IRREGULAR CURVES AND FUNCTIONAL EQUATIONS†

W. Wunderlich (Vienna)

Starting from a common geometric principle of generation upon which the construction of quite a number of well known irregular curves may be based, e. g., the curves of Takagi, Steinitz, Bolzano, v. Koch, Knopp, Sierpinski, Peano, Hilbert, de Rham, etc., a simple and clear arithmetic description of these curves will be derived, making use of appropriate non-decadic number systems. In connection with this point of view, it becomes evident that the curves considered can be completely characterized by certain functional equations which correspond to automorphic transformations of these curves. It seems that this principle has not been sufficiently noticed and used. In the present note it will be illustrated by applying it to the special plane curves mentioned above.

1. *Principle of generation.* Let Γ be a simply connected, bounded and closed domain of a Euclidean space E, and $A_1, \dots, A_r$ an ordered set of $r \geqslant 2$ "fundamental transformations", i. e., topological point-mappings in E satisfying the following conditions:

(I) The fundamental transformations A_k map the fundamental domain Γ on proper subdomains $A_k\Gamma$ of Γ, which have no inner points in common.

(II) The distance between any two points p, q of Γ is decreased by applying a fundamental transformation, so that dist $(A_k p, A_k q) \div$ dist $(p, q) \leqslant \theta < 1$ for all k.

From this condition—which could be weakened*—it follows that the sequence of domains $A^n{}_k\Gamma$, produced by iteration of A_k, converges towards a single point p_k, since each

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* For instance by limiting it to point-pairs of the domains $A_k\Gamma$.

domain includes the following and has a diameter tending to zero. The points p_k are invariant points of the respective transformations A_k.

(III) The invariant points $p_1=a$ and $p_r=b$ are distinct, and the points $A_k b$ and $A_{k+1} a$ coincide in the same point $a_k (k=1, \dots, n-1)$.

As the point a_k belongs to both $A_k \Gamma$ and $A_{k+1} \Gamma$, it must be a common boundary point of these two domains. It follows that a and b are boundary points of Γ.

Now let us use the number system of base r, writing the numbers of the interval $0 \leqslant \tau \leqslant 1$ in the form

$$\tau = 0, \tau_1 \tau_2 \dots = \sum_{k=1}^{\infty} \tau_k r^{-k} \text{ with } \tau_k = 0, 1, \dots, r-1. \qquad (1\cdot1)$$

Corresponding to each τ we define a certain point t by

$$t = \lim_{n\to\infty} T_1 T_2 \dots T_n \Gamma \text{ with } T_k = A\tau_{k+1}. \qquad (1\cdot2)$$

As in the sequence of domains $T_1 \dots T_n \Gamma$ $(n=1, 2, \dots)$ each one contains the next (Condition I), and as the diameter is indefinitely decreasing (Condition II), the point t is well determined in each case.

This correspondence between the interval of numbers τ and the set of points t is unique and continuous.

To be sure of the uniqueness, we have to consider those numbers τ which are represented by a finite r-adic fraction, since such numbers may also be written in a second way as an infinite fraction :

$$\tau = 0, \tau_1 \dots \tau_m 00 \dots = 0, \tau_1 \dots \tau_{m-1} \tau'_m \, r' r' \dots$$
$$\text{with } \tau'_m = \tau_m - 1, \; r' = r-1. \qquad (1\cdot3)$$

The points t, $\bar{t}$ corresponding to the first or second representation are

$$t = \lim_{n\to\infty} T_1 \dots T_m A_1^n \Gamma = T_1 \dots T_m a, \qquad (1\cdot4)$$
$$\bar{t} = \lim_{n\to\infty} T_1 \dots T_{m-1} T'_m A_r^n \Gamma = T_1 \dots T_{m-1} T'_m b$$

where $T_m' = A_{k-1}$ when $T_m = A_k$. According to Condition III we have $T_m a = T'_m b$ and the two expressions mean actually the same. The continuity follows again from Condition II, as two points t, t', corresponding to values τ, τ' which have the first n digits in common, are situated in the same domain $T_1 \dots T_n \Gamma$ and therefore have a distance not surpassing $\theta^n \gamma$, where γ denotes the diameter of Γ. Consequently this distance may be made arbitrarily small, if only τ and τ' differ sufficiently little.

The set of points t thus defined forms a certain continuous curve C, beginning in a and ending in b. It is contained in all "chains" C_n of r^n consecutive domains $T_1 \dots T_n \Gamma$, produced by applying all possible combinations of n fundamental transformations to the fundamental domain Γ. Such a chain C_n is very suitable for approximating the curve C, if only the index n is large enough. The next chain C_{n+1} is produced by effecting the transformations $A_1, \dots, A_k$ on C_n; this construction leads to a rather clear idea of the nature and appearance of the curve C. Another way of approximation may be derived from the set of points t, associated with the r^n values τ which are representable by finite r-adic fractions with not more than n figures, completed by the point b (corresponding to $\tau=1$). Connecting these points in their natural succession (given by the natural order of the numbers τ) by line segments, we get an approximating polygon P_n of r^n sides, none being longer than $\theta^n \delta$, where $\delta = \text{dist}\,(a, b)$. Of course the points used may be replaced by arbitrary points within the successive elements of the chain C_n.

2. *Analytic representation.* Translating the definition (1·2) into analytic form, based upon the use of an appropriate system of coordinates, we arrive at an arithmetic representation of the curve C, using τ as parameter.

The points t which belong to finite r-adic fractions τ will explicitly be given by application of the formulas (1·3) and (1·4). They form an everywhere dense set C^* of points of C, and this set is quite sufficient to determine the curve C, since C is continuous.

It may be remarked that in most cases it will be also possible to determine explicitly the coordinates of the points t which belong to any rational values of τ. In general such a value is represented by a periodic fraction

$$\tau=0,\ \tau_1\ldots\tau_m\overline{\tau_{m+1}\ldots\tau_{m+p}}\ldots, \qquad (2\cdot 1)$$

the bar indicating the period . The corresponding point t is then defined by

$$t=\lim_{n\to\infty} T_1\ldots T_m(T_{m+1}\ldots T_{m+p})^n\,\Gamma=T_1\ldots T_m\,c, \qquad (2\cdot 2)$$

where c denotes the invariant point of the transformation $U=T_{m+1}\ldots T_{m+p}$. In the case of linear transformations which are chiefly used, the determination of the invariant point makes necessary only the solving of linear equations.

The inverse problem of finding the parameter value τ, associated with a certain point t which is known to belong to the curve C, can be solved in the following way. At first we have to decide to which of the r subdomains $A_k\Gamma$ the given point t belongs; its index k indicates the first digit $\tau_1=k-1$ of τ. Now we apply the inverse transformation A_k^{-1} to t and then determine the subdomain $A_l\,\Gamma$ which contains the point $t_1=A_k^{-1}t$ thus obtained; its index l gives the second digit $\tau_2=l-1$. Proceeding in this way by noting the index $m=\tau_3+1$ of the subdomain $A_m\Gamma$ which contains the point $t_2=A_l^{-1}t_1$ and so on, we successively get the digits of the required value $\tau=0,\ \tau_1\tau_2\ldots$. As it may happen that a point t_n belongs to two or more subdomains $A_i\,\Gamma$, for certain points of C there may be found two or more values of the parameter. A similar principle can be used to determine the points in which a curve C is cut by a given line.

3. *Automorphisms and functional equations.* It is evident that the segment of the curve C which corresponds to an

interval

$$0,\ \tau_1\ldots\tau_n\leqslant\tau\leqslant 0,\ \tau_1\ldots\tau_n r'r'\ldots \quad (r'=r-1),$$

of length r^{-n}, is obtained by applying the transformation $T_1\ldots T_n$ to the whole curve C. It follows that an arbitrary segment of C, however small, always contains an infinity of subsegments which are "equivalent" to the whole curve, being regular maps of C.

This extraordinary property of the curve C causes in general a certain "irregularity"—for instance the non-existence of tangents or curvature— for any singularity in a or b appears reproduced in each point of the everywhere dense set C^*. This phenomenon will be sufficiently illustrated by the examples below. Nevertheless, it should be pointed out that any irregularity of C is not simply a consequence of the conditions I-III alone. As counter-example take for C an arc ab of a regular conic and remember that C can be mapped by projective transformations on any other arc of a conic, in particular on any arbitrarily small sub-arc of C.

Now let us consider the special automorphisms of the curve C which are induced by the single transformations A_k. They map C on its r segments belonging to the intervals $(k-1)/r\leqslant\tau\leqslant k/r$. The corresponding relations

$$t\left(\frac{\tau}{r}+\frac{k-1}{r}\right)=A_k\,.\,t(\tau) \quad (k=1,\ldots,r), \qquad (3\cdot 1)$$

translated into analytic form, lead to a set of functional equations for the coordinates in their dependence on τ. These functional equations are completely sufficient to characterize the curve C. For $\tau=0$ and $k=1$, the relation $t(0)=A_1\,.\,t\,(0)$ makes it possible, in the first place, to determine the initial point $t(0)=a$, and systematic application of the further relations (3·1) then leads to the points $t(k/r)=A_{k+1}\,a$ (for $k=1,\ldots,r-1$) and, successively, to all points of C^* (except b). It has already been pointed out that this everywhere dense set completely determines the curve C.

In special cases—particularly in the presence of symmetries—the number of fundamental equations may be reduced by introducing other relations.

4. *Curve of Takagi.* As first example let us consider the curve representing in Cartesian coordinates one of the most familiar continuous functions without (finite) derivative, due to T. Takagi (1903) and later generalized by K. Knopp [13, 4]. The equation is given by the convergent series

$$y = \sum_{n=0}^{\infty} 2^{-n} g(2^n x), \tag{4·1}$$

where $g(z)$ stands for the absolute difference between z and the nearest integer. This curve will be generated by our iteration principle in the following way.

Take as fundamental domain Γ the unit square $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$, and as fundamental transformations the $r=2$ affinities

$$A_1 \ldots \qquad x' = \tfrac{1}{2}x, \qquad y' = \tfrac{1}{2}(x+y); \tag{4·2}$$

$$A_2 \ldots \qquad x' = \tfrac{1}{2}(1+x), \; y' = \tfrac{1}{2}(1-x+y)\,.$$

They map the square Γ upon two adjacent parallelograms contained in it, so that Condition I is satisfied. To examine Condition II, we only have to calculate the major axis θ of the ellipse $4x'^2+4(y'-x')^2=1$, onto which the unit circle $x^2+y^2=1$ is mapped by A_1 (A_2 producing a congruent ellipse); we get $\theta=(1+\sqrt{5})/4=0{,}809\ldots<1$. The invariant points $a(0, 0)$ of A_1 and $b(1, 0)$ of A_2 are found by solving the respective equations $x'=x$, $y'=y$. As required by Condition III, the points $A_1 b$ and $A_2 a$ coincide in a point $c(1/2, 1/2)$.

The parametric representation $x=x(\tau)$, $y=y(\tau)$ of the curve C generated will be characterized by the functional equations

$$x(\tau/2) = \tfrac{1}{2}x(\tau), \; x\{(1+\tau)/2\} = \tfrac{1}{2}[1+x(\tau)];$$

$$y(\tau/2) = \tfrac{1}{2}[x(\tau)+y(\tau)], \; y\{(1+\tau)/2\} = \tfrac{1}{2}[1-x(\tau)+y(\tau)]. \tag{4·3}$$

From the first pair we deduce $x=\tau$; hence, on introducing, for the sake of simplification, a symmetry relation, the curve C is also determined by

$$y(x/2) = \tfrac{1}{2}[x+y(x)], \; y(1-x) = y(x). \tag{4·4}$$

As these equations are satisfied in the interval $0 \leqslant x \leqslant 1$ by the function (4·1) too, it follows that the curve C is identical with the corresponding arc of Takagi's curve.

It is easy to see that the curve C has a tangent, in its beginning point a coinciding with the y axis; in consequence of the automorphisms, corresponding tangents (parallel to the y axis) exist in all points of the everywhere dense set C^*, where C possesses a sort of cusps. In all other points, C has no tangents.

The general curves of Knopp, defined by the convergent series

$$y = \sum_{n=1}^{\infty} a^{-n} g(\beta^n x) \; (a>1, \; \beta = \text{integer}), \tag{4·5}$$

in particular the curve of van der Waerden ($a=\beta=10$), can be treated in a very similar way by means of a number system of base $r=2\beta$.

5. *Curves of Steinitz.* A wide class of continuous functions without derivative was constructed by E. Steinitz [12]. He starts with a function $f_1(x)$ in the interval $0 \leqslant x \leqslant 1$, linear in each of the r equal parts of this interval and having the values $y_k = c_k$ for $x_k = k/r$ ($k=0, 1, \ldots, r$; $c_0=0$, $c_r=c$). The next step consists —speaking already geometrically—in replacing each of the r sides of the polygonal line P_1 representing f_1 graphically, by a polygon affine to P_1; this operation leads to the graph P_2 of a function f_2. Proceeding in this way, always substituting for each side of a polygon P_n a polygon affine to P_1, the sequence of associated functions f_n converges towards a continuous function f, provided that

$$|c| > |c_k - c_{k-1}| > 0 \quad (k=1, \ldots, r). \tag{5·1}$$

This construction is just made for our interpretation. The fundamental transformations are represented by the above-mentioned affinities

$$A_k \ldots \qquad x'=(k-1)/r+x/r,\ y'=c_{k-1}+(d_k/c)\,y, \qquad (k=1,\ldots,r;\ d_k=c_k-c_{k-1}). \tag{5·2}$$

As fundamental domain Γ we take a rectangle between $x=0$ and $x=1$, large enough to contain all rectangles $A_k\Gamma$—which will be possible because of (5·1). It is not difficult to see that Conditions I-III are satisfied.

The approximating polygons P_n are obtained by applying all possible combinations of n transformations A_k to the line segment P_0 between a (0, 0) and $b(1, c)$. The kind of irregularity of the limit curve C depends on the choice of the c_k's. In the case $c_0=c_2=0$, $c_1=c_3=c_5=1$, $c_4=c_6=2$, for instance, discussed by H. Hahn [2], the curve possesses no tangents whatsoever.

The characteristic functional equations—introducing again $(\tau=x)$—are

$$y\{(x+k-1)/r\}=c_{k-1}+(d_k/c)\,y(x) \qquad (k=1,\ldots,r). \tag{5·3}$$

6. *Curve of Bolzano-Kowalewski.* This oldest example of a continuous function without (finite) derivative, given in 1830 by B. Bolzano and slightly simplified by G. Kowalewski [7], is based upon a certain operation which replaces every occuring line segment by a four-sided zig-zag polygon: on both halves of the segment, having the slope m, triangles are erected upwards, whose new sides have the slopes $\pm 2m$ (cf. A. N. Singh [11]). Starting from an initial line segment P_0 between the points $a(0, 0)$ and $c(1, 1)$, the successive application of this operation leads to a continuous curve whose irregularity is similar to that of Takagi's curve. The generation principle however is nearer to that of Steinitz, the fundamental transformations consisting in $r=4$ affinities

$$\begin{array}{lll} A_1 \ldots & x'=\tfrac{3}{8}x, & y'=\tfrac{3}{4}y; \\ A_2 \ldots & x'=\tfrac{1}{2}-\tfrac{1}{8}x, & y'=\tfrac{1}{2}+\tfrac{1}{4}y; \\ A_3 \ldots & x'=\tfrac{1}{2}+\tfrac{3}{8}x, & y'=\tfrac{1}{2}+\tfrac{3}{4}y; \\ A_4 \ldots & x'=1-\tfrac{1}{8}x, & y'=1+\tfrac{1}{4}y. \end{array} \tag{6·1}$$

As fundamental domain we may take the rectangle $\Gamma(0\leqslant x\leqslant 1,\ 0\leqslant y\leqslant 2)$. Then Conditions I and II hold $(\theta=3/4)$, but not Condition III, since $b(8/9,\ 4/3)\neq c(1,1)$.

Nevertheless the chain C_n of adjacent rectangles $A_{k_1}\ldots A_{k_n}\Gamma$ tends with increasing n towards Bolzano-Kowalewski's curve C, which appears approximated by the polygon P_n, consisting of the elements $A_{k_1}\ldots A_{k_n}P_0$.

So it is only the mode of parametric representation which needs a certain alteration, possible indeed by use of an auxiliary operator I which interchanges a transformation A_k with its "complementary" A_{4-k}; thus $I^n A_k$ means A_{4-k} if n is odd, and A_k if n is even. To any parameter value τ, written as tetradic fraction

$$\tau=0,\ \tau_1\tau_2\ldots \qquad (\tau_k=0, 1, 2, 3), \tag{6·2}$$

we associate now—instead of as in (1·2)—the curve point

$$t=\lim_{n\to\infty} T_1T_2\ldots T_n\Gamma$$
$$\text{with } T_k=I^{\tau_0+\tau_1+\ldots\tau_{k-1}}A_{\tau_k+1}\ (\tau_0=0). \tag{6·3}$$

According to (6·1) the characteristic functional equations are

$$y(3x/8)=\tfrac{3}{4}y(x),\ y\left(1-\frac{x}{8}\right)=1+\tfrac{1}{4}y(x),$$
$$y\left(\frac{1+x}{2}\right)=\tfrac{1}{2}+y(x/2) \tag{6·4}$$

A rather similar example with a very simple arithmetic definition—requiring equality or inequality of consecutive digits of the dyadic representation of the ordinate y, according as the corresponding digits of the triadic representation of the abscissa x are equal or not—is due to P. Finsler and was recently discussed by the author [15]. As mentioned there, the non-differentiable continuous function $y(x)$ in $0\leqslant x\leqslant 1$ is characterizable by functional equations quite analogous to (6·4), although the parametric description of the curve would be difficult.

7. *Curve of von Koch.* This well known tangentless curve—sometimes presented as "snowflake"—was invented by H. v. Koch in 1903 [6] and later generalized by K. Knopp [5]. The essential operation consists in erecting upon the central third of any oriented line segment an equilateral triangle to the left, thus changing the segment into a four-sided polygon (cf. A. N. Singh [11]). Starting from an initial segment $P_0=ab$ and successively applying the operation mentioned, one finally arrives at a Jordan curve without any tangents.

This process seems to require (as in the previous example) a number system of base four, but as remarked already by Knopp, one can do with the base $r=2$. Taking as fundamental domain an isosceles triangle $\Gamma=abc$, with the angle $\alpha<\pi/4$ at the base ab, we use as fundamental transformations A_1, A_2 the similitudes $ab\to ac$ resp. $ab\to cb$ with change of orientation. Placing Γ in the Gaussian plane of complex numbers $z=x+iy$ with $a=0$, $b=1$, $c=\frac{1}{2}\sec\alpha.\, e^{i\alpha}$, the transformations can be written as

$$A_1\cdots \quad z'=c\bar{z}, \qquad A_2\cdots \quad z'-1=\bar{c}(\bar{z}-1), \tag{7·1}$$

the bar indicating the conjugate. It is evident that for $\alpha=\pi/6$ we arrive at the approximating polygons P_n of Koch's curve by applying all possible combinations of $2n$ transformations A_k to the segment $P_0=ab$.

As Conditions I-III are satisfied $(\theta=\frac{1}{2}\sec\alpha,\ A_1b=A_2a=c)$, all developments of 1-3 may be used. The fact that for $t\to a$ the chord at continually oscillates within the sector $bac=\alpha$, is responsible for the lack of tangents in all points of the set C^*. Any other point t of C lies in the interior of an infinite sequence of nested triangles $T_1\cdots T_n\Gamma$, the bases of which form an indefinitely decreasing sequence of chords tending towards t; as consecutive bases always form the angle α, there exists no limit direction. Thus C has nowhere a tangent.

The characteristic equations for the function $z(\tau)$ are, according to (7·1),

$$z(\tau/2)=c.\bar{z}(\tau), \quad z(1-\tau)+\bar{z}(\tau)=\tfrac{1}{2}. \tag{7·2}$$

In general $z(\tau)$ describes a curve of Koch's type. For $\alpha=\pi/4$, however, there results a curve of quite another type, filling out the whole area of the triangle Γ; this very simple example of a Peano curve is due to Knopp.

8. *Curve of Sierpinski.* Another tangentless curve, possessing, however, an infinity of double points, was invented by W. Sierpinski [10].

Let us use triangular coordinates x, y, z, measuring the distances of a point from the sides of the fundamental equilateral triangle Γ, with altitude 1; they are all positive for an interior point and satisfy the condition

$$x+y+z=1. \quad . \quad . \quad . \tag{8·1}$$

The lines $x=\frac{1}{2}$, $y=\frac{1}{2}$, $z=\frac{1}{2}$ divide Γ into four similar triangles, and the essential operation of Sierpinski consists in deleting the central one. Proceeding in the same way with the remaining and all further triangles, one is finally led to a point set which can be considered as a curve, since it may be obtained also as the limit of a sequence of approximating polygons.

We effect just the same operation by considering as fundamental transformations the $r=3$ similitudes

$$\begin{array}{llll} A_1\cdots & x'=\frac{1}{2}(1+x), & y'=\frac{1}{2}z, & z'=\frac{1}{2}y; \\ A_2\cdots & x'=\frac{1}{2}x, & y'=\frac{1}{2}y, & z'=\frac{1}{2}(1+z); \\ A_3\cdots & x'=\frac{1}{2}z, & y'=\frac{1}{2}(1+y), & z'=\frac{1}{2}x. \end{array} \tag{8·2}$$

Conditions I-III are satisfied with $\theta=1/2$, $a=(1,0,0)$, $b=(0,1,0)$

For the limit curve C, the non-existence of tangents can be proved in a way similar to that for Koch's curve. The middle point c of ab is a double point (belonging to the parameter values $\tau=1/6$ and $5/6$), and of the same kind are all other points derived from c by any combinations of A_k's.

The coordinate functions $x(\tau), y(\tau), z(\tau)$ may be characterized by the following equations :

$$x\left(\frac{1+\tau}{3}\right)=\tfrac{1}{2}x(\tau), y\left(\frac{\tau}{3}\right)=\tfrac{1}{2}z(\tau), z\left(\frac{\tau}{3}\right)=\tfrac{1}{2}y(\tau); \qquad (8\cdot3)$$

$$x(\tau)+y(\tau)+z(\tau)=1, x(\tau)=y(1-\tau).$$

For numerical evaluation it would be suitable to write the coordinates as dyadic fractions; it is then possible to associate with each digit of τ corresponding digits of x, y, z.

Other curves of the same type could be derived from divisions of Γ into p^2 similar triangles, by deleting an appropriate aggregate of these.

9. *Curve of Hilbert.* The most familiar example of *a* "Peano curve", passing through every point of a square, is due to D. Hilbert [3]. His purely geometric construction is based upon the continued division of a square into four smaller squares and the numbering of the subsquares in such a way that consecutive numbers belong to neighbouring parts (cf. A. N. Singh [11]).

This process is quite in line with our principle. Let us take as fundamental domain the unit square $\Gamma(0\leqslant x\leqslant 1, 0\leqslant y\leqslant 1)$. The fundamental similarity transformations

$$\begin{array}{lll} A_1\cdots & x'=\tfrac{1}{2}y, & y'=\tfrac{1}{2}x; \\ A_2\cdots & x'=\tfrac{1}{2}x, & y'=\tfrac{1}{2}(1+y); \\ A_3\cdots & x'=\tfrac{1}{2}(1+x), & y'=\tfrac{1}{2}(1+y); \\ A_4\cdots & x'=\tfrac{1}{2}(2-y), & y'=\tfrac{1}{2}(1-x). \end{array} \qquad (9\cdot1)$$

map the square Γ onto its four subsquares into which it is divided by the medians $x=1/2$, $y=1/2$. As Conditions I-III are satisfied with $\theta=1/2$, $a=(0,0)$, $b=(1,0)$, all developments of 1-3 may be applied without difficulty. Since each point of the square is the limit of at least one (and at most four) sequences of nested squares $T_1\cdots T_n\Gamma$, each point of Γ belongs to the generated "curve" C (as simple, double, threefold or fourfold point). It is almost evident that C has no tangent.

The usual approximating polygons P_n, connecting the centers of the 4^n squares of the n-th subdivision, may be considered as chord polygons belonging to the curve points with parameter values $\tau=0, \tau_1\cdots\tau_n 2$ $(\tau_k=0, 1, 2, 3)$. The next polygon P_{n+1} consists of the four similar maps $A_kP_n(k=1, 2, 3, 4)$, completed by three connecting line segments of length 2^{-n-1}.

The following may serve as appropriate set of characteristic equations for the coordinate functions $x(\tau), y(\tau)$:

$$x\left(\frac{\tau}{4}\right)=\tfrac{1}{2}y(\tau), \; y\left(\frac{\tau}{4}\right)=\tfrac{1}{2}x(\tau),$$

$$x\left(\frac{1+\tau}{4}\right)=\tfrac{1}{2}x(\tau), \; y\left(\frac{1+\tau}{4}\right)=\tfrac{1}{2}[1+y(\tau)], \qquad (9\cdot2)$$

$$x(\tau)+x(1-\tau)=1, \; y(\tau)=y(1-\tau).$$

Numerical evaluation will be facilitated if the coordinates are written in dyadic form. An arithmetic description of Hilbert's curve, making use of such a representation, was recently given by E. Borel [1].

The original curve presented by G. Peano [8] in purely arithmetic form (cf. A. N. Singh [11]), can be generated geometrically in a quite analogous way, using repeated square divisions into nine parts. It may be remarked that, while Hilbert's quadrisecting construction is essentially unique, the partition of the square into nine subsquares opens a considerable variety of different possibilities for the iteration. Arranging the nine squares in a serpentine manner (as Peano does), the repetition can be defined in $2^9=512$ different ways; considering symmetric results as equivalent, there still remain 272 essentially different resulting Peano curves (whose approximating polygons offer a remarkably nice ornamental impression). Another possible arrangement of the nine subsquares in a meandering way allows (like Hilbert's) only one mode of repetition, if inversions (as appearing in the case of Bolzano's construction) are excluded.

10. *Curve of de Rham.* G. de Rham [9] recently revealed interesting number-theoretic relations connected with a special curve, derived from a square by continued application of an operation which cuts off the corners of a polygon in such a way that the central thirds of the sides remain untouched. The resulting continuous curve has in every point a tangent, but infinite curvature in the points of an everywhere dense set.

To generalize this construction, let us consider the fundamental triangle Γ with the vertices $0(0, 0)$, $a(1, 0)$, $b(0, 1)$, and let us apply the fundamental affinities

$$A_1 \dots \qquad x'=2\lambda+(1-2\lambda)x-\lambda y,\ y'=\lambda y; \tag{10·1}$$

$$A_2 \dots \qquad x'=\lambda x,\ y'=2\lambda-\lambda x+(1-2\lambda)y,$$

with $0<\lambda<1/2$. Conditions I-III being satisfied, the process leads to a continuous curve C between the vertices a and b, which are the invariant points of A_1 and A_2. Characteristic functional equations are

$$\left.\begin{array}{l} x(\tau/2)=2\lambda+(1-2\lambda).x(\tau)-\lambda.\,y(\tau), \\ y(\tau/2)=\lambda.\,y(\tau),\ x(1-\tau)=y(\tau). \end{array}\right\} \tag{10·2}$$

To discuss the behaviour of the curve in the initial point a, let us introduce new coordinates ξ, η by means of

$$\xi=(1-3\lambda)(x-1)-\lambda y,\ \eta=\lambda y. \tag{10·3}$$

Then the transformation A_1 takes the simple form

$$\xi'=(1-2\lambda)\,\xi,\ \eta'=\lambda\eta. \tag{10·4}$$

Any point of C, when transformed by repeated application of A_1, moves along a trajectory of the continuous group generated by A_1, i.e. a general parabola

$$\xi/\eta^q=\text{const, with } q=\lg(1-2\lambda)/\lg\lambda. \tag{10·5}$$

This trajectory touches the axis $\eta=y=0$ when $0<q<1$, that is for $0<\lambda<1/3$, while it touches the line $\xi=0$ when $q>1$, that is for $1/3<\lambda<1/2$. For that reason the first class of curves C consists of smooth curves, touching the axis of x in a, and the axis of y in b, while the second class has corners with different half-tangents in all points of the everywhere dense set C^*. The limit case $\lambda=1/3$ —just the curve of de Rham —belongs yet to the first class, but needs a special treatment, since the determinant of the auxiliary transformation (10·3) vanishes.

The smooth class $0<\lambda\leqslant 1/3$ may be divided into two sub-classes. According to (10·5) the curves for $0<\lambda<1/4$ have the curvature zero in a and thus in all points of C^*, while the curves with $1/4<\lambda\leqslant 1/3$ possess infinite curvature in these points. The limit case $\lambda=1/4$ leads to the ordinary parabola $\xi^2=\eta$, demonstrating in a non-trivial way that the generation principle discussed in this paper need not produce irregular curves under all circumstances.

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