 3). With respect to $OP = PA = PA' = l$ this mechanism of freedom 2 may be called a generalized Chebyshev dyad; its original form appears with $\kappa APA' = 2\pi = \pi(n = 2, \mu)$, a plane in $0 \leq \theta \leq \delta$. The mapping $\chi^0$, which is also simple to execute graphically, is an algebraic two-to-two transformation of order 4: When the point $A$ is led along a straight line $g$ (not passing through $O$), then the image point $A'$ describes a (symmetric) coupler quartic $g'$ of the slider-crank mechanism $gAA'P-O$. In Fig. 4 the initial arc $\tilde{c}_1$, chosen as a part of a circle (center $C_1$), generally, the derived arc $\tilde{c}'_1 = \tilde{c}_1$ is part of a (symmetric) computer sextic generated by the four-bar $C_1 AA'P-O$.

Repeating the transformation $\chi^0$, we obtain the mapping $\chi^2: A \rightarrow A''$ which is, due to $OA = OA''$ and $\kappa AOA'' = 2\pi$, a rotation about the center $O$ through the angle $2\pi$. It is effected by means of a particular

**plagograph of Sylvester**, composed of two Chebyshev dyads (Fig. 3). Consequently the next portion $\tilde{c}_2 = \tilde{c}'_2 = \tilde{c}_1$ of the profile $\tilde{c}$ is directly congruent with $\tilde{c}_1$ (Fig. 4). Analogously we have $\tilde{c}_4 \simeq \tilde{c}_5 \simeq \tilde{c}_1 \simeq \tilde{c}_3$, etc. Thus, we state that the working profile $c$ of the cam also admits the rotation with angle $2\pi$ about $O$, and likewise consists of alternating arcs of two kinds.

Figure 5 shows a possible form of a single-disk cam mechanism with an oscillating double roller follower, based on the scheme of Fig. 4 ($n = 3, \alpha = 60^\circ$). The arrangement is somewhat unconventional, as the follower pivot $P$ lies within the cam disk.

### 4. ANALYTICAL SOLUTION

Introducing polar coordinates $r, \theta$ in the cam system $\Sigma$, it may be convenient for the observer in $\Sigma$, to count the polar angles in opposite sense (Fig. 3). The important mapping $\chi: A(r, \theta) \rightarrow A'(r', \theta')$ is then determined—applying the cosine theorem to triangle $AOA'$—by

$$r^2 + r'^2 - 2rr' \cos \alpha = 4l^2 \sin^2 \alpha, \quad \theta' = \theta + \alpha. \quad (4.1)$$

The rotation $\chi^0: A(r, \theta) \rightarrow A'(r', \theta')$ is described by

$$r'^2 = r^2, \quad \theta' = \theta + 2\pi. \quad (4.2)$$

Provided we know the polar equation $r = r(\theta)$ of the chosen arc $\tilde{c}_1$ of the pitch profile $\tilde{c}$ of the cam (Section 3), we find the equation $r' = r(\theta')$ of the corresponding arc $\tilde{c}_1 = \tilde{c}_2$ by solving the quadratic eqn (4.1), thus, obtaining

$$r' = r \cos \alpha + \sqrt{(4l^2 - r^2)} \cdot \sin \alpha. \quad (4.3)$$

With respect to condition (2.2) and the consequences revealed in Section 3, the function $r(\theta)$ is supposed to be defined in an interval of length $\delta = \pi/\lambda = \pi/\mu$, for instance in $0 \leq \theta \leq \delta$. The associated function $r'(\theta')$ is then obtained in the interval $\alpha \leq \theta' \leq \alpha + \delta = (\mu + 1)\pi/\mu$. For reasons of continuity eqn (4.1) has to be satisfied with $r = r(0)$ and $r' = r(\delta)$. All of the additional portions $\tilde{c}_3, \tilde{c}_4, \ldots$ of the profile $\tilde{c}$ might be found by successively con-

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**Fig. 4. Overconstrained linkage.**

**Fig. 5. Single-disk cam mechanism with oscillating double roller follower.**
tinuing the procedure, but we know already that they are congruent either with \( \hat{\psi} \) or with \( \hat{\psi} \). This fact is in accordance with eqns (4.2), although the latter are not the only possible consequences of eqn (4.1); see [3], eqn (3.5). Unifying now all the partial functions defined in different intervals of length \( \delta \), we obtain a periodic function describing the profile \( \hat{\psi} \) as a whole.

In order to find the transmission law \( \psi(\phi) \) associated with the given profile function \( r(\theta) \), we take from Fig. 3:

\[
\cos (\phi - \theta) = r(\theta)/2l, \quad \psi + \alpha = 2(\phi - \theta). \tag{4.4}
\]

By solving these equations we obtain \( \phi \) and \( \psi \) as functions of \( \theta \), hence a parametric representation of the required law.

Moreover, we see from Fig. 3 that the rotation angles \( \phi \) and \( \phi' \), belonging to adjacent follower positions \( ABP \) and \( A'P'B' \), are related by

\[
\phi + \phi' = 2\theta = 2\theta + 2\alpha = 2\phi - \psi + \alpha. \tag{4.5}
\]

With attention to the corresponding oscillation angles \( \psi \) and \( \psi' = -\psi \) we state a linear transformation

\[
\phi' = \phi - \psi + \alpha, \quad \psi' = -\psi. \tag{4.6}
\]

Repeating this step, we attain

\[
\phi'' = \phi' - \psi' + \alpha = \phi + 2\alpha, \quad \psi'' = \psi. \tag{4.7}
\]

This means that \( \psi(\phi) \) is a periodic function with period \( 2\alpha = 2\pi n = 2\pi n/\lambda \). Taking into account the natural period \( 2\pi \), it follows that any linear combination \( u + v \cdot 2\alpha \) with integers \( u \) and \( v \) is also a period of \( \psi(\phi) \).

Choosing for \( u, v \) a solution pair of the Diophantine equation \( \mu u + \lambda v = 1 \), we find the fundamental period \( 2\delta = 2\pi l/\lambda \). Hence it holds

\[
\psi(\phi) = \psi(\phi + 2\delta) \quad \text{with} \quad \delta = \pi/\lambda = \pi/\mu. \tag{4.8}
\]

Introducing now \( \alpha = \mu \beta \) in eqn (4.6) and remembering that \( \mu \) is odd (Section 2), we find the following pair of values also corresponding in the transmission law:

\[
\phi = \phi - \psi + \delta, \quad \psi = -\psi. \tag{4.9}
\]

The geometrical interpretation of this linear mapping \((\phi, \psi)\rightarrow(\hat{\phi}, \hat{\psi})\) means that the Cartesian graph of the law \( \psi(\phi) \) admits a certain affinity. Its effect is illustrated in Fig. 6 by applying it to the rectangle with the vertices \((0, 0), (\delta, 0), (0, \epsilon), (\delta, \epsilon)\); the image under (4.9) is the parallelogram with the vertices \((\delta, 0), (2\delta, 0), (2\delta - \epsilon, -\epsilon), (\delta - \epsilon, -\epsilon)\).

Consequently, the transmission law \( \psi(\phi) \) can be prescribed only in a relatively small interval of length \( \delta = \pi/\lambda; \) the rest is completed by application of eqn (4.9) and repetition with the period \( 2\delta \). In choosing the initial part it is necessary to take care to avoid discontinuities and corners of the graph.

Fig. 6. Transmission law of the cam mechanism of Fig. 5.

To construct a cam for a given transmission law \( \psi(\phi) \) which satisfies the condition (4.9), the polar function \( r(\theta) \) of the pitch profile \( \hat{c} \) may be derived from eqn (4.4). We obtain

\[
\theta = \phi - \frac{1}{2}(\psi + \alpha), \quad r = 2l \cdot \cos \frac{\psi + \alpha}{2}. \tag{4.10}
\]

where \( \phi \) serves as the independent variable.

5. CONNECTION WITH THE PROBLEM OF FLAT-FACED FOLLOWER PAIRS

The solution expounded in the preceding sections depends on a suitable choice of a part of the pitch profile \( \hat{c} \) or of the corresponding part of the transmission law \( \psi(\phi) \). Hence the complete cam profile is "patched up" of components of different kind: Compare the example of Figs. 4 and 5, where the profile \( \hat{c} \) is composed of circular arcs \( \hat{c}_1, \hat{c}_2, \hat{c}_3 \) and of parts \( \hat{c}_4, \hat{c}_5, \hat{c}_6 \) of coupler sextics. From the mathematical point of view an analytically closed solution would be more welcome. This means that the function \( r'(\theta') \) should be the analytic continuation of \( r(\theta) \). Hence

\[
\frac{r^2 + r'^2 - 2rr' \cos \alpha = 4l^2 \sin^2 \alpha \quad \text{with} \quad r' = r(\theta + \alpha). \tag{5.1}
\]

This equation is of just the same type as eqn (4.1) in [2] which is the key equation for the similar problem of single-disk cam mechanisms with a flat-faced follower pair. Indeed there exists a close connection between the two problems: Erecting in Fig. 3 the perpendiculars \( p \perp OA \) at \( A, p' \perp OA' \) at \( A' \) etc., we obtain an equiangular polygon inscribed in a fixed circle with radius \( 2l \). Since its angles have the common constant value \( \pi - \alpha \), the sides \( p, p', \ldots \) represent adjacent positions of a rigidly connected pair of flat-faced followers operated by a single cam disk, whose profile \( c^* \) is the envelope of the varying positions of the polygon sides. Hence \( c^* \) is a curve with the isotropic circle \((O, 2l)\); this characteristic property of \( c^* \) was mentioned in Section 1. Conversely we state: Our pitch profile \( \hat{c} \) is a pedal curve of \( c^* \), i.e., the locus of the bases \((A, A', \ldots)\) of the perpendiculars dropped from the center \( O \) onto the tangents \((p, p', \ldots)\) of \( c^* \).

Instead of adopting the analytical solution (4.9) from [2] and duly adapting it for the present purpose,
we prefer to develop the solution of eqn (5.1) directly.
First we introduce an auxiliary Cartesian frame 
\((O; X, Y)\) which rotates about the origin \(O\) in such a manner that the symmetrical \(X = Y\) always coincides with the bisector of the angle \(AOB\) (Fig. 7). Hence the coordinate axes form equal angles \(\pm \angle XOY = \sigma\) with the rays \(OA\) and \(OB\); their constant value is
\[
\sigma = (\pi/4) - (\pi/2).
\]
(5.2)

The radial distances \(r = OA\) and \(r' = OB\) are linear combinations of the coordinates \(X\) and \(Y\) of the pivot \(P\), namely
\[
r = 2(aX + bY), \quad r' = 2(bX + aY)
\]
with \(a = \cos \sigma, \quad b = \sin \sigma\).
(5.3)

Evidently the quantities \(X\) and \(Y\) are periodic functions of \(\theta\), with the same period \(2\pi\) as \(r(\theta)\) and \(r'(\theta + \pi)\). Moreover they satisfy the relations
\[
X(\theta + \pi) = Y(\theta), \quad Y(\theta + \pi) = X(\theta).
\]
(5.4)

These relations correspond to the passage from \(APB\) to \(A'B'P'B\)' and may be formally checked by means of the inversions
\[
X = \frac{ar - br'}{2\sin \alpha}, \quad Y = \frac{ar' - br}{2\sin \alpha}.
\]
(5.5)

Inserting now the expressions (5.3) in eqn (5.1), this functional equation reduces to
\[
X^2 + Y^2 = l^2 \quad \text{with} \quad Y(\theta) = X(\theta + \pi),
\]
(5.4)
a simpler form of the condition \(OP = l\). Due to eqns (5.4) the variable \textit{difference}
\[
X^2 - Y^2 = \Delta
\]
(5.7)
is a periodic function of \(\theta\) satisfying the relation
\[
\Delta(\theta + \pi) = -\Delta(\theta).
\]
(5.8)

For an \textit{arbitrary function} \(\Delta(\theta)\) we find
\[
X^2 = \frac{1}{2}(l^2 + \Delta), \quad Y^2 = \frac{1}{2}(l^2 - \Delta).
\]
(5.9)

Making use of eqns (5.3), we finally attain the required \textit{solution} of the functional eqn (5.1):
\[
r = \sqrt{2[a\sqrt{l^2 + \Delta(\theta)} + b\sqrt{l^2 - \Delta(\theta)}]}
\]
with \(a = \cos \left(\pi - \frac{\pi}{4}\right), \quad b = \sin \left(\frac{\pi}{4} - \frac{\pi}{2}\right)\).
(5.10)

Arguing as in Section 4, we state that the auxiliary function \(\Delta(\theta)\), possessing the commensurable periods \(2\pi\) and \(2\pi\), has the fundamental period \(2\delta\) and satisfies the condition
\[
\Delta(\theta - \delta) = -\Delta(\theta) \quad \text{with} \quad \delta = \pi/\lambda = \pi/\mu.
\]
(5.11)

Consequently, \(\Delta(\theta)\) can be chosen only in an interval of length \(\delta\).

To secure real values of the square roots in formula (5.10), the restriction \(|\Delta(\theta)| < l^2\) is necessary. The associated transmission law \(\psi(\phi)\) is to be calculated as indicated in eqns (4.4).

Another way to solve the problem has been used in [5]: Fastening to the follower \(APB\) a third arm \(PQ\) of length \(l\) and which is positioned along the bisector of the angle \(APB\), its endpoint \(Q\) will describe a certain path \(q\) in the cam system \(\Sigma_1\). Since \(Q\) repeatedly passes through the center \(O\), this path \(q\) has the shape of a \textit{rosette}. Choosing one of the congruent leaves of \(q\) (within rather narrow limits), and repeating it, the motion \(\Sigma_2/\Sigma_1\) is completely determined by leading \(P\) along the circle \((O, l)\) and simultaneously \(Q\) along the rosette \(q\). Hence the pitch profile \(\bar{c}\) of the cam is kinematically generated as the common path of the points \(A\) and \(B\). The corresponding eqn (6.2) in [5], describing the profile \(\bar{c}\), is easy to derive, but not so advantageous as eqn (5.10) above.

6. ALGEBRAIC EXAMPLES

The simplest choice of a suitable auxiliary function \(\Delta(\theta)\), satisfying condition (5.11), is
\[
\Delta = m^2 \cos \lambda \theta \quad \text{with} \quad 0 < m < l.
\]
The resulting profile \(\bar{c}\), represented by its polar eqn (5.10) or in Cartesian coordinates by
\[
x = r \cdot \cos \theta, \quad y = r \cdot \sin \theta,
\]
is \textit{algebraic}, but in general not rational.

In the case \(n = 2\), for instance, we have \(\lambda = 2, \quad \alpha = \delta = \pi/2, \quad a = 1, \quad b = 0\). The pitch profile \(\bar{c}\), represented by
\[
r^2 = 2(l^2 + m^2 \cos 2\theta)
\]
or
\[
(x^2 + y^2)^2 = 2(l^2 + m^2)x^2 + 2(l^2 - m^2)y^2,
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a simpler form of the condition \(OP = l\). Due to eqns (5.4) the variable \textit{difference}
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\[
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\]
or
\[
(x^2 + y^2)^2 = 2(l^2 + m^2)x^2 + 2(l^2 - m^2)y^2,
\]
is a bicircular quartic known as "Booth's lemniscate" (Fig. 8).

Corresponding to a remark in Section 5, it is the central pedal of Goldberg's ellipse [4]. This curve \( \tilde{c} \) and also the equidistant cam profile \( c \) are ovals, if \( m^2 < l^2/3 \). The follower now has stretched arms, since \( \pm APB = 2\pi = \pi \). Considering the closed chain of adjacent follower positions around the cam, we detect an apparently unknown property of Booth's lemniscate: It contains a movable chord rhombus \( AA' A" A" \) with constant side length \( s = 2l \) (Fig. 8).

The order \( N \) of a curve \( \tilde{c} \) belonging to the family (6.1) is the number of intersection points (real and imaginary) with any straight line, and may be found by considering the line equation \( Ax + By = C \), after inserting there the quantities (6.2), (5.10) and (6.1), and then passing, by means of Euler's formula, to the algebraic parameter \( t = e^{i\theta} \). The degree \( N \) of the resulting equation in \( t \) has (with exception of the case \( n = 2 \)) the value

\[
N = \begin{cases} 
4(\lambda + 2) & \text{for } \lambda \text{ odd}, \\
2(\lambda + 2) & \text{for } \lambda \text{ even}. 
\end{cases} \tag{6.4}
\]

Thus the pitch profile \( \tilde{c} \) in Fig. 9 which illustrates the case \( n = 4/3 \) is of order 12; here we have \( \lambda = 4 \), \( \alpha = 135^\circ \), \( \sigma = -22.5^\circ \), \( a = 0.924 \), \( b = -0.383 \). Since the double follower is right-angled and the pivot \( P \) is outside of the cam disk, the arrangement looks more familiar than in those devices which are obtained under the restrictive condition of integer values \( n \) in [1].

A vast variety of algebraic cam profiles is obtainable by combining several terms of the kind (6.1) in a trigonometric polynomial

\[
A = \sum_j (a_j \cos j\lambda \theta + b_j \sin j\lambda \theta), \quad j \text{ odd}. \tag{6.5}
\]

Suitable choice of coefficients \( a_j, b_j \) will allow approximation to any wanted transmission law, and that without discontinuities of derivatives.

Similar but less simple algebraic solutions are obtained by means of the other method, indicated at the end of Section 5, by choosing an algebraic rosette \( q \) (e.g. a suitable hypotrochoid, see Fig. 6 in [5]).

In 1939 Fischer [6] raised the problem concerning the existence of plane curves (apart from the circle) in which an equilateral chord polygon with a certain side length \( s \) always closes, wherever it begins. Obviously the solution of our cam problem offers a family of special solutions of the geometric problem: The chain of adjacent follower segments \( AB \) of length \( s = 2l \sin \alpha \) is a movable (and deforming) chord polygon of the pitch profile \( \tilde{c} \); see the rhombus in Fig. 8, the hexagon in Figs. 4 and 5, and the star-shaped octagon in Fig. 9. Although special, as distinguished by the fact that all chords subtend the same constant angle \( \alpha \) from the center \( O \), our solutions are insofar remarkable, as they contain an infinity of algebraic examples, whereas Fischer’s solutions, if analytical at all, are transcendent. By the way, for the purely geometric question the restrictions \( \mu \) odd and \( | \lambda | < l^2 \) may be dropped.

7. CONCLUSION

Planar single-disk cam mechanisms with an oscillating double roller follower—whose possibility was doubted in [2], but proved in [1]—exist under three conditions:

1. The common length \( l \) of the follower arms \( PA \) and \( PB \) must be equal to the distance between the rotation center \( O \) of the cam and the pivot \( P \) of the follower.

2. The angle \( \pm APB = 2\alpha \) of the follower arms must be commensurable with the full angle \( 2\pi \).

3. The rational fraction \( \pi/\alpha = n = \lambda/\mu > 1 \) must have an odd denominator \( \mu \). (For practical purposes only small integers \( \lambda \) and \( \mu \) are useful.)

The cam disk shows a certain periodicity (cyclic symmetry): it admits the rotation about the center \( O \) through the angle \( 2\delta = 2\pi/\lambda = 2\pi/\mu \). The polar equa-
tion $r = r(\theta)$ of the pitch profile $\bar{c}$ (equidistant to the working cam profile $c$) satisfies the functional equation

$$r^2(\theta) - r'(\theta + \alpha) - 2r(\theta)r(\theta + \alpha) \cos \alpha = 4l^2 \sin^2 \alpha.$$  \hspace{1cm} (7.1)

Analytical solutions of this equation are represented by formula (5.10). They depend on the arbitrary choice of an auxiliary periodic function $A(\theta)$ obeying the law $A(\theta + \delta) = -A(\theta)$. Appropriate functions $A(\theta)$, e.g. (6.1) or (6.5), provide algebraic solutions; these curves are also of purely geometric interest, as they possess movable closed chord polygon (Section 6). There exists a close connection between the present problem and the similar problem of flat-faced follower pairs operated by a single-disc cam (Section 5).

The transmission law $\psi(\phi)$, relating the oscillatory motion of the follower with the rotation of the cam, has the period $2\alpha$ and satisfies the conditions

$$\bar{\phi} = \phi - \psi + \delta, \quad \bar{\psi} = -\psi.$$  \hspace{1cm} (7.2)

This means that the Cartesian graph of $\psi(\phi)$ admits a certain affine mapping onto itself.

For the synthesis of the cam mechanism with a given follower angle $2\alpha$ either of two possibilities is indicated:

(a) A section of the pitch profile $\bar{c}$, subtending the angle $\alpha$ from the center $O$, may be suitably chosen. The rest of the profile is then completely determined and simply derivable (Section 3). The corresponding transmission law is obtained by means of eqns (4.4).

(b) Prescribing the transmission law $\psi(\phi)$ in an interval of length $\delta$, the rest is completely determined by the relations (7.2). The pitch profile $\bar{c}$ of the associated cam is found by means of eqns (4.10).

A remarkable overconstrained linkage, connected with the cam mechanism, is discussed in Section 3.

REFERENCES


