1 Math Lectures GG – Elementary

▸▸▸ **Application: measuring distances through GPS**

Using GPS (Global Positioning System), it is possible to determine one's position at any point on Earth up to a few meters. We will discuss in Application p. 36 how these remarkably precise positions are calculated. In summary, it is necessary to take accurate measurements of one's distance to three satellites. These satellites continuously transmit characteristic signals that propagate at the speed of light. In order to determine the distance of the satellite, one must measure the time that it takes for the signal to reach the receiver, and multiply it by the speed of light. How accurately must such time measurements be taken so that the accuracy of position would fall within $1\,\mathrm{m}$?

Solution:

The speed of light $300,000\,\frac{\mathrm{km}}{\mathrm{s}} = 3\cdot 10^5\,\frac{\mathrm{km}}{\mathrm{s}} = 3\cdot 10^8\,\frac{\mathrm{m}}{\mathrm{s}}$ implies that the time for the signal to cross a meter is $\frac{1}{3}\cdot 10^{-8}\,\mathrm{s} \approx 3.3\cdot 10^{-9}\,\mathrm{s}$.

One should be able to measure time to an accuracy of a nanosecond. Even in a microsecond, the GPS signal will already have travelled a distance of $300\,\mathrm{m}$.

⊕ *Remark*: Even atomic clocks (which are used on satellites) can "only" measure microseconds reliably (the magnitude of error of such a clock equates to about 1 second in 6 million years). To overcome this difficulty, the GPS employs a trick: The characteristic signal encodes a description within the wave itself: Since it has a period of $300\,\mathrm{m}$ in length, the exact position within the oscillation at the time the wave is received can be observed. Thus, it is possible to say how many nanoseconds have passed since the last microsecond. ⊕ ◂◂◂

▸▸▸ **Application: if the Antarctic ice melts . . .**

The Antarctic (the mainland has a surface area of about 12 million km^2) has only relatively recently become frozen – but now it is all the more so! The major part of the fresh water on Earth is bound in its ice sheet, which is about 2 km thick on average. By how much will the sea level rise if the ice melts completely?

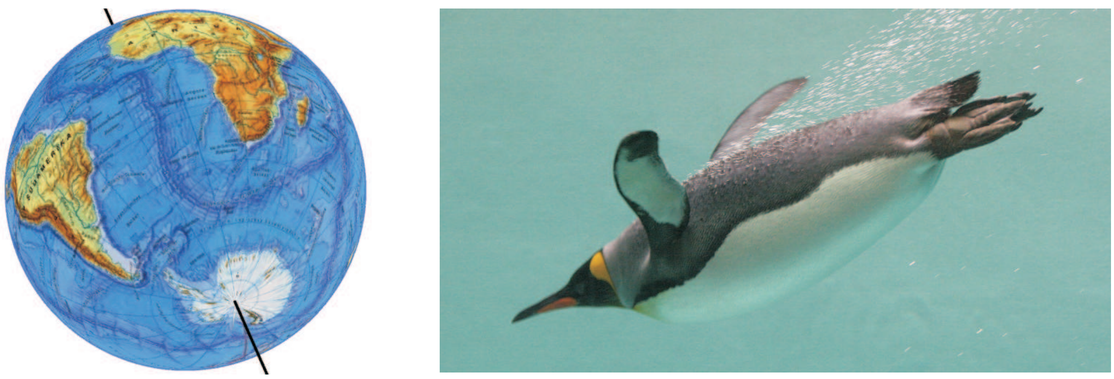

Fig. 1.1. the Antarctic and one of its inhabitants – the emperor penguin

Solution:

We will set the unit to kilometers. The volume of ice on the mainland is $24\cdot 10^6\,\mathrm{km}^3$. When ice melts, it loses about 10% of its volume, so that about $21\cdot 10^6\,\mathrm{km}^3$ remains.

The Earth has a surface area of about $500 \cdot 10^6\,\mathrm{km}^2$, of which $70\% \approx 350 \cdot 10^6\,\mathrm{km}^2$ is water. Let Δ be the difference in the heights of the sea level, then

$$350 \cdot 10^6\,\mathrm{km}^2 \cdot \Delta = 21 \cdot 10^6\,\mathrm{km}^3 \Rightarrow \Delta \approx \frac{21}{350} = 0.06\,\mathrm{km} = 60\,\mathrm{m}.$$

⊕ *Remark*: 60 m is quite a lot. With this rise of sea level, whole groups of islands would disappear, large parts of Florida would be flooded, etc. The emperor penguins probably would die out – not because the temperature would become too high, but because once again – as before – mammals and reptiles would start to live on the Antarctic. These mammals and reptiles would then steal the eggs of emperor penguins and eat their defenceless pups.

Fig. 1.2. The last glacial maximum ($\approx 24,500$ BC): Vast ice sheets covered large parts of northern Europe, North America, Siberia, and also parts of the southern hemisphere.

Conversely, the sea level rose to this level of 60 meters about 15 million years ago (because the Antarctic was free of ice). During the ice ages that have occurred several times in the last $100,000$ years (where, for example, thick ice was superimposed over central and northern Europe), the sea level was 125 m (!) lower than today. Today divers in Southern France have found underground entrances of caves where Stone Age people lived! The low sea allowed a small group of people to walk across the Bering Strait from Siberia to Alaska. During that time, they had more than $22,000$ years to migrate to the two American continents (they joined together only 3 million years ago). ⊕ ◂◂◂

▸▸▸ Application: continental drift

Pangea broke up into two roughly equal continents called *Laurasia* and *Gondwana* roughly 280 million years ago. About 150 million years ago, *Gondwana* further broke and has been drifting apart since then – as seen, for example, in the movement of South America and Africa away from each other. What used to be a daring theory (Alfred *Wegener*, 1912) can be confirmed through measurement today. Assume that the drift velocity v was reasonably constant and calculate how far Africa and South America drift apart every year or how far they drift apart every second when their current distance is about $5,000\,\mathrm{km}$. ◂◂◂

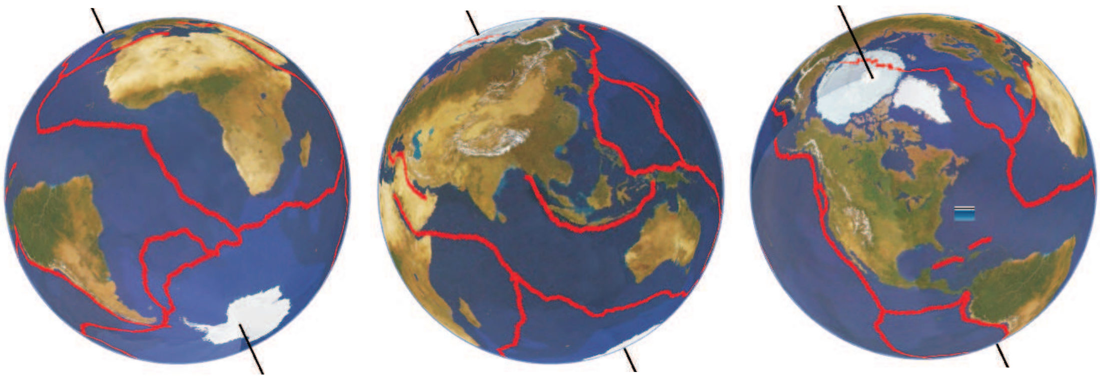

Fig. 1.3. drifting continents, tectonic plates

Solution:

$$v = \frac{5,000\,\mathrm{km}}{150 \cdot 10^6\,a} = \frac{5 \cdot 10^6\,\mathrm{m}}{150 \cdot 10^6\,a} \approx \frac{3 \cdot 10^{-2}\,\mathrm{m}}{a} = \frac{3 \cdot 10^{-2}\,\mathrm{m}}{365 \cdot 24 \cdot 3,600\,\mathrm{s}} = \frac{3 \cdot 10^{-2}\,\mathrm{m}}{3 \cdot 10^7\,\mathrm{s}} = \frac{1 \cdot 10^{-9}\,\mathrm{m}}{\mathrm{s}}.$$

We have a drift velocity per year of 3 cm.

⊕ *Remark*: We can measure the original distance using the drift velocity of 3 cm per year by looking at the current positions (currently at 1 m accuracy) of many places in Africa or South America over large time intervals by means of GPS (Application p. 1, Application p. 36). Then, the drift velocity is obtained by calculating the average, and this becomes more accurate given the more measurements one has taken and the longer the time interval is (for example, 5 years). ⊕

▸▸▸ Application: a global conveyor belt as an air motor

The enormous amount of 20 million cubic meters of salt water (which is almost half the amount of freshwater on Earth) flows every second, most of it at great depths. It flows in streams at a rate of 1 to 3 km per day repeatedly around the world (Fig. 1.4). In certain places, for example, in the Gulf of Mexico, this stream is "caught" and rises so that it heats up, quickly reaching the polar regions (the Gulf Stream!), where it cools down rapidly and descends again. Looking at the six projections in Fig. 1.4, the question arises: How long will a full cycle take?

Solution:

An important preliminary remark: The circulation belt is usually depicted on a "rectangular projection" where each estimation of length – especially near the poles – leads to incorrect results.The sphere is doubly curved and cannot be unfolded distortion-free into the plane. The multiple belts encircling the South Pole (bottom right) is clearly visible in the image. It is in total no more than a huge loop in the Pacific (bottom left). If we estimate the length of all flows thoroughly, then the length is roughly three times the Earth's circumference (120,000 km). Assuming the flow covers a distance of 1 to 3 km daily, then assuming a yearly distance of 600 km – just to have a "nice number" – the cycle lasts for 200 years. ◂◂◂

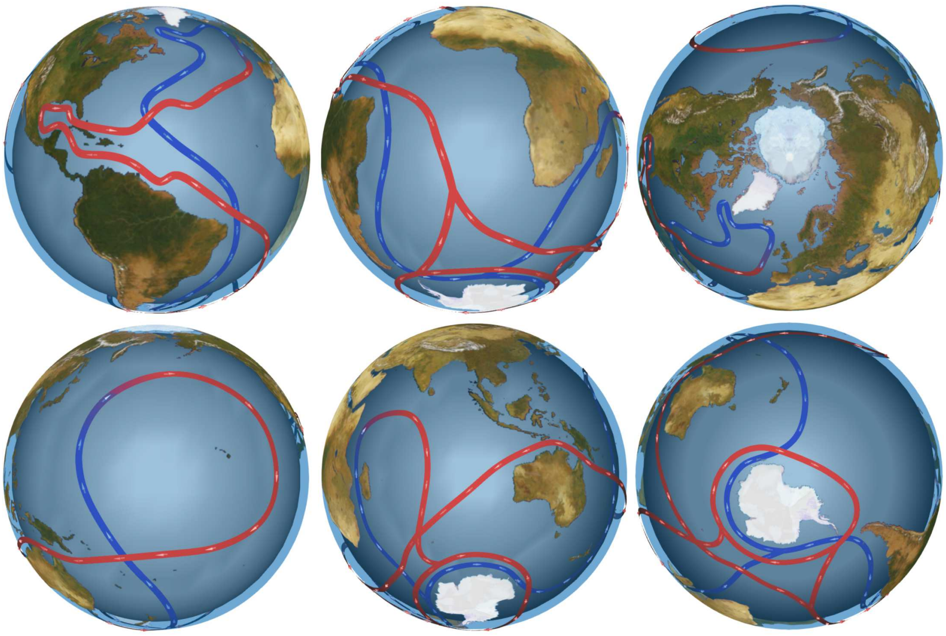

Fig. 1.4. thermohaline circulation, colloquially shown as a global conveyor belt in different views of the globe. Images using a single view of the globe can easily lead to misinterpretations of the currents (particularly around the Antarctic).

▸▸▸ Application: a crown of pure gold

Legend says that King Hiero confronted the great all-round scientist Archimedes with a tricky question, and the answer was a matter of life or death for his goldsmith: He wanted to know if his crown was made of pure gold, but this should be determined without destroying the crown. Archimedes's answer: He brought a balance beam (Fig. 1.5) to balance between a piece of pure gold on one side and the crown on the other side. It changed upon immersion within water and was not at equilibrium, which spelt bad news for the goldsmith ...

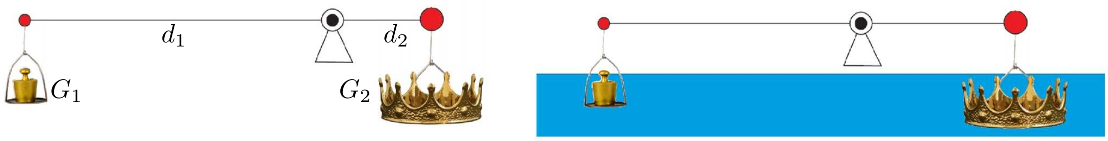

Fig. 1.5. the gold test of *Archimedes*: $G_1 : G_2 = d_2 : d_1$

Solution:

Here Archimedes combined two of his most important inventions: the law of the lever and Archimedes's principle concerning buoyancy. Suppose a calibrated weight of pure gold (weight G_1, mass $M_1 = G_1/\varrho_1$; the density $\varrho_1 = \varrho_{Au}$ being as given in the previous example). With a volume of V_2 and a density of ϱ_2, the crown has the weight $G_2 = V_2{\cdot}\varrho_2{\cdot}g$. If the beam balance (Fig. 1.5, left) is in equilibrium, then $G_1 : G_2 = d_2 : d_1$ from which G_2 can be deduced.
If we now submerse the two weights completely in water, their buoyancy forces reduce by about the weight of either displaced amount of water. Thus, the respective density decreases indirectly by about the density of water, i.e., about 1 kg per dm^3:
$G_1^* = M_1 \cdot (\varrho_1 - 1) = G_1 \cdot (\varrho_1 - 1)/\varrho_1$ and $G_2^* = M_2 \cdot (\varrho_2 - 1) = G_2 \cdot (\varrho_2 - 1)/\varrho_2$.

With the same density $\varrho_1 = \varrho_2$, the balance remains in equilibrium. However, if $\varrho_2 < \varrho_1$, then the buoyancy of the crown is slightly higher, and as a result, the crown rises. Equilibrium is reached again only when the bearing point is moved to the left by a (measurable) distance of Δ (Fig. 1.5, right). With a little skill, you can calculate ϱ_2 from all known values, and it subsequently determines a possible silver content in the crown. This works well when using a weight G_1 that is not made of pure gold. ◂◂◂

▸▸▸ Application: swarm rules (Fig. 1.6)

At times, shoaling fish or flocking birds seem to "dance" in the water or in the sky. One might be excused for thinking that there is a complicated and deliberate choreography behind it. Animals often gather or travel together in large numbers. In many cases, this swarming behavior serves as a defense against predation. Yet, how can we explain the complex and intriguing motions of swarms, as these collections of animals change directions within a fraction of a second, split up into groups and then reunite?

Solution:

One might assume the existence of an alpha specimen that determines the motion of the swarm. However, how is it possible that this individual always stays at the front of the pack? In fact, there is no such leader of the pack. All members of the swarm are equal while they are in motion and merely follow three very simple rules:
1. Move in a common direction.
2. Always keep a certain distance to your neighbours.
3. If a predator is approaching, escape.

In moments of danger, the distances between neighbours increase due to reaction time.

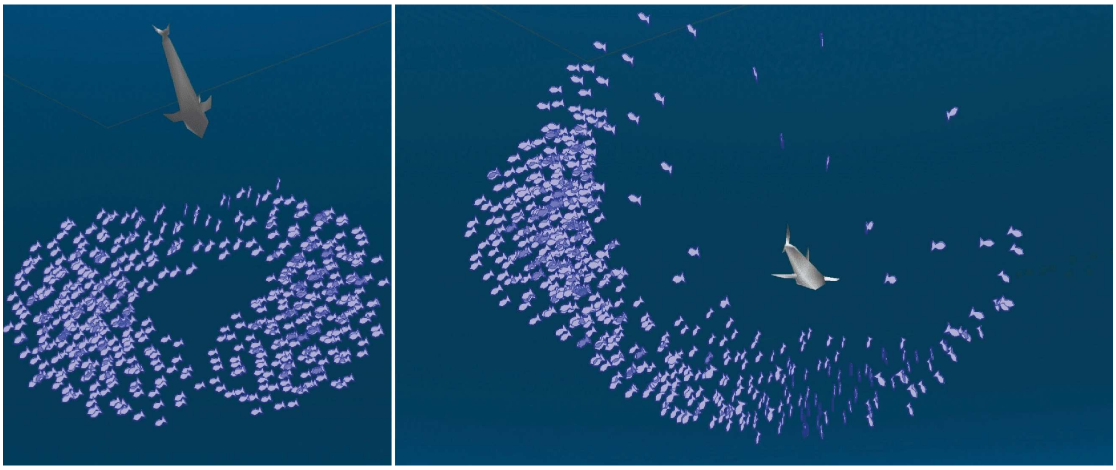

Fig. 1.6. three major rules ...

The swarm becomes wider and may even "tear apart", but as soon as the predator has left the scene, the remaining group usually reunites. As a virtual shark attacks a swarm in a computer simulation, all individuals obey the abovementioned rules. This simulation yields extremely realistic behaviour and may, thus, be taken as heuristic "proof" that the swarm rules actually exist. What is more, predators are usually distracted by swarming behaviour, and the chances of survival are larger for the individual. ◂◂◂

2 Math Lectures GG – Scaling

Objects are considered to be *similar* if they differ only in scale but not in form. Angles are identical, and pairs of lengths are at a constant proportion k with respect to each other.

In the plane, the following important theorem applies:

If the lengths of a two-dimensional object are enlarged by a factor k, then its area is enlarged by the factor k^2.

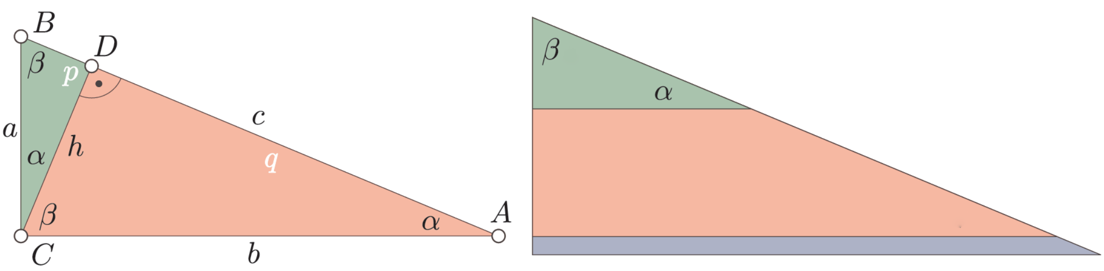

Fig. 2.1. a remarkable proof

▸▸▸ Application: Einstein's proof of the Pythagorean theorem
Almost everyone knows the formula $a^2 + b^2 = c^2$ for the lengths of the sides of a right triangle. There are hundreds of different proofs for it. Particularly noteworthy is the proof that the eleven year-old *Albert Einstein* (18791955) discovered.
Solution:
Einstein imagined a triangle ABC as being composed of the similar (since the angles are the same) component triangles CBD and ACD (Fig. 2.1). All three similar triangles (they have hypotenuses a, b, c) can be obtained from a prototype with a hypotenuse of length 1 by multiplying its sides by the factors a, b, c. Let F be the area of this prototype. Since the area of a triangle grows as the square of the scaling factor, we have $F \cdot c^2 = F \cdot a^2 + F \cdot b^2$, and one has only to cancel F to obtain the Pythagorean theorem. ◂◂◂

In *space*, the analogous and important result on scales reads:

Let an object and a similar copy of it be given. Corresponding measures L in lengths may be in a ratio of $1 : k$ (scaling factor k), then corresponding areas (surfaces) are in a ratio $1 : k^2$ and corresponding volumes are in ratio $1 : k^3$:

$$L_1 : L_2 = 1 : k, \quad S_1 : S_2 = 1 : k^2, \quad V_1 : V_2 = 1 : k^3. \tag{2.1}$$

When you magnify an object, the volume increases faster than the surface. Specifically: The ratio $V : S =$ *Volume* : *Surface* is proportional to the scaling factor k.

▸▸▸ Application: construction of the pyramids of Giza (Fig. 2.2)
What percentage of the mass of the Great Pyramid of Giza (pyramid of Chephren,

Menkaure) was installed at the time when the pyramid reached one-third (one-half, three-fourths) of its final height? What percentage of the surface was done at this time?

Fig. 2.2. the pyramids of Giza at the outskirts of Cairo

Solution:
The following statement applies to each of the pyramids:
We expect the missing upper part of the pyramid to be similar to the totally built pyramid. This residual pyramid had 2/3 (1/2, 1/4) of the final height. The missing mass was, therefore, $(2/3)^3 = 8/27$ $((1/2)^3 = 1/8$, $(1/4)^3 = 1/64)$ of the total mass. Thus, $1 - 8/27 = 19/27$ $(1 - 1/8 = 7/8, 1 - 1/64 = 63/64)$ of the mass has already been installed, or about 70% (87.5%, 98.4%). For the surfaces, the square of the scaling factor matters. Accordingly, for the time in question, there is already $1 - (2/3)^2 = 5/9$ $(1 - (1/2)^2 = 3/4$, $1 - (1/4)^2 = 15/16)$ of the surface finished, this is about 56% (75%, 94%).

⊕ *Remark*: The surface of the great pyramids was originally smoothly polished and reflected sunlight. Thus, the pyramids offered a completely different impression than today. Only about 500 years ago, the precious surface material was almost completely removed and used for the construction of buildings in Cairo. Only the uppermost part of Khafre's Pyramid is still reasonably intact. ⊕ ◂◂◂

▸▸▸ Application: weight comparison
A man of 1.60 m height has a mass of 50 kg. What is the mass of another man who is 2 m tall and has a similar figure?
Because of the same density, the masses are in ratio $M_1 : M_2$ and so are the volumes $V_1 : V_2$,

$$k = 2 : 1.6 = 1.25 \;\Rightarrow\; k^3 \approx 2 \Rightarrow\; M_2 \approx 2 \cdot M_1 = 100\,\text{kg}.$$

In practice, however, larger people often have different proportions. In fact, they tend to have muscles of a relatively smaller gauge.[1] ◂◂◂

▸▸▸ Application: the surface of the Moon and Mars
The Moon has a fourth of the diameter of the Earth, and therefore, a sixteenth of the surface, i.e. approximately 31 million square kilometers. This is almost exactly the area of Africa. Mars has a diameter that amounts to a little more than half the diameter of the Earth (0.532). Thus, the surface is little more than a quarter of the Earth's surface ($0.532^2 = 0.283$), which is the area of all continents together. ◂◂◂

[1]The exact mathematical result, therefore, does not necessarily coincide with the various "BMI tables" (*body mass index*), which can be found on the Internet.

▸▸▸ Application: subjective size of the Moon
The distance of the Moon from the Earth varies between 356,410 km and 406,760 km. How many times greater in these extreme cases do the following appear: a) the surface of the crescent, b) the supposed volume c) the brightness of the Moon?

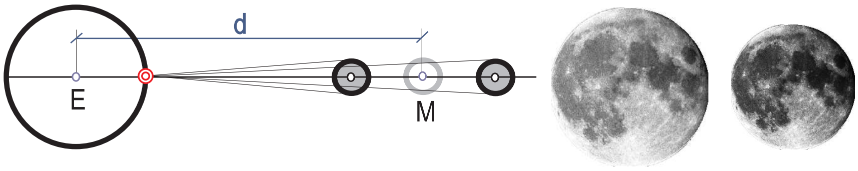

Fig. 2.3. subjective Moon size at varying distances

Solution:
The given circumstances relate the respective centers of the Earth and the Moon. In fact, we have to subtract the Earth's radius $R = 6,370\,\text{km}$, though the result is, of course, hardly affected (Fig. 2.3). We have

$$1 : k = (356,410\,\text{km} - R) : (406,760\,\text{km} - R) \approx 1 : 1.144$$

$$\Rightarrow A_1 : A_2 = 1 : 1.31, \quad V_1 : V_2 = 1 : 1.50.$$

The area of Moon's disc in the firmament fluctuates by almost a third, and with it, the radiation strength of the Earth fluctuates through the Moon as well. The Moon's volume subjectively varies by as much as 50%. However, these extreme values are only achieved at intervals of several months. ◂◂◂

▸▸▸ Application: divers fears (Fig. 2.4)
Viewed through a diving mask, submerged objects appear $\frac{4}{3}$ times their actual size (based on the length scale). So, a 1.3 m long reef shark appears to have a length of 1.7 m, but with an alleged mass increasing by a factor of $(\frac{4}{3})^3 \approx 2.4$ (240%). This may be seen as a possible contributing reason behind reported sightings of gigantic sharks. ◂◂◂

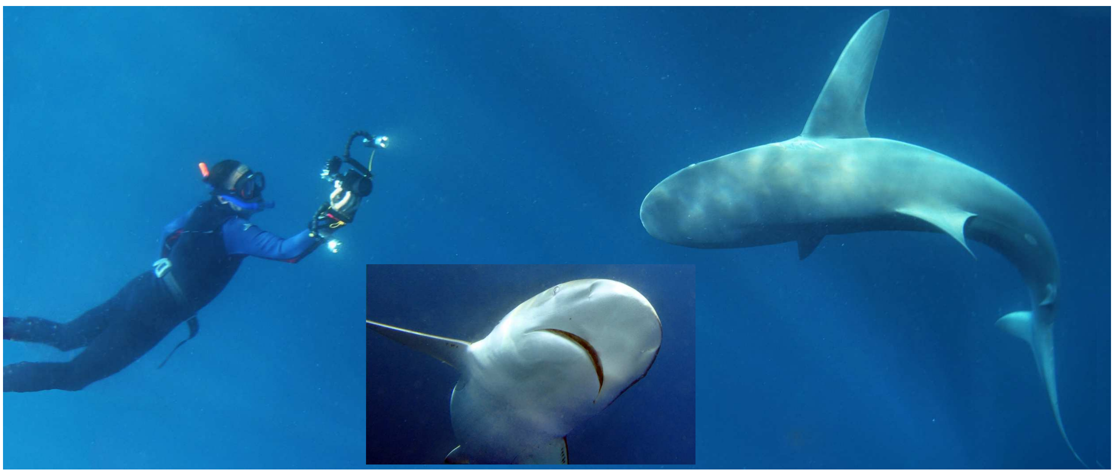

Fig. 2.4. Diver's fears: The author faces a 2.6 m dusky shark (*Carcharhinus obscurus*), photo from above: Sean Hedger.

▸▸▸ Application: diver's reality
The greatest danger to divers is not sharks. It is the so-called decompression sickness ("caisson disease" or "the bends"). It occurs when a diver stays at a great depth for a long

time and then emerges too fast. Due to the increased external pressure, more nitrogen is accumulated in the blood and in the tissues, while the oxygen in the respiratory air is exhausted. During ascent, the reduction of external pressure (*Henry*'s law) causes nitrogen bubbles to form in the blood and tissues. If they are not "exhaled" in time by a very slow ascent, they expand and are "caught" in the joints and tissues. This initially results in extreme strong joint pain ("bends") and can be fatal if the diver is not brought up in time for recompression in a hyperbaric chamber (decompression chamber).

Numerical example: For each 10 m depth increase in water, the external pressure increases by 1 bar. In a water depth of 30 m, there is a pressure of 1 bar + 3 bar = 4 bar. The diver now makes a rapid ascent (> 20 m per minute), for example because the diver's air supply has run out. The resulting rise in nitrogen bubbles are then not fully exhaled and they increase in volume due to the pressure reduction. The surface volume is now a quarter of the external pressure; hence, it has (4 bar → 1 bar) quadrupled. The radius – and thus the diameter – of each bubble has increased by a factor of $\sqrt[3]{4} \approx 1.6$. Diagnosis: The "bends" get worse the longer the diver has been exposed to high pressure. ◂◂◂

▸▸▸ **Application**: **floating needles** (Fig. 2.5)
It is well-known that a water strider is supported by the surface tension of water. One can hardly imagine that this is also true for a steel needle. Although steel has almost eight times the density of water ($\varrho = 7.8\,\mathrm{g/cm^3}$), a sufficiently small steel needle – if it is very carefully placed on the surface of water – "swims". Find the reason for this. How much bigger/heavier could an aluminium needle be ($\varrho = 2.7\,\mathrm{g/cm^3}$)? Is a gold needle ($\varrho = 19.4\,\mathrm{g/cm^3}$) able to float?

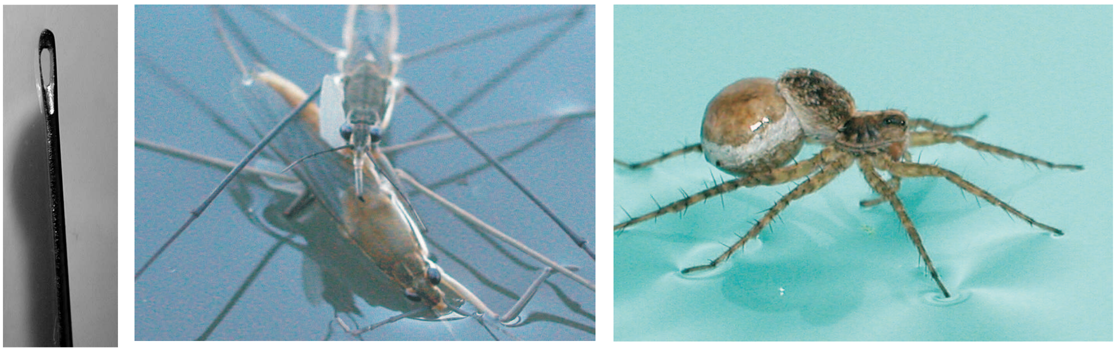

Fig. 2.5. floating steel needle, water striders, spider (not a water spider!)

Solution:
The smaller the steel needle, the greater its surface area relative to its volume or weight. From a certain critical length (depending on the needle's shape), the needle's surface, and thus the surface tension of the water, is large enough to carry the steel needle.
A similarly shaped aluminium needle can be $k = 7.86/2.7 \approx 3$ times as long. The volume is then k^3 times as large, but the weight is only k^2 times as large, and thus, the surface has increased as well.
Without much ado, the needle can be made from pure gold if it is $k = 7.86/19.4 \approx 1/2.5$ shorter. ◂◂◂

▸▸▸ **Application**: **less heat loss through a larger body** (Fig. 2.7).
Explain why large whales are much more likely to live in Arctic waters than small whales (lengths: dolphin . . . 2m, blue whale . . . 20m).

Solution:
Since the body shapes are fairly similar, the ratio $\frac{V}{S}$ of the whale is $k = 10$ times as large

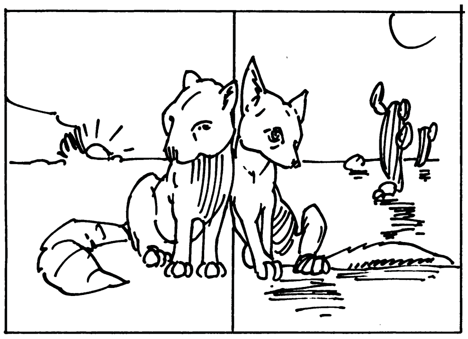

Fig. 2.6. Arctic fox and desert fox (ear size)

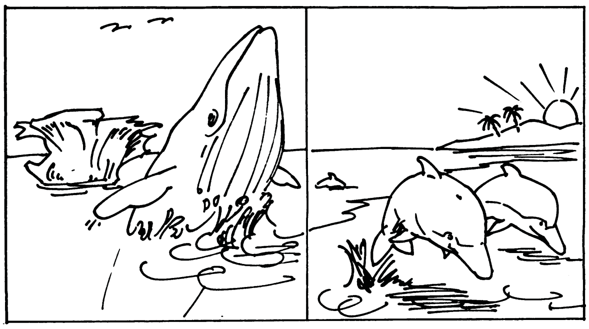

Fig. 2.7. blue whale and dolphin (larger bodies have comparatively less surface)

as the dolphin's ratio. The surface of the blue whale can be kept warm by its circulation of blood. Therefore, it is 10 times easier to keep it warm.

⊕ *Remark*: The Arctic waters have lots of oxygen, and therefore, they are more nutritious. The largest whales are, thus, more likely to be found there. They only swim in warmer waters so that their newborn calves do not freeze in the colder waters. ⊕

⊕ *Remark*: African elephants have cooling problems because of their body size. That is why they have big ears (surface area and of course "fan function").
The "cold-blooded" great white shark, which is constantly in motion (Application p. 34), loses so little warmth to the outside that its body temperature can be up to ten degrees more than the water temperature. ⊕

⊕ *Remark*: Arctic foxes and desert foxes almost have the same shape. It is striking that desert foxes have much larger ears (Fig. 2.6). These are used for listening to the often almost noiseless and odorless preys (scorpions, etc.), as well as for cooling. Naturally, the Arctic fox has to forgo both. ⊕ ◂◂◂

Fig. 2.8. The swan is one of the largest flying birds (up to 14 kg mass).

▸▸▸ Application: weight limit for volant animals

Why do volant animals (especially birds) have a weight limit (10 – 15kg)?

Solution:

Buoyancy while flying is related to the wing-span ("tearing edge") and wing surface. The change of both will be slower compared to the change of the volume when increasing the length scale. The larger the bird, the more difficult it is for it to take off. Large raptors launch themselves from rocks to reach the initial velocity required. The non-volant ostrich (50 – 100 kg) would need a start-up speed of several hundred kilometers per hour in order to take off.

⊕ *Remark*: *Pteranodon* (the biggest flying dinosaur) probably had a wingspan of about seven meters, but only a mass of approximately 15 kg. It would never have been able – like in *Jurassic*

Fig. 2.9. bee-eaters, wasp with prey at departure, a stowaway (1 mm)

Park III – to drag an object of human size through the air. A similar performance can only be achieved by much smaller creatures, like the wasp on the right of Fig. 2.9, which has paralyzed a spider and is transporting the spider in flight. ⊕ ◂◂◂

▸▸▸ Application: cooling of the planet

After a huge explosion in space, two planets are produced whose diameter have a ratio of 1 : 2. Which planet will cool faster?
The volumes have a ratio of 1 : 8; the surfaces have a ratio of only 1 : 4. That is, the larger planet has half as much surface area in proportion to its volume, and therefore, cools down more slowly.

Fig. 2.10. odd cocktail "on the rocks" and ice transportation at 35° in the shade

⊕ *Remark*: Our Moon is probably 4.5 billion years old[2] and is believed to have merged from 20 smaller "moonlets" that spalled off following an asteroid strike on the Earth. The fact that the density of the Moon is less than that of the Earth may be taken as an indication supporting this theory. The larger density of the Earth is mainly due to the nickel-iron core. The Moon, with a diameter of only about a quarter of the Earth's diameter, has, therefore, already cooled by the above considerations, while the Earth is still liquid inside (the frozen continents float like a milk skin on the surface of coffee). Due to their own gravity, all comets or moons with a diameter of more than 500 km attain a spherical shape as long as they are liquid inside. For example, the two quite small moons of Mars are not spherical.
What applies to cooling generally also applies to temperature compensation. Fig. 2.10 illustrates how much the size of a frozen block influences its melting time. Towards the end of the austral winter, the giant icebergs on the sea ice of the South Atlantic cover an area which is twice as large as Canada. The enormous amounts of ice have to be thawed only once! ⊕ ◂◂◂

The fact that during enlargement or reduction, the ratio *Volume* : *Surface* changes, had an enormous impact on the respective development of insects and warm-blooded animals:

[2]In National Geographic, issue September 2001, there is a clear explanation for the relatively precise dating of the formation of the Earth.

 

Fig. 2.11. an elongated and a "rotund" insect

▸▸▸ **Application**: **maximum size of insects** (Fig. 2.11)
In general, insects are shorter than 10–15 cm (the heaviest, though not the largest, insect is the up to 12 cm-long Goliath beetle, see Fig. 2.11, right). Their oxygen supply is the so-called trachea, which is a system of tubes in their exoskeleton. The oxygen exchange takes place via the surface of this system. For a length over 15 cm, this oxygenation is no longer enough: There is a mismatch in the ratio *volume (weight) : tracheal surface* which is getting too large. The ideal size of an insect is clearly in the range of 1 cm or less. Smaller insects are much more resistant than bigger ones. This is also due to the deteriorating ratio between body weight and muscle strength (compare Application p. 14).

⊕ *Remark*: As always, exceptions prove the rule: There is an up to 30 cm-long stick insect (*Palophus titan) (Fig. 2.11, left), and the giant dragonflies in carbon rocks have a wing-span of up to* 75 *cm! For an extremely elongated body shape, only a very short trachea is required, and such a trachea may then have a relatively large diameter.* ⊕ ◂◂◂

▸▸▸ **Application**: **optimal oxygen supply** (Fig. 2.12)
In contrast to insects, the bodies of higher developed animals are supplied with oxygen via their blood (Fig. 2.12). More blood allows more oxygen to be transported. The amount of blood increases in proportion to the volume and there is no mismatch. The oxygenation by blood thus limits the size of the living being but it is not the only factor – there are other factors at play as well (see Application p. 14).

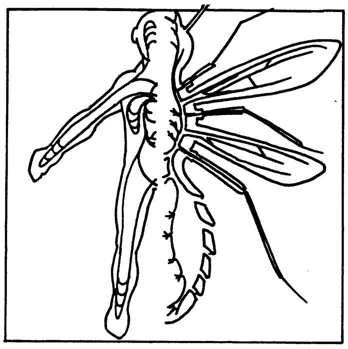

Fig. 2.12. veins and trachea

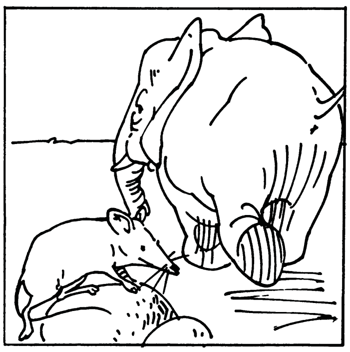

Fig. 2.13. mammals

⊕ *Remark*: The only problem is the accumulation of blood in the lungs: The surface of the lungs has to increase disproportionately, which is made possible by a rapid increase in the number of alveoli in large animals. Infants initially have fewer alveoli, and they disproportionately increase their number until the age of eight, to be ready by "maturity". ⊕ ◂◂◂

▸▸▸ **Application**: **minimum size of warm-blooded animals** (Fig. 2.13)
For animals with constant body temperature, there is now a clear lower limit for the height, namely, about the size of a pygmy shrew (Fig. 2.13) in mammals and the bee hummingbird (a hummingbird species with only $2.5\,g$ of body weight) in birds. For both, there is an extremely small ratio of weight (and thus warming amount of blood) to surface area over which these animals constantly lose heat (energy). To make up for this loss, the shrew has to eat constantly. Smaller shrews from earlier geological periods, therefore, subsisted partially on nectar, extremely high-energy food. The same holds true for the hummingbird. ◂◂◂

For similar reasons, one can explain why large animals in relation to their mass are much weaker than smaller animals: The strength of a muscle is not dependent on its volume but on the cross-section:

Fig. 2.14. an ant and an elephant at work

▸▸▸ **Application**: **relative body strength** (Fig. 2.14, Fig. 2.15)
An ant can carry many times its weight. Although an elephant is objectively incomparably stronger, it can only carry much smaller loads relative to its weight. In addition, an elephant already needs very thick legs (due to a disproportional cross-sectional enlargement of the muscles), in order to bear the enormous weight.

Fig. 2.15. Left: It took minutes to overtake the millipede – the short legs of the victim are strong levers. Right: An ant tries to escape a meat-eating sticky plant (sundew, *Drosera capensis*).

⊕ *Remark*: This is even true when comparing relative strengths of small and large articulate animals: 5 mm long jump spiders can jump much further (in relation to their size) than the quite tremendous (and $10,000$ times heavier) tarantulas. ⊕

Fig. 2.16. One cannot get much smaller. Left: Escaped after being trapped in a cat's mouth!

Numerical example: Pygmy Shrew: 4.3 – 6.6 cm (without the tail) with 2.5 – 7.5 g, elephant: up to 3.5 m and 4, 000 kg. If the elephant were as slim as the shrew, it would weigh less than 2, 000 kg. ◂◂◂

If the ratio $V_2 : V_1$ of the volumes is given, then the similarity factor is calculated by taking the cube root:

$$k = \sqrt[3]{V_2/V_1}. \tag{2.2}$$

Then, one can again use the formulas (2.1).

▸▸▸ Application: allometries in the animal kingdom

Biologists are understandably cautious when using the term "similarity" and instead use the term "allometry" in this context, when certain organs are not in the same proportion as most others. Nevertheless, a domestic cat (5 kg) and a leopard (50 kg) are not too allometric in spite of their similar shape (Fig. 2.17). How do their shoulder heights respectively relate to their coat surface or their tread surface (i.e. the surface of their paw prints) ?

Fig. 2.17. similar cats

Solution:

Since their bodies have the same density, the ratio of the masses equals that of the volumes.

$$k^3 = 10 \Rightarrow k = \sqrt[3]{10} = 2.15, \;\; k^2 \approx 4.6.$$

The shoulder height has a ratio of 1 : 2.15. The coat surfaces or the treading surfaces have a ratio of 1 : 4.6. ◂◂◂

Fig. 2.18. similar white rhinos (low allometry)

▸▸▸ Application: similarity between a pup and a dam
Relatively little allometry can be seen in the two rhinos in Fig. 2.18 and the tigers in Fig. 2.19. In the middle image of Fig. 2.19, it can be seen only at second glance that a "baby tiger" slurps water. The difference in size to the mother can only be estimated from the image on the right. How can the masses of the young animals be guessed?

Fig. 2.19. similar Siberian tigers (low allometry)

Solution:
Fig. 2.18: Since the rhinos in both images are standing in parallel, length measurements (for example, the length of the spine) can be compared quite well. The scaling factor is about $k \approx 0.75$. The mass of the baby animal is k^3 of the mothers mass, so at a good 40%. Since adult rhinos have a mass of $1,500 - 2,000$ kg it is likely that the cub would weigh around $600 - 650$ kg. On the right, the shoulder height of the baby animal may be $k = 0.4$ of his mother's shoulder height, resulting in a mass of about $k^3 = 0.4^3 = 0.064 \approx 1/16$, which equals about 100 kg.
Fig. 2.19: Here, the length dimensions are not directly comparable. If the shoulder height of the young tiger were 45% of the shoulder height, the masses would be in the ratio $0.45^3 : 1^3 \approx 1 : 10$. Siberian tigers are the largest living carnivores. Even the females have a mass of $100 - 170$ kg. So, young tigers must have a mass of about 15 kg.

⊕ *Remark*: The ants in Fig. 2.20 are directly comparable. The larger guardian ant is about 1.5 times as long as the worker ant, and in absolute terms it is $1.5^2 = 2.25$ times stronger. Relatively speaking, it has a mass that is $1.5^3 \approx 3.4$ times as large as the mass of the worker ants. It is, thus, weaker than the smaller ant. ⊕ ◂◂◂

Fig. 2.20. two ants of the same species (worker, guardian)

▸▸▸ **Application: the gravitation on the surface of a celestial body**
On Earth, we have a gravitational acceleration of 1 g. What is the acceleration on another planet or on the Sun with k times the diameter and d times the density?

Solution:
The mass of the body increases with $d\,k^3$. Gravitation decreases at the same time with the square of the distance from the center (factor $1/k^2$). Altogether, the increase is, thus, $d\,k$.
Since Mars, for instance, has a lower density than the Earth ($d \approx 0.7$) and its diameter is about half ($k = 0.532$), we can expect little less than 40% of the gravitational force on its surface. On our Moon ($d \approx 0.8$, $k = 0.25$), we have 20%. For both Jupiter ($k = 10$) and the Sun ($k = 100$), we have $d \approx 1/4$. Thus, we can expect 2.5 g or ≈ 25 g, respectively. ◂◂◂

▸▸▸ **Application: ostrich and chicken eggs compared** (Fig. 2.21, left)
An ostrich's egg looks like a large chicken egg, but has about 24-times the mass of a chicken egg. What is the ratio of the diameters and what is the ratio of the surface areas? Why do ostrich eggs need extreme heat for hatching?

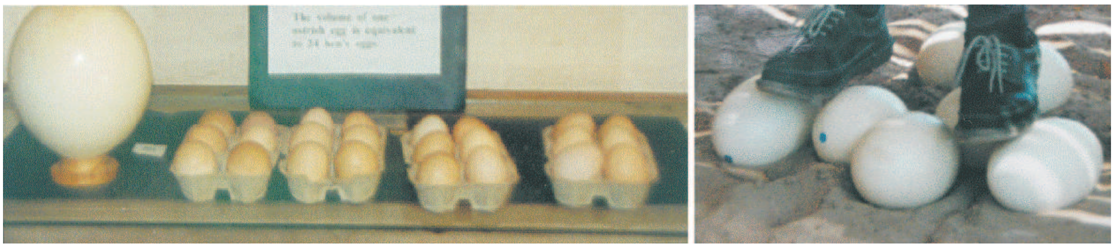

Fig. 2.21. ostrich and chicken eggs in comparison

Solution:

$$M_1 : M_2 = 1 : 24 \Rightarrow V_1 : V_2 = 1 : 24 \text{ (equal consistency)}$$
$$\Rightarrow d_1 : d_2 = 1 : \sqrt[3]{24} \approx 1 : 2.9 \Rightarrow S_1 : S_2 = 1 : (\sqrt[3]{24})^2 \approx 1 : 8.3.$$

The ratio of the surfaces: The volume of the chicken egg is much smaller than that of the ostrich egg (using the factor $k = 2.9$, see Application p. 10). In order to heat the entire interior of the egg, a long exposure to a higher outdoor temperature is required. Therefore, ostriches live mainly in the hot semi-deserts of southern Africa.

⊕ *Remark*: How long do you have to cook an egg? This question depends not only on whether you want to have the egg soft-boiled or hard-boiled, but also on the size of the egg. Large eggs take longer! ⊕ ◂◂◂

▸▸▸ Application: **Why do big air bubbles ascend faster?** (Fig. 2.22)
Every diver knows: Big air bubbles ascend faster than small ones (Fig. 2.22). The rule of thumb is the following: Ascending faster than 20 m per minute should be avoided – and this also applies somehow to small air bubbles.

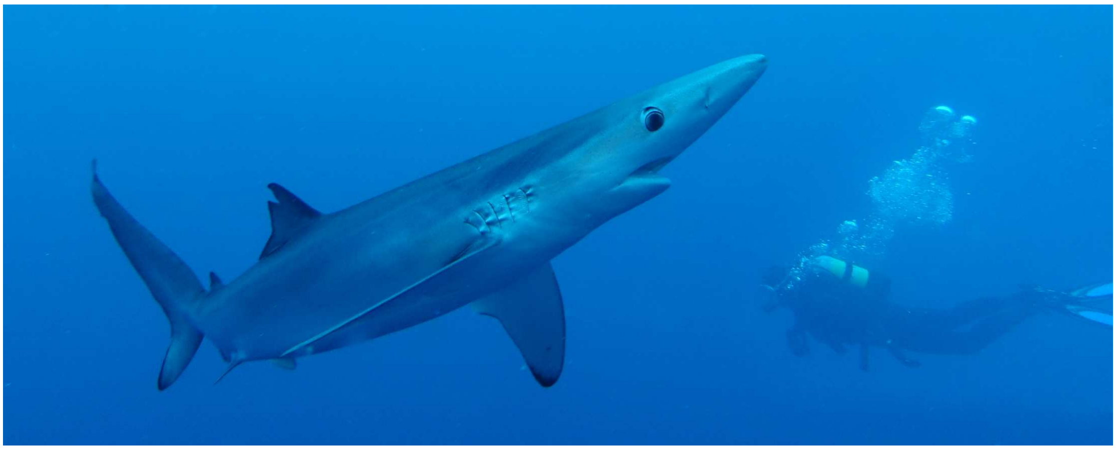

Fig. 2.22. One major, but simplified, rule for divers: Do not ascend faster than the *small* bubbles – no matter what happens.

Solution:
We first simplify the situation and assume that bubbles are spherical. Then, larger bubbles paradoxically have – in relation to their volume – less surface and less cross-section. Due to Archimedes's principle (Application p. 5), the buoyancy force equals the weight of the displaced water bubble, whereas the cross-section is mainly responsible for the friction force. One could compare the situation with the free fall of small and big spheres of the same material with *extreme* air-resistance: The big spheres, then, are faster than the small ones.

Fig. 2.23. Ascending air bubbles change their shape all the time. The big bubbles roughly develop the shape of spherical caps and resemble the shape of jellyfish.

When looking at photo series or videos of ascending water bubbles (Fig. 2.23), one notices two things: Firstly, big bubbles look like spherical caps, and secondly they "explode" and form new smaller spheres every once in a while as they ascend. This might have the following explanation: Small bubbles have a spherical shape due to their surface tension. Since they ascend at a lower speed, they are barely deformed. When the bubbles get bigger due to decreasing pressure, the streaming dents the sphere on the back.

⊕ *Remark*: Application p. 9 explains that air (80% Nitrogen) is dissolved under pressure into the blood system of the diver. If the diver ascends too quickly, the air cannot be fully exhaled by the lungs and nitrogen embolism can cause serious damage. ⊕ ◂◂◂

▸▸▸ Application: Do large aircraft need relatively less fuel?

Solution:

Assuming that both large and small commercial aircraft seem fairly similar, the question is quickly answered: Yes, they need a relatively smaller amount of fuel. A plane that can carry 320 passengers needs to be only twice as large as one that carries 40 people ($40 \cdot 2^3 = 320$). Then, it has only four times the air resistance, and the per capita consumption is half as much for the small model.

⊕ *Remark*: Large commercial aircraft can fly even higher than small ones and they have a lower air resistance. On the other hand, the ascent requires an enormous effort, which is why the situation is clearly only suitable for long haul flights.

Numerical example: Lufthansa has indicated a fuel consumption of an average of 4.4 liters per person on a 100 km route. The wide-bodied Airbus A380-800, however, consumes 3.4 liters. The planned new Boeing 787 is supposed to get through with as little as 2.5 liters. ⊕ ◂◂◂

Fig. 2.24. length runs . . .

▸▸▸ Application: in shipping: "length runs" (Fig. 2.24)

Solution:

One criterion for water resistance is the width of the hull, and it increases in similar constructions only with the root of the submerged hull.

Another criterion is the area of the surface that is in contact with the water, and this decreases proportionally with increasing body length L. Occasionally, one even finds the formula for the maximum velocity $v_{max} \approx 2.5\sqrt{L}$ nautical miles/hour (= knot).

⊕ *Remark*: In a downhill race, much longer skis are used than in the technical disciplines, again because "length runs". For the interview with the winner, the skis are replaced by shorter ones so that the brand is clearly visible inside the frame of TV screens. ⊕ ◂◂◂

▸▸▸ Application: transmission matters (Fig. 2.25)

Explain how the depicted electric screwdriver works and determine its angular velocity if the electric motor makes 3,500 revolutions per minute.

Solution:

Let us consider the inside of the screwdriver. A 3.6 volt battery (Fig. 2.25, left) provides the necessary electricity to turn the small electric motor with a speed of about 3,500 revolutions per minute. The motor shaft propels a small gear with only six teeth (Fig. 2.25, middle). This so-called sun gear transfers its rotational momentum to three rotationally-symmetric positioned planetary gears. The planetary gears are positioned so that they are connected to a fixed outer gear with 48 teeth. (Their number of teeth – in this case 19 each – depends on the sizes of the sun gear and outer gear and has no direct impact.) This causes the centers of the planetary gears to rotate with one eighth $(6:48)$ of the drive shaft's angular velocity.

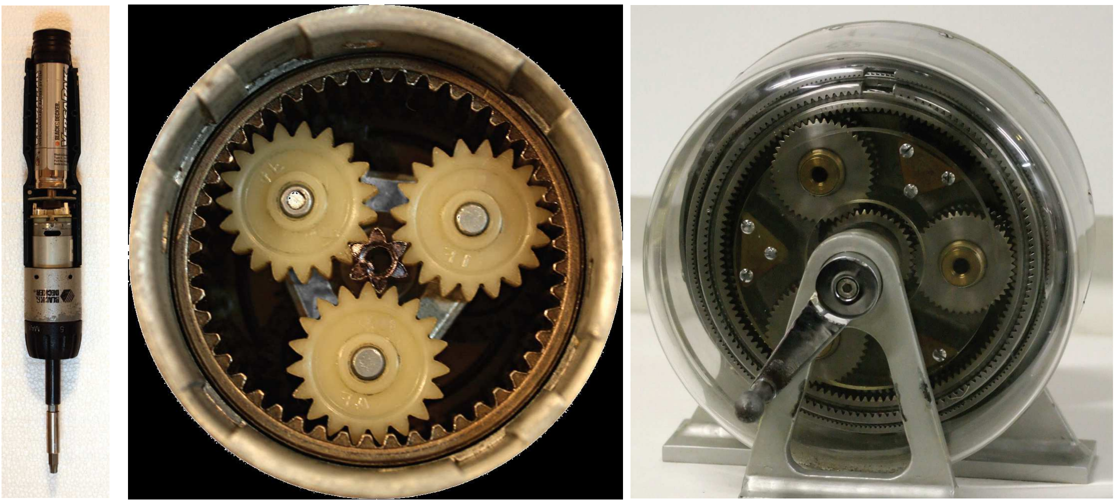

Fig. 2.25. left: electric screwdriver, middle: the small sun gear transfers its rotational momentum to three planetary gears, right: the axes are fixed, and the outer gear is allowed to rotate.

The centers of the planetary gears form a fixed equilateral triangle. On its reverse, in a second step, a further 6-tooth sun gear propels three other planetary gears with the same transmission ratio as before. The connecting triangle between the planetary gears in the second layer is finally connected to the axis of the screwdriver. The $8 \cdot 8 = 64$ revolutions of the electric motor thus cause a single turning of the screw (roughly $3,500/64 = 55$ revolutions per minute).

⊕ *Remark*: This explains the large angular momentum M. Given the motors constant power P, we can apply the equation $P = M \cdot \omega$, where ω is the angular velocity. Planetary gears have many applications in technology, such as in gearboxes, cable winches, and bicycle hub gears.

Another variation on the same principle is also notable: If the axes of the planetary gears are fixed and the outer gear is allowed to rotate, then it rotates significantly slower than the drive shaft (Fig. 2.25, right). ⊕ ◂◂◂

3 Math Lectures GG – More interesting things

▸▸▸ **Application**: **How often does Greenland fit into Africa?** (Fig. 3.1)
If you ask people to estimate estimate, the solutions offered to this question tend to be of the order of "four to six times" or so. The correct answer, however, is 15 times! This has to do with the fact that Africa is "around the equator" and Greenland is far North. The rectangular maps of the Earth distort Greenland much more than Africa.

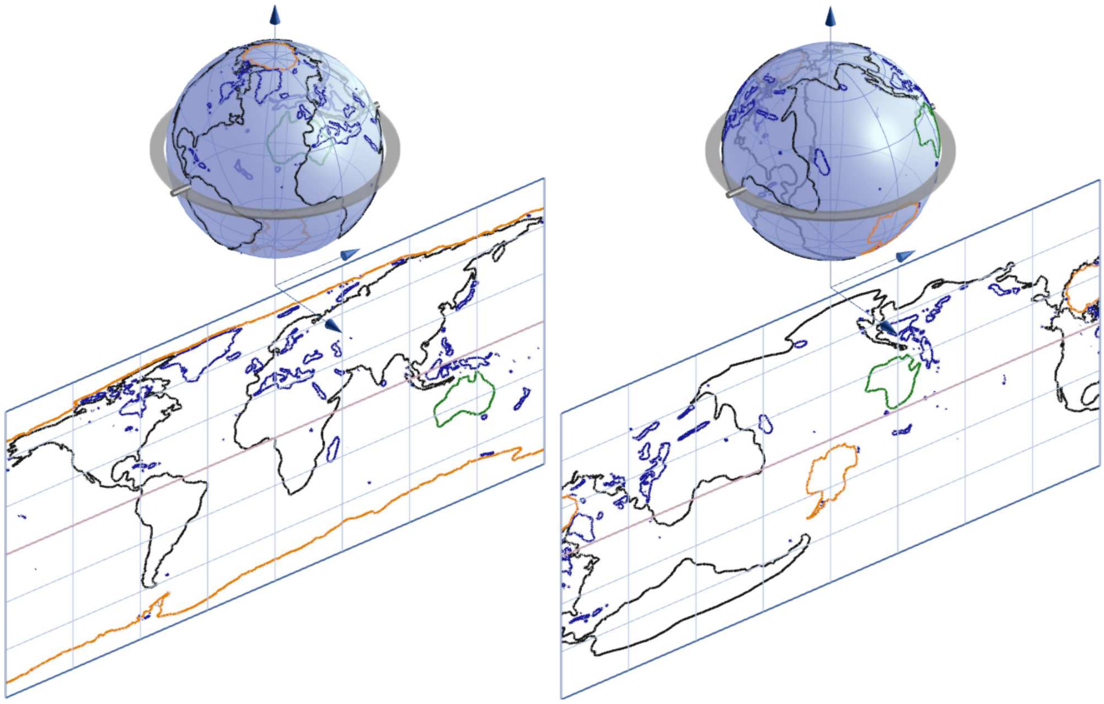

Fig. 3.1. The distortion in the rectangular map of the Earth makes area comparisons hard. In the right image, another great circle takes over the role of the equator. Both Africa and Greenland are close to the equator, and the size comparison is more realistic. Compare also Australia and Antarctica in the left and the right rectangular map.

⊕ *Remark*: Even on the "usual" rectangular map, one can see that Africa and South America were once joined together. This, however, is only possible since both continents are close to the equator. Australia and Antarctica used to be one continent as well, but this cannot be seen on such extremely distorted maps. ⊕ ◂◂◂

▸▸▸ **Application: rolling cone** (Fig. 3.2)
When a cone of revolution is rolling on the base plane, it will stay inside a circle around the apex. We now choose two parallel circles on the cone. Let the ratio of their radii be $1 : k$, and d be their distance measured on the cone's generating line. What is the radius s of the inner blue circle in Fig. 3.2? Compare also Application p. 22.

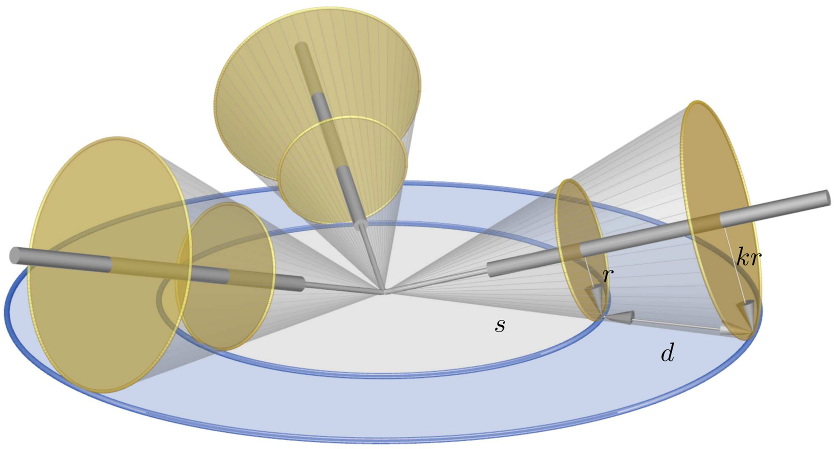

Fig. 3.2. The rolling of the cone can be substituted by the rolling of two rigidly connected parallel circles of the cone.

Solution:
Due to the intercept theorem, we have $s : r = (d + s) : (kr)$, and thus, $s = d/(k - 1)$. ◂◂◂

▸▸▸ **Application: Why does a train not derail?**
This application does not need calculations, but has something to do with arc lengths. The answer "The train does not derail, because it runs on rails" is too simple.

Fig. 3.3. conical wheels

Solution:
The train would continually derail if pairs of co-axial wheels were not conical. Even the steel ring on the inside of the wheel cannot prevent derailment. It is mainly the slightly conical shape of the wheels (Fig. 3.3) that can do so: If the train drifts to the right, then the contact circle on the right wheel apparently increases its diameter due to the conical shape of the wheel. At the same time, the diameter of the contact circle on the left wheel decreases (causing a tilt of the axis). This is exactly what forces the axis to move back into the right and stable position.

⊕ *Remark*: When the train has to take a curve, the conic wheels are also of great advantage. Imagine the left curve in Fig. 3.4: The tracks have to be inclined, otherwise the centrifugal forces would tip the train over. These centrifugal forces will still shift the train to the right. This causes the circle of contact with the right track to increase, and the corresponding circle on the left to decrease. This is good, because now the train automatically tilts to the left (Fig. 3.2). Inclining

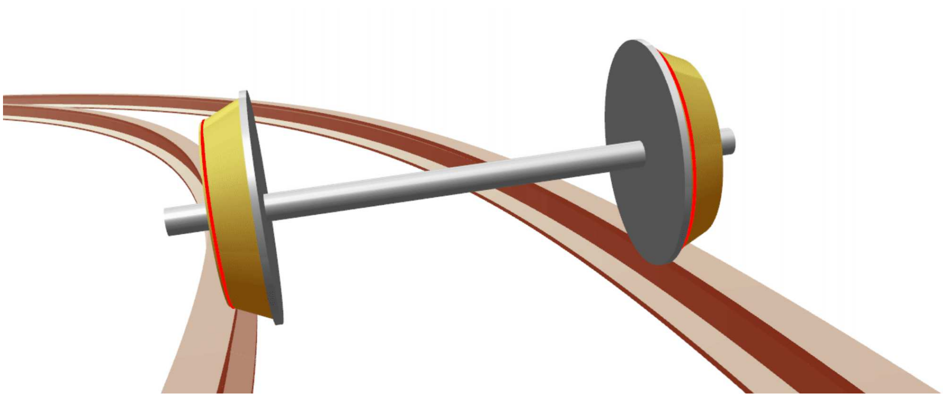

Fig. 3.4. No problem with taking the left curve: The smaller circle of contact on the left and the larger to the right make the train stay on the tracks.

not enough to the right will push the train further right, whilst inclining too far to the right will push it back to the left. The perfect equilibrium lies somewhere in between. ⊕ ◂◂◂

▸▸▸ **Application**: **Snell's law** (Fig. 3.5, Fig. 3.6)

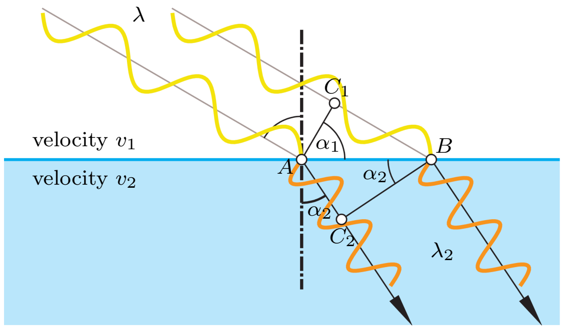

Fig. 3.5. The wave front plausibly explains the law of refraction of light rays (the wave front tilts upon hitting the layer of separation).

From Fig. 3.5, we obtain

$$\frac{\overline{BC_1}}{\overline{AC_2}} = \frac{v_1}{v_2} = n \Rightarrow \frac{\overline{BC_1}}{\overline{AB}} = n \cdot \frac{\overline{AC_2}}{\overline{AB}} \Rightarrow \sin\alpha_1 : \sin\alpha_2 = n.$$

According to the law of refraction (Snell's law), the angle α of the incoming ray and the angle β of the outgoing ray (measured to the normal of the surface) are related by $\sin\alpha : \sin\beta = 4 : 3$ for the transition from air to water. There is, due to the atmosphere, no total reflection. That is, all light rays are at least partially deflected into water. Conversely, there is a critical angle β_0 at which no light passes from the water through the surface. The critical angle β_0 is obtained from the maximum value of 1 from $\sin\alpha$:

$$\frac{1}{\sin\beta} = \frac{4}{3} \Rightarrow \sin\beta_0 = \frac{3}{4} \Rightarrow \beta_0 \approx 48.6°.$$

⊕ *Remark*: A seal that floats on the blind spot (Fig. 3.6) cannot see the Inuit who is waiting for it at the air hole in the ice: The only light rays that lead to the hunter are shielded by the ice. Even the Inuit sees the seal only when its nostrils stretch out of the water (Fig. 3.7)! ⊕ ◂◂◂

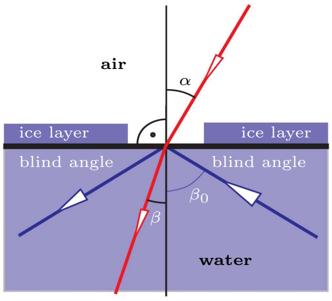

Fig. 3.6. total reflection

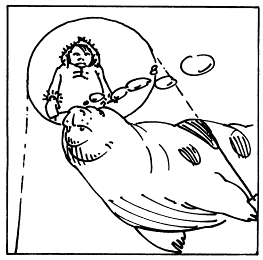

Fig. 3.7. situation in the water ...

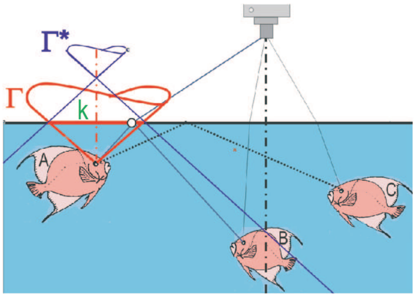

Fig. 3.8. Who sees whom now? This question is by no means trivial!

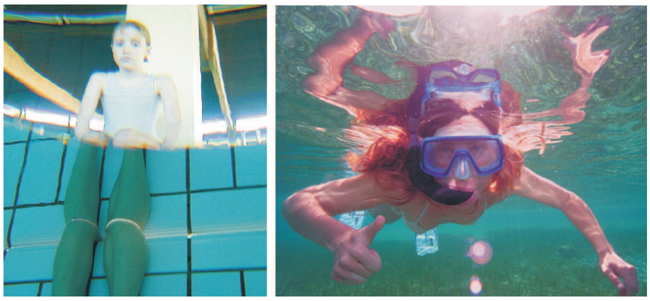

Fig. 3.9. total reflection outside of Γ^* (on the left the track circuit of Γ^* is seen)

▸▸▸ Application: Who sees whom?
In the calm pool of water (Fig. 3.8, left), the fish A (the diver) sees:

- "everything" outside the pool, albeit it is distorted greatly. The refracted image is inside a circle k on the surface. This circle is the intersection of a cone Γ of revolution with an aperture angle of $2 \times 48.6°$;
- the total reflections of those parts of the basin which are outside the reflected cone Γ^*, e.g. the fish C (particularly clear in Fig. 3.9, left);
- reflections of the rest of the basin (Fish B) in the interior of k – as a result of partial reflection (the calmer the surface of the water, the clearer the image);
- the fishes B and C, even directly!

On a photograph from outside the basin (for instance, from a diving board) you can see all the fishes – to some extent severely distorted. ◂◂◂

▸▸▸ Application: How does a rainbow form? (Fig. 3.11)
A good moment to see a rainbow is when the Sun is low after a downpour (Fig. 3.10). Why? How does a rainbow form?
Solution:
The wavelengths of the spectrum of visible light range from 380 to 780 nanometers, i.e. slightly less than one thousandths of a millimeter. The individual spectral colors are less broken during the transition to an optically denser medium if their frequencies are smaller. Therefore, violet is refracted more strongly than red. Beyond red, we have the warming infrared rays. Beyond violet, we have the harmful ultraviolet rays.
Now, after a downpour, the air is full of (spherical) water droplets (Fig. 3.11). These, we usually see in a diffused form as mist or cloud. Let r be the direction of the incoming light rays. At any point of the illuminated hemisphere, a light ray is partially reflected at the hemisphere, while the other part is partially refracted inside the hemisphere (→ ray

Fig. 3.10. Two concentric rainbows (primary and secondary). Left: two real rainbows were captured from a terrace with a fish-eye lens during a low sun position. These shimmering water droplets are many kilometres away! Right: this photo was taken with a 30 mm wide angle lens. No rainbow was present in the sky rather, a jet of water was shooting closely past the camera in the shape of a parabolic trajectory, producing two clearly visible rainbows.

r_1). The ray is "fanned" into the different spectral colors due to the different refractive indices. The ray r_1 meets the sphere from the inside and is partially refracted, partially reflected (→ ray r_2). The ray r_2 again meets the spherical wall and is partially refracted (→ ray r_3), partly reflected where the fanning is partly undone, but is still maintained. The outgoing ray r_3 is a further fanned bundle of rays. The reflected residual light "wanders further around", whereby the light ray loses intensity by splitting into reflected and refracted light.
It now seems that r_3 is mainly responsible for the rainbow effect. To be more precise, it is responsible for the *primary rainbow*. A further reflection and subsequent emergence produces a *secondary rainbow* (Fig. 3.10). This was known by René *Descartes* (1596 – 1650) almost 400 years ago!

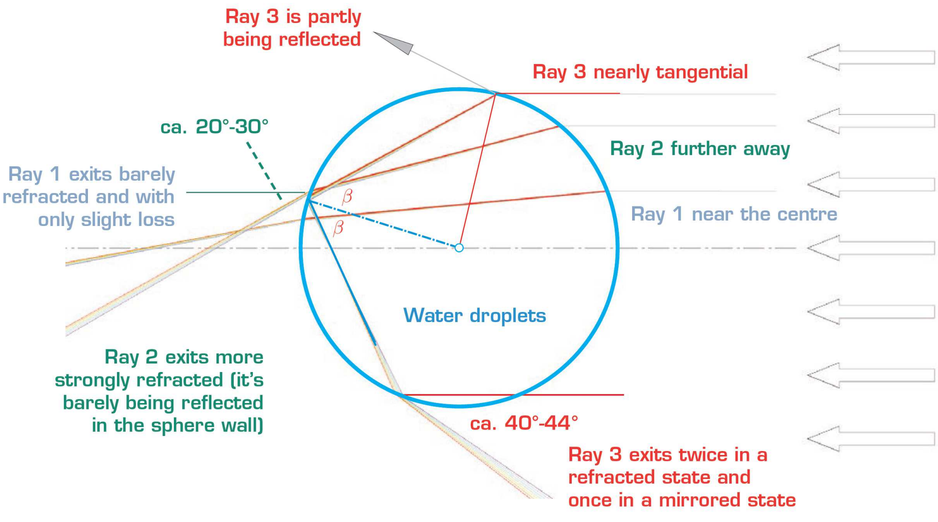

Fig. 3.11. emergence of the rainbow

If we send not just one but an infinite number of parallel rays of light into the sphere (Fig. 3.11), most of the rays – being refracted in every direction – will leave the water droplets at the back. Those rays that arrive at the backside of the droplet approximately at the critical angle of total reflection will return relatively highly fanned towards r_3. Fig. 3.11 illustrates that, at a certain exit angle ($\alpha \approx 43°$), there is a distinct maximum

of such rays r_3.[1] For each partial spectrum in the entire spectral range – for each color – this angle is slightly different (Violet ... 42°, Red ... 44°). Viewing the droplet at this angle, an appropriate amount of color prevails.
Thus, all rain drops that "dispatch" rays r_3 at the angle α are seen in the corresponding spectral color. All these droplets are distributed on a cone of revolution whose apex is the eye and its half aperture angle equals α. Thus, this cone of revolution appears in a "projecting" manner, and therefore, it appears as an arc (Fig. 3.11). The human eye is able to distinguish seven essentially different colors, with red on the *margin*. In the secondary rainbow – it occurs for $\alpha \approx 51°$ – the order turns back because of the additional reflection!

⊕ *Remark*: Depending on the position of the Sun, one can see at most a semicircle. Looking out of a plane (above the clouds), you can see a whole circle if the Sun is high. Since the cone is dependent on the position of the observer, one always sees a new rainbow when walking. So, it makes no sense "to look for the treasure that is buried at the foot of the rainbow". ⊕ ◂◂◂

Fig. 3.12. red sunset with delay of five minutes ...

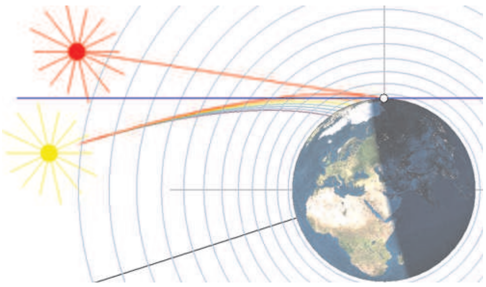

Fig. 3.13. The underlying theory shows: We can see around the bend!

▸▸▸ Application: Why does the setting Sun appear to be Red?

Solution:
If the Sun is low on the horizon, the light rays, which contain the entire visible spectrum (see Application p. 24), transit the atmosphere at equal angles. The atmosphere becomes denser as it gets closer to the Earth's surface. The rays are, thereby, increasingly fanned into the different colors, with violet being the most bent towards the front and red the least. Theoretically, the portion of light that is blue reaches us for a longer period of time than the red component (Fig. 3.12). However, due to its shorter wavelength, this blue portion is scattered across the atmosphere (which is why the sky is blue). So, when the Sun goes down under the horizon, you can see the color red for about five minutes (Fig. 3.13)!

⊕ *Remark*: Thus, for about five minutes, we have the impression that an object is located in the extension of the light rays reaching the eye. Eventually, the red portion of light is no longer able to make it "around the bend" and the Sun goes down, seemingly deformed to an oval. ⊕ ◂◂◂

[1] Start the demo program `rainbow.exe` to see the animated situation. For more theoretical details on this issue, see the following interesting website:
`https://plus.maths.org/content/rainbows`
See also Steven *Janke*: *Modules in Undergraduate Mathematics and its Applications*, 1992.

▸▸▸ **Application**: **Fata Morgana?** (Fig. 3.14)
This particular sunset deserves a closer explanation. Both images to the left look relatively familiar. From the third image onwards, however, something like a reflection appears to emerge. This reflection is then increasingly distorted in the subsequent images, even when the Sun has already disappeared completely.

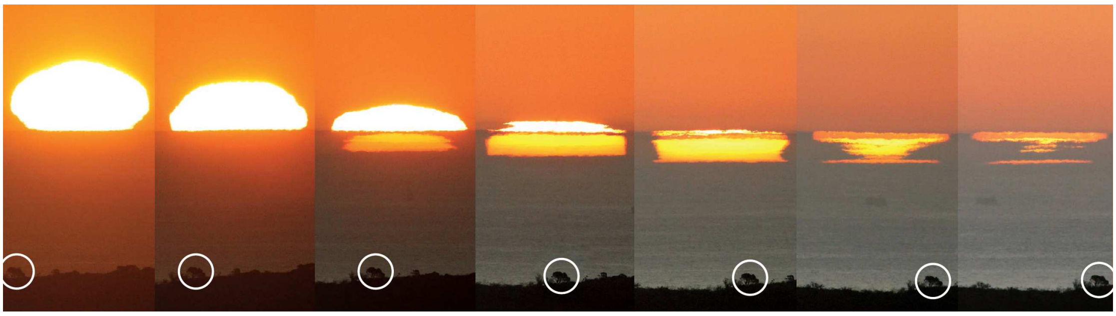

Fig. 3.14. 3.5 exciting minutes

Solution:
A Fata Morgana is an optical illusion caused by the refraction of light in layers of air of differing temperature and humidity. As previously mentioned (Application p. 26), the Sun should already have disappeared from sight several minutes before the picture was taken. Due to the low angle of incidence, the blue components of light with their short wavelength were already fully absorbed by the atmosphere, while the red components still managed to go around the bend. Yet, what causes the strange reflection? The specular points are not infinitely far away, but are instead located on the surface of the water. In fact, due to the curvature of the Earth, they can be much closer than one might expect. Depending on one's height above water level, it is possible to see anywhere from several to 100 kilometres ahead, though the latter distance is only possible during clear atmospheric conditions. The images on the right were taken a few seconds after the Sun had disappeared completely. The fisherman at sea, however, would still have been able to see it. The sunrays which hit the surface of the water at the fisherman's position could have been detected by us, although they might have been slightly refracted, owing to the different atmospheric conditions of temperature and humidity. The series of photos was taken on the Cape of Good Hope at the end of February. On the northern hemisphere, at the same latitude, the Sun sets clockwise in the direction west-north-west. On the same day on the southern hemisphere, the Sun sets in anti-clockwise direction towards the west-south-west. Note the encircled tree which appears to move one Sun diameter to the right within a span of 3.5 minutes. ◂◂◂

▸▸▸ **Application**: **the common center of gravity of the Earth and the Moon** (Fig. 3.16)

With $M_1 = 81M_2$, we have $\vec{s} = \frac{1}{82}(81\vec{s}_1 + \vec{s}_2)$. Because $d \approx 384,000\,\text{km}$, the common center of gravity, thus, lies at a distance $\frac{d}{82} \approx 4,700\,\text{km}$ from the center of the Earth, that is still within the Earth (Earths radius, $r \approx 6,370\,\text{km}$). The Earth and the Moon rotate around this common center of gravity, that is, the Earth "wobbles".

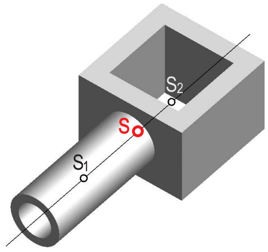

Fig. 3.15. multiple blocks

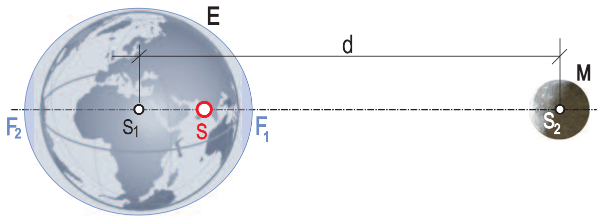

Fig. 3.16. common center of gravity of the Earth and the Moon, *two* tidal bulges

⊕ *Remark*: Analogously, the Sun "wobbles" around the common center of our planetary system. We now know that there are other suns (="stars") which also "wobble".
With the fact that the double planet Earth-Moon rotates around the common center of gravity S, one can also explain why there are high tides and low tides *twice* a day (Fig. 3.16): The first tidal bulge is at the point F_1 that is closest to the Moon. This point on Earth is affected the most by lunar attraction, while it is only minimally affected by the centrifugal force during the rotation around S. The second tidal bulge occurs at the exact opposite point F_2 – we get the maximum centrifugal force due to the rotation around S because it is so far away from this and more than 6 times as far as the Moons next point. The two tide mountains move due to the rotation of the Earth over the course of almost 25 hours (the Earth has to "overturn" in order to "catch" the next migrated moon. ⊕ ◂◂◂

Multiple reflections

▸▸▸ **Application**: **specular corners** (Fig. 3.17, Fig. 3.19)

If you look into a mirror, you see yourself "reversed". In a two-way mirror with an *acute* angle α (Fig. 3.17: $\alpha = 60°$), you can see how others see you. What happens if $\alpha = 90°$ or $\alpha > 90°$.

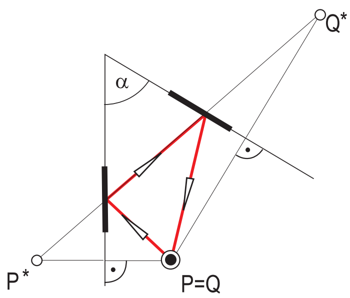

Fig. 3.17. double reflection

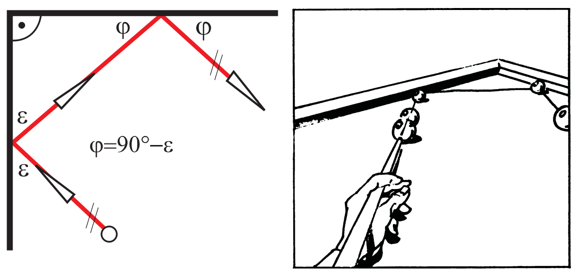

Fig. 3.18. $\alpha = 90°$, billiards

Solution:
If $\alpha = 90°$, then the doubly-reflected ray is "exactly parallel to the incident ray" (Fig. 3.18). This has the consequence that one eye sees the other, and, vice versa. The same applies to the two halves of the face. The distance from the mirror does not matter. When looking at the edge of a rectangular double mirror, one sees – separated by the edge – both sides of the face, and they do not appear mirror-inverted.

At obtuse angles $\alpha > 90°$ and with increasing distance, you will not see yourself when looking into the mirror's corner.

⊕ *Remark*: The above example with $\alpha = 90°$ and $P \neq Q$ can be interpreted in terms of billiards (rail shot). ⊕ ◂◂◂

Fig. 3.19. Confusion results from using three mirror planes, each constituting approximately 60°. Since the angles are not accurate, some artefacts come into being.

▸▸▸ **Application**: **reflecting "cuboid corner"** (Fig. 3.21)
The special case $\alpha = 90°$ has an important practical application: Consider a reflecting "cuboid corner", which consists of three concurrent faces of a cube (Fig. 3.21). Prove that if one shoots a laser ray into this corner, the ray is redirected parallel to the incoming ray.

Fig. 3.20. cuboid corner in shipping

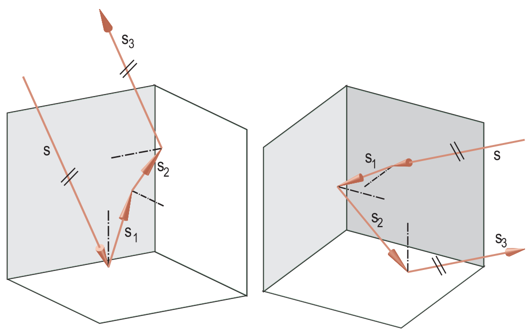

Fig. 3.21. reflecting cuboid corner

Proof: We look at the situation from above to get a top view: There, the reflections in the vertical planes appear as planar reflections in the respective normals. However, the reflection in the horizontal plane is not seen in the top view because the normal "projects" to the base plane. Thus, following the above (Fig. 3.18) considerations in two-dimensions, the incoming ray appears in the top view parallel to the outgoing ray. The same holds for the views from the front and right (front view, right-side view). So, the outgoing ray is parallel to the incoming ray in space.

Another proof is derived from the fact that, with each reflection in a coordinate plane, the sign of the corresponding component of the direction vector switches. Since all coordinate planes are affected, the vector is just reoriented. ⊙

⊕ *Remark*: This property of the specular cuboid corner allows us to measure distances. Reflecting prisms are, for example, mounted on the Moon to carry out certain experiments (verification of light speed and distance measuring). Such devices are also mounted on bridges to allow ships to determine their distance from the bridge's pillars. ⊕

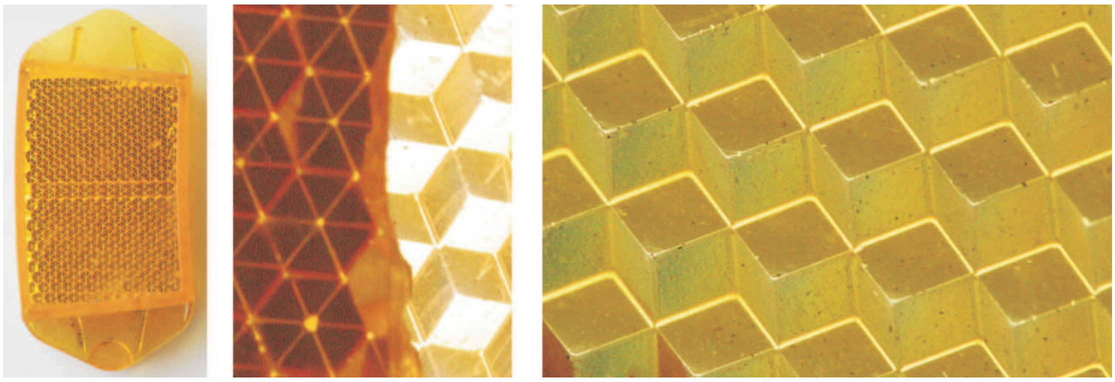

Fig. 3.22. Reflectors for bicycles consist of many small corners!

⊕ *Remark*: Fig. 3.22 shows how reflecting cuboid corners are applied in daily life: The reflectors which are mounted on bicycles – viewed under a microscope – consist of nothing but small cuboid corners. When they are illuminated by a car, they accurately reflect back to the car driver. Along with the rotation, the reflectors produce an extremely effective and bright warning. ⊕ ◂◂◂

▸▸▸ Application: multiple reflections in two orthogonal mirrors
With multiple reflections in two perpendicular planes, a remarkable effect occurs, whereby left and right – and not, as in the ordinary mirror, the front and the rear – are interchanged. What exactly is happening here?

Solution:
For the sake of simplicity (Fig. 3.23), the two planes ξ and η are assumed to be vertical coordinate planes with the equations $x = 0$ and $y = 0$. Each of the planes initially generates a virtual – usually reversed – counterpart Ω_x (change of signs in the x-values) or Ω_y (change of sign in the y-values). The two objects are indirectly congruent and – from the standpoint of perception – are of equal rank as the original Ω.

Fig. 3.23. Several mirrors in two mutually perpendicular planes of symmetry

Consequently, Ω_x has a virtual counterpart Ω_{xy} through reflection in η, and Ω_y has a virtual counterpart Ω_{yx} through reflection in ξ. However, because of the special position of the mirror planes ($\xi \perp \eta$), the two new objects are identical: $\Omega^* = \Omega_{xy} = \Omega_{yx}$. Furthermore, they can only be partly seen in the respective "mirror window". The part labeled with Ω_{yx} is produced through reflection in the right mirror window of the visible virtual object Ω_y in the left mirror window.

The two mirror windows are touching each other along the edge $s = \xi \cup \eta$ (according to our choice, the z-axis), so they merge into a single window, which consists of two mutually perpendicular rectangles. This allows us to view our object Ω^*. It was created by double-reflection and is, therefore, directly congruent to Ω. The two directly congruent objects Ω and Ω^* can be transformed into each other by a half-turn about the intersection s of the mirror planes or by an axial reflection in s. Thereby, left and right are interchanged firstly, and the characters are still readable as usual.
In the photo in Fig. 3.23, a wide-angle lens was used in order to make the "auxiliary objects" Ω_x and Ω_y entirely visible.
In practice (also in human vision), the visual angle is smaller, and then, one only sees Ω^* in the image – comparable to an ordinary mirror image, but *left and right are interchanged.* ◂◂◂

▸▸▸ **Application**: **What exactly happens between two parallel mirrors?**
Consider the confusion with two parallel mirrors in Fig. 3.24: The photographer was obviously able to focus through different distance settings on any of the available originals or reflections. Visually, one is unable to distinguish between original and virtual objects, so that it is not even guaranteed that the person depicted is actually seen as such – this depends on the focal length that is actually used. That is to say, it depends essentially on the aperture of the viewing cone.

Fig. 3.24. Three similar scenes, but in each another reflection is in focus. Can you see where "the original" is?

Solution:
Let A be the position of the observer (Fig. 3.25) standing between two mirrors σ_1 and σ_2 (distance d apart). The camera, which is directed towards the axis a, can only capture those things that are within the cone of vision Δ.

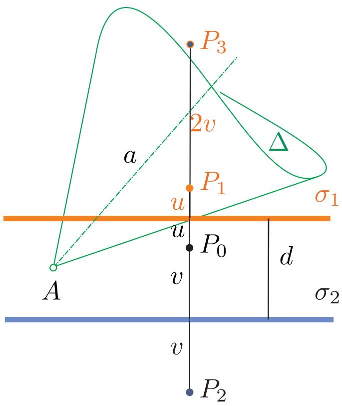

Fig. 3.25. For each virtual point a new one arises at the "speed of light".

Now, assume that another person is standing (symbolized by a point P_0) between the mirrors, at a distance u from σ_1 and at a distance v from σ_2: $u + v = d$. Thus, two virtual mirror images P_1 and P_2 arise at the distances u and v behind σ_1 and behind σ_2. The two points are a distance $2u + 2v = 2d$ apart. P_2 is visually indistinguishable from a real point and is reflected in σ_1 to a point P_3. This is at distance $2d$ from P_0. If one continues, one obtains two congruent series of points $P_0, P_3, \dots$ and $P_1, P_5, \dots$ whose points are separated by a constant distance of $2d$.

In Fig. 3.24, these series are clearly visible (in one of them, the person can be seen from the front, in another from behind). In both Fig. 3.24 and Fig. 3.25, the observer and the multiple reflections are not contained in the cone of vision. It is noteworthy that the autofocus of the camera has no problem focusing on any reflection, which again demonstrates that a purely optical mirror image is indistinguishable from reality. The distance to which the autofocus adjusts equals exactly that of the "reflected" person in the room – and not the distance to the mirror. The repeated reflection causes a loss of brightness, so that further apart reflections appear weaker (darker). ◂◂◂

▸▸▸ **Application**: **Why does an airplane fly?** (Fig. 3.26)

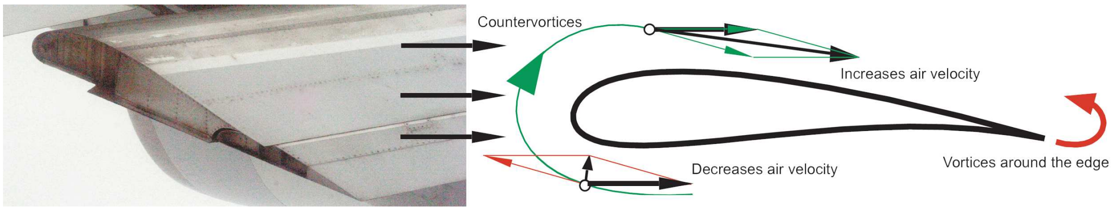

Fig. 3.26. A non-trivial explanation of how lifting forces make an aircraft fly.

Solution:
The airfoil (usually almost symmetrical in the case of faster aircraft and curved upward in the case of slower ones) has a sharp edge in its rear. When "pitching" upwind, there arises a counter-clockwise velocity that is strongly dependent on the speed v. (Aircraft with symmetrical profiles have extendible flaps which enhance the formation of vortices used for slow flight phases.)

Fig. 3.27. wing motions of a butterfly (average frequency)

According to the *law of conservation of angular momentum*, a vortex arises, rotating clockwise around the profile (speed $|\Delta v|$). If the angle of approach is too large, the flow breaks off and the aircraft descends uncontrollably. At the opposite vortex, there are velocity vectors $\overrightarrow{\Delta v}$ (by definition directed and oriented) assigned to each point. The sum vector $\vec{v} + \overrightarrow{\Delta v}$ is, therefore, greater at the top than at the bottom.
Due to the *aerodynamic paradox*, negative pressure acts on the side where the air flows at greater speed, and this will generate a buoyant force that causes the aircraft to stay in the air.

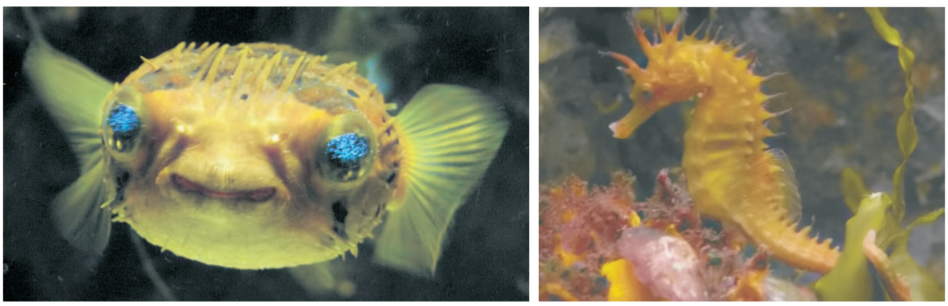

Fig. 3.28. fin motions at relatively low frequencies

⊕ *Remark*: A simplified explanation that is occasionally found reads as follows: The path of the air at the top is longer than the path at the bottom, so that the air at the top must flow at a greater velocity. This explanation is too simple.

Flying animals – like butterflies – produce the necessary vortex by synchronously twisting their wings (Fig. 3.27). At a very high wing-beat frequency, air behaves like a denser medium, and flying insects can – similar to how water animals work with water resistance (Fig. 3.28) – repel compressed air to create cushions. Aircrafts glide at high speed on air cushions. ⊕ ◂◂◂

▸▸▸ **Application**: **Why does a shark not sink?** (Fig. 3.29)
The following question seems to be related to that of Application p. 33: Why does an airplane fly? Surprisingly, the answer is very similar.

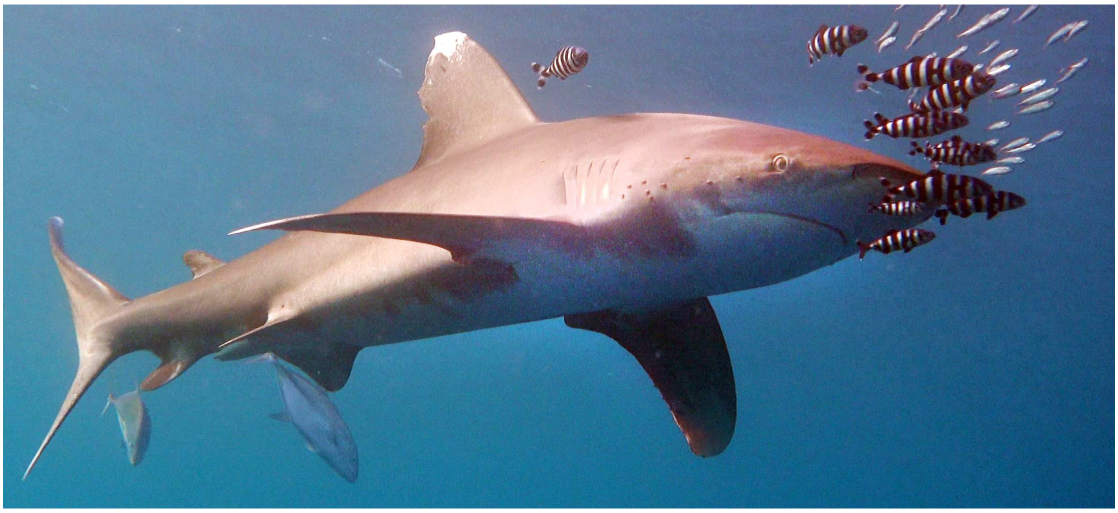

Fig. 3.29. An oceanic shark bears a striking resemblance to an aircraft. The pectoral fins work like the wings of an aircraft.

Solution:
Big sharks like the oceanic whitetip shark (*Carcharhinus longimanus*) have a density greater than water. So, if the shark stops swimming, it will sink. As it swings, the stiff pectoral fins, which are slanted upwards, generate lift like an aircraft wing. Since the pectoral fins are positioned before the animals center of mass, the generation of lift results in a torque that tilts the sharks head upward.
To repeat it again (this time adapted to the fluid water): The sharp edges of the pectoral fins create vortices that, according to the *law of conservation of angular momentum*, generate counter-rotating vortices around the profile (speed $|\Delta v|$). The vector sum $\vec{v} + \overrightarrow{\Delta v}$, therefore, has greater magnitude at the top than at the bottom.
Because of the *hydrodynamic paradox*, negative pressure is generated on the side where the water flows at greater speed, which will, in turn, produce a buoyant force that causes the shark to ascend.
The sharks asymmetrical dorsal fin, however, simultaneously produces a torque that tilts the head downwards. Both torques compensate each other, and thus, it is possible for the shark to swim straight ahead without sinking. ◂◂◂

▸▸▸ **Application**: **positive or negative buoyancy** (Fig. 3.31)
A hull sinks into water, and depending on the weight of the ship, it will go down to a certain level. How far does it sink? Which forces occur, and when are they balanced? What will happen if the body topples over?

Solution:
We simplify the reasoning by only looking at a typical cross-section of a ship. The implementation of the following calculations, of course, is left to a computer.

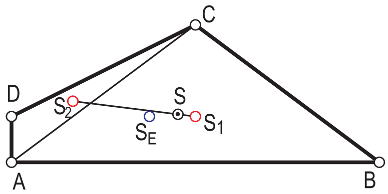

Fig. 3.30. center of gravity of a quadrangle

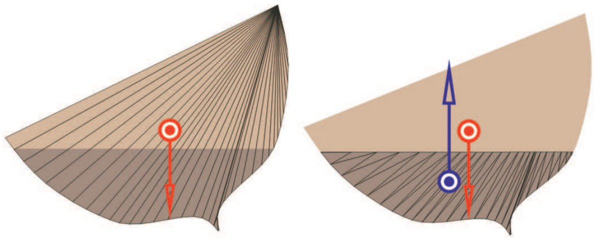

Fig. 3.31. positive or negative buoyancy

- One can now consider the weight force acting at the center of gravity (Fig. 3.31, left). For the computation, we triangulate the cross-section, and the weight is proportional to the total area.

- According to *Archimedes*, the buoyancy is equal to the weight of the displaced fluid. The force of gravity acts on the center of gravity of the displaced water (Fig. 3.31, right). We intersect the partial triangles of the cross-section with the surface of the water. If a triangle is completely outside the water, it does not count. If it is completely under the water, the area and centroid remain unchanged. For a triangle that intersects the surface of the water such that the part under the water is again a triangle, one can immediately calculate the area and the centroid. Otherwise, the part underwater is a quadrangle, which can be decomposed into two triangles.

- When the buoyant force is greater than the weight force, the ship is lifted. If the centers of gravity are not lying precisely above each other, they induce a torque. The cross-sections of vessels are designed so that the ship is automatically lifted “upright”.

◂◂◂

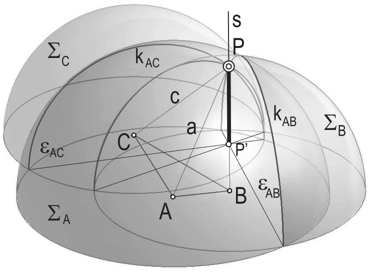

Fig. 3.32. positioning

Fig. 3.33. GPS for planes, ships, and cars

▸▸▸ **Application**: **Global Positioning System – GPS** (Fig. 3.33)
The GPS has brought a revolution in the determination of locations. By means of handy devices, one can know his/her exact position at any time with an accuracy of up to a few meters (!). This is made possible by the fact that a relatively small number (24) of strategically distributed satellites constantly transmits signals containing their current position and the corresponding exact time. How can the device determine its own position?

Solution:
In principle, the GPS operates as follows: The current positions A_1, B_1, and C_1 of (at least) three satellites are obtained by radio. For a first calculation, a position Q close to the actual position P is chosen (e.g., the last calculated position).
This yields the distances $a_1 = \overline{A_1Q}$, $b_1 = \overline{B_1Q}$, and $c_1 = \overline{C_1Q}$. Now, we intersect these three spheres centered at A_1, B_1, and C_1 and obtain a solution. However, the new positions A_2, B_2, and C_2 of the quickly moving satellites are assumed and the old distances are calculated as a result. This results in a "mixed position" Q_{12}. This, of course, does not agree with Q or P for $P \neq Q$. The distance $d = \overline{QQ_{12}}$ is a measure of the error: the smaller d, the closer the actual position P to Q.
Now, we do the following: We repeatedly carry out the above-described "calculation using a sample" by systematically varying the coordinates of $Q(q_x/q_y/q_z)$ and observing the direction in which the distance d decreases: Then, for example, the values of d are obtained only by varying the x-value. Next, we can find a better q_x-value through relatively coarse interpolation. Now, we change the q_y-value and then the q_z-value to get the result. In the meantime, we will have to include new satellite positions into the calculation in order to account for possible changes in position.

⊕ *Remark*: Optimizing the step-by-step (*iterative*) search for the solution is a very challenging task, but we will not discuss this here. ⊕ ◂◂◂