Shaky polyhedra of higher connection[1]

By WALTER WUNDERLICH (Vienna)

Summary. Families of closed polyhedra admitting infinitesimal deformation are constructed for any genus $g>0$ (connection $h=2g+1$).

1. Introduction

By the term "shaky" polyhedron, we understand a closed polyhedron in Euclidean 3-space which admits an infinitesimal deformation conserving the forms of all faces (and thus the lengths of all edges too). A model built up of rigid plates with moveable hinges along the edges (for instance, stiff cardboard connected by means of thin paper strips) shows this deformability very distinctly in a certain instability of shape. A rich variety of examples is available, beginning with the shaky octahedra of BLASCHKE [1], and completed by various contributions of GOLDBERG [2] and the author [3] (antiprisms, dipyramids, dodecahedra, icosahedra).

All of these examples are topological spheres (genus $g=0$). The existence of shaky polyhedra belonging to the topological type of the torus $(g=1)$, occasionally doubted, was proved in [4]. The corresponding examples, nevertheless, have the disadvantage of possessing certain pairs of adjacent faces which are coplanar. To remove this inconvenience, which causes a snapping noise accompanying the deformation of the model, we are going to modify the construction and, at the same time, generalize it for arbitrary genus $g \geqq 1$.

2. Construction

An orientable surface of genus g (connection $h=2g+1$) may be represented by two spheres connected by $n=g+1$ tubes. We now intend to realize this prototype by a system of connected tubular polyhedra.

Using Cartesian coordinates x, y, z we start with a plane kite *1234* determined by the vertices

(2.1) $$1(p,0,0),\ 2(r,s,0),\ 3(q,0,0),\ 4(r,-s,0),$$

[1] Paper presented at the 4th Austro-Hungarian Conference on Geometry, held in Sopron on May 18—22, 1987.

with $q>r>p>0$ and $s>0$. A second symmetric quadrangle *5678* defined by the points

(2.2) $$5(0, 0, u),\ 6(\alpha t, \beta t, w),\ 7(0, 0, v),\ 8(\alpha t, -\beta t, w),$$

where

(2.3) $$\alpha = \cos\omega,\ \beta = \sin\omega,\ \omega = \pi/n,\quad n = g+1,$$

and $v>w>u>0$, $t>0$ will be skew if $g>1$. Adding then to the eight sides of the deltoids the eight segments *15, 26, 37, 48* and *25, 27, 45, 47*, we obtain the edges of an open tubular polyhedron consisting of eight triangular faces (Fig. 1).

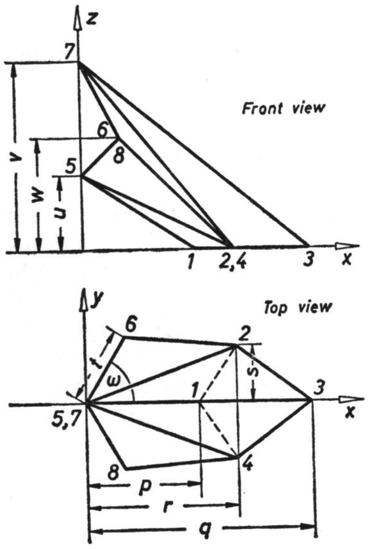

Fig. 1
Polyhedral tube element

Joining now this tube element with its mirror image with respect to the base plane $z=0$, we obtain a broken tube contained in the space sector between the planes $\beta x \pm \alpha y = 0$ if $\alpha s \leqq \beta r$. Combining finally this double tube with its congruent copies generated by repeated turns through the angle 2ω about the z-axis, we arrive at a closed polyhedron of genus $g=n+1$, provided $\alpha s<\beta r$. It consists of $f=16n$ triangular faces with $e=24n$ edges, and has $k=6n+4$ vertices (or knots in the equivalent rodwork). These numbers are related by Lhuilier's law

(2.4) $$k-e+f = 2(1-g).$$

For our polyhedron, determined by the prescribed genus g and the eight form parameters p, q, r, s, t, u, v, w, the nine different edge lengths $\overline{ij}=a_{ij}$ — fixing the shape of the object — can be calculated by means of the distance formulas

$$\begin{aligned} a_{12}^2 &= (p-r)^2+s^2, & a_{15}^2 &= p^2+u^2,\\ a_{23}^2 &= (q-r)^2+s^2, & a_{25}^2 &= r^2+s^2+u^2,\\ a_{56}^2 &= t^2+(u-w)^2, & a_{27}^2 &= r^2+s^2+v^2,\\ a_{67}^2 &= t^2+(v-w)^2, & a_{37}^2 &= q^2+v^2,\\ & & a_{26}^2 &= (r-\alpha t)^2+(s-\beta t)^2+w^2. \end{aligned} \tag{2.5}$$

3. Conditions for shakiness

In general, the polyhedra constructed in Section 2 will be rigid. Under certain circumstances, however, namely for appropriate values of the form parameters $p, q, \ldots, w$, such a polyhedron may be shaky.

To establish the corresponding conditions, we consider dislocations of the vertex system which are caused by variation of the form parameters. Due to the assumed symmetry, it is sufficient to investigate the single tube element of Fig. 1. If the displacement of the vertices (2.1) and (2.2) is described by certain functions $p=p(\tau)$, $q=q(\tau)$, etc. of a time variable τ, then their distances a_{ij} depend on τ in a definite way determined by eqs. (2.5). Stationary distances a_{ij} may be replaced, for a moment at least, by rigid rods. Differentiability supposed, we have to study the system of equations $\mathrm{d}a_{ij}/\mathrm{d}\tau=0$. From eqs. (2.5) we obtain:

$$\begin{aligned} (p-r)(\mathrm{d}p-\mathrm{d}r)+s\,\mathrm{d}s &= 0, & p\,\mathrm{d}p+u\,\mathrm{d}u &= 0,\\ (q-r)(\mathrm{d}q-\mathrm{d}r)+s\,\mathrm{d}s &= 0, & r\,\mathrm{d}r+s\,\mathrm{d}s+u\,\mathrm{d}u &= 0,\\ (u-w)(\mathrm{d}u-\mathrm{d}w)+t\,\mathrm{d}t &= 0, & r\,\mathrm{d}r+s\,\mathrm{d}s+v\,\mathrm{d}v &= 0,\\ (v-w)(\mathrm{d}v-\mathrm{d}w)+t\,\mathrm{d}t &= 0, & q\,\mathrm{d}q+v\,\mathrm{d}v &= 0,\\ (r-\alpha t)(\mathrm{d}r-\alpha\,\mathrm{d}t)+(s-\beta t)(\mathrm{d}s-\beta\,\mathrm{d}t)+w\,\mathrm{d}w &= 0. \end{aligned} \tag{3.1}$$

Thus we have nine linear homogeneous equations for the eight increments $\mathrm{d}p, \mathrm{d}q, \ldots$ $\ldots, \mathrm{d}w$. For the existence of non-trivial solutions the matrix of coeffcients must have the rank 7. Hence we have to expect two conditions.

The right-side block in the system of eqs. (3.1) may be written as follows:

$$p\,\mathrm{d}p = -u\,\mathrm{d}u = r\,\mathrm{d}r+s\,\mathrm{d}s = -v\,\mathrm{d}v = q\,\mathrm{d}q. \tag{3.2}$$

By means of these substitutions the first two of the equations on the left side of (3.1) take the form

$$\begin{aligned} (2p-r)\,u\,\mathrm{d}u+p^2\,\mathrm{d}r &= 0,\\ (2q-r)\,u\,\mathrm{d}u+q^2\,\mathrm{d}r &= 0. \end{aligned} \tag{3.3}$$

These relations are equivalent if the determinant of the coefficients vanishes, hence if

$$2pq = (p+q)\,r. \tag{3.4}$$

This is the first condition expected for shakiness. Geometrically it means that the vertices 1 and 3 are harmonically separated by the diagonal 24 and the origin 0 (Fig. 1). From the difference of eqs. (3.3) it follows that

$$\mathrm{d}r = -\frac{2u}{p+q}\,\mathrm{d}u = -\frac{ru}{pq}\,\mathrm{d}u. \tag{3.5}$$

Further more, with attention to (3.2), we find

$$\mathrm{d}s = \frac{r^2-pq}{pqs}\,u\,\mathrm{d}u. \tag{3.6}$$

The third and fourth equations on the left side of the crucial system (3.1), written in the form

$$\begin{aligned}(w-u)\,\mathrm{d}w+t\,\mathrm{d}t &= (w-u)\,\mathrm{d}u,\\ (w-v)\,v\,\mathrm{d}w+vt\,\mathrm{d}t &= (w-v)\,u\,\mathrm{d}u,\end{aligned} \tag{3.7}$$

lead to

$$\mathrm{d}w = \frac{w}{v}\,\mathrm{d}u, \quad \mathrm{d}t = \frac{(w-u)(v-w)}{tv}\,\mathrm{d}u. \tag{3.8}$$

Passing now to the last equation in (3.1), and replacing there the increments dr, ds, dt, dw by the expressions (3.5), (3.6) and (3.8) which relate them all with du, we obtain the second condition of shakiness:

$$(r-\alpha t)\left[\frac{ruv}{pq}+\frac{\alpha}{t}(w-u)(v-w)\right]+ \tag{3.9}$$

$$+(s-\beta t)\left[\frac{uv}{pqs}(pq-r^2)+\frac{\beta}{t}(w-u)(v-w)\right] = w^2.$$

Summarizing, we state: The polyhedra constructed in Section 2 admit infinitesimal deformation if the eight form parameters $q, p, \ldots, w$ satisfy the two conditions (3.4) and (3.9).

Note that the foregoing developments and results stay valid also in the limiting case $g=0$ $(n=1,\ \omega=\pi,\ \alpha=-1,\ \beta=0)$.

4. Examples

To construct a particular shaky polyhedron of prescribed genus g — which by eqs. (2.3) defines the coefficients α and β — we start with suitable values of the parameters p and q. Condition (3.4) then determines the third parameter r.

After appropriate choice of the parameters s and t, condition (3.9) yields a homogeneous quadratic equation in u, v, w which can easily be satisfied. For simplicity, it seems suitable to choose s and t in such a way that

$$s = \beta t. \tag{4.1}$$

This resctriction, which considerably reduces eq. (3.9), implies that the edges 26 and 48 are parallel to the plane of symmetry $y=0$.

As a first example, let us consider the case $g=1$ $(n=2, \omega=\pi/2, \alpha=0, \beta=1)$. Starting with $p=10$ and $q=15$, we find $r=12$ from condition (3.4). Choosing then $s=t=4$ and $w=12$, we have to satisfy the second condition $uv=150$, for instance by $u=6$, $v=25$. The resulting annular polyhedron, constructed by following the rules of Section 2, is shaky, but without the inconveniences of the model in Fig. 4 of [4]. Such a frame-like polyhedron would be obtained now by taking $t=s$, $u=p$, $v=q$. For our new model, depicted in Fig. 2, the infinitesimal dislocations of

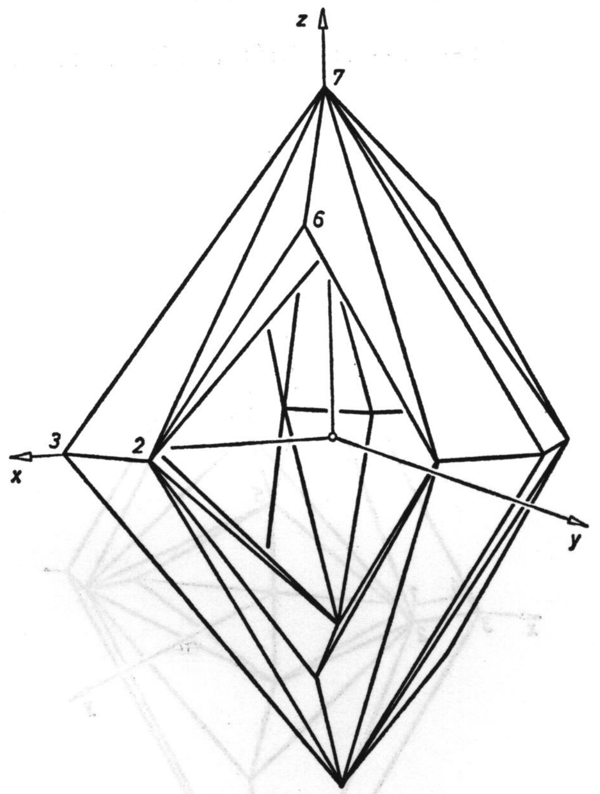

Fig. 2
Shaky polyhedron of genus 1

the vertices, determined by eqs. (3.2), (3.5), (3.6) and (3.8), are characterized by

$$\text{(4.2)} \qquad dp:dq:dr:ds:dt:du:dv:dw = $$
$$= -30:-20:-24:-3:39:50:12:24.$$

The second example concerns the case $g=2$ $(n=3,\ \omega=\pi/3,\ \alpha=1/2,\ \beta=\sqrt{3}/2)$. We start again with $p=10$, $q=15$ and $r=w=12$. Taking $s=3\sqrt{3}$ and $t=6$, we obtain for u and v the condition

(4.3) $$uv-300(u+v)+8400=0.$$

It may be satisfied with $u=10$ and $v=540/29\approx 18{,}62$. The resulting shaky polyhedron is shown in Fig. 3. Its infinitesimal deformation is characterized by

(4.4) $$\mathrm{d}p:\mathrm{d}q:\mathrm{d}r:\mathrm{d}s:\mathrm{d}t:\mathrm{d}u:\mathrm{d}v:\mathrm{d}w=$$

$$=-15:-10:-12:-1{,}15:1{,}78:15:8{,}06:9{,}67.$$

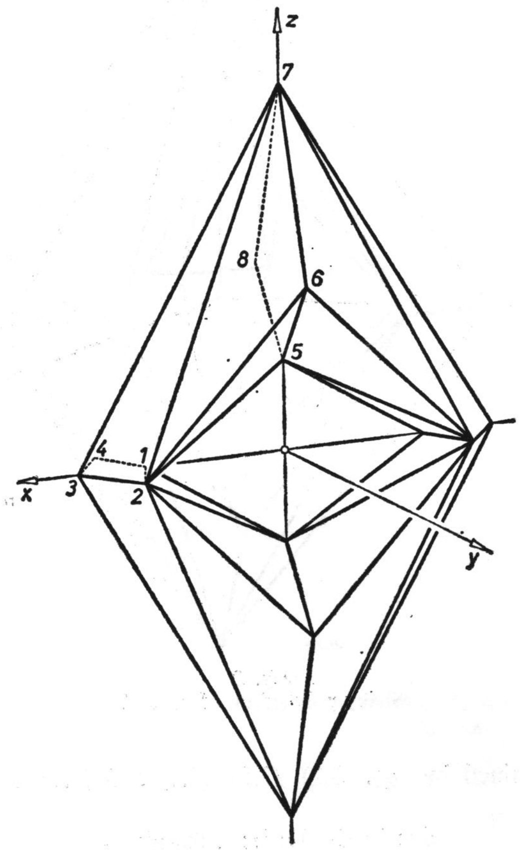

Fig. 3.
Shaky polyhedron of genus 2

References

[1] W. BLASCHKE, Wackelige Achtflache. *Math. Z.* **6** (1920), 85—93.
[2] M. GOLDBERG, Unstable polyhedral structures. *Math. Magaz.* **51** (1978), 165—170.
[3] W. WUNDERLICH, Snapping and shaky polyhedra. *Math. Magaz.* **52** (1979), 235—236. — Neue Wackelikosaeder. *Anz. Österr. Akad. Wiss.* **117** (1980), 28—33. — Wackelige Doppelpyramiden. *Anz. Österr. Akad. Wiss.* **117** (1980), 82—87. — Wackelikosaeder. *Geom. Dedicata* **11** (1981), 137—146. — Wackeldodekaeder. *Elem. Math.* **37** (1982), 153—163.
[4] W. WUNDERLICH, Ringartige Wackelpolyeder. *Anz. Österr. Akad. Wiss.* **119** (1983), 71—77.

PROF. DR. W. WUNDERLICH
INSTITUT FÜR GEOMETRIE
TECHNISCHE UNIVERSITÄT
WIEDNER HAUPTSTRASSE 8—10
A—1040 WIEN, AUSTRIA

(Received July 31, 1987)